Colouring complete bipartite graphs from random lists

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Abstract

Let $K_{n,n}$ be the complete bipartite graph with n vertices in each side. For each vertex draw uniformly at random a list of size k from a base set $\mathcal{S}$ of size $s = s(n)$. In this paper we estimate the asymptotic probability of the existence of a proper colouring from the random lists for all fixed values of k and growing n. We show that this property exhibits a sharp threshold for $k \geq 2$ and the location of the threshold is precisely $s(n) = 2n$ for $k = 2$, and approximately $s(n) = \frac{n}{2^{k-1} \ln 2}$ for $k \geq 3$.

1 Introduction

Let G be a simple and undirected graph. Assign to each vertex x of G a set $L(x)$ of colours (positive integers). Such an assignment L of sets to vertices in G is referred to as a *colour scheme* for G. An *L-colouring* of G is a mapping f of $V(G)$ into the set of colours such that $f(x) \in L(x)$ for all $x \in V(G)$ and $f(x) \neq f(y)$ for each $(x, y) \in E(G)$. If G admits an L-colouring, then G is said to be *L-colourable* . In case of $L(x) = \{1, \dots, k\}$ for all $x \in V(G)$, we also use the terms *k-colouring* and *k-colourable* respectively. A graph G is called *k-choosable* if G is L-colourable for every colour scheme L of G satisfying $|L(x)| = k$ for all $x \in V(G)$. The *chromatic number* $\chi(G)$ (*choice number* $ch(G)$) of G is the least integer k such that G is k-colourable (k-choosable). The choosability concept was introduced independently by Vizing [15] and by Erdős, Rubin and Taylor [8].

Assign colour lists to the vertices of G by choosing for each vertex v its colour list, $L(v)$, uniformly at random from all k-subsets of a ground set $\mathcal{S} = \{1 \dots s\}$. Intuitively, the larger is s, the more spread are the colours, and the easier it is to colour G from the chosen random lists. The question is how large should be the value of $s = s(k, G)$ to guarantee the almost sure choosability from random lists.

In [13] the authors solve the problem for the d-th power of a cycle on n vertices, denoted by C_n^d (i.e. the vertices of C_n^d are those of the n-cycle, and two vertices are connected by an edge if their distance along the cycle is at most d). In this case the authors prove that for $d \geq k$, the threshold for choosability occurs at n^{1/k^2} (note that the threshold does not depend on d), i.e. that if the $s(n) = \omega(n^{1/k^2})$ almost surely a proper choice exists and if $s(n) = o(n^{1/k^2})$ almost surely a proper choice does not exist. The threshold in this case is coarse. That is to say that if $s(n) \sim tn^{1/k^2}$ for fixed t then the probability that there exists a proper choice tends to $\phi(t)$, where $\phi(t)$ is an increasing positive function which tends to 0 when t tends to 0, and tends to 1 when t tends to infinity. This

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is achieved in [13] by showing that the choosability property is controlled by a very local property: the appearance of a clique of size $k+1$ with identical lists drawn for each vertex.

In this paper we study the problem for the complete bipartite graph with n vertices in each side, denoted by $K_{n,n}$. Let $\mathcal{S}$ be a set of colours of size $s(n)$. For each vertex of $K_{n,n}$ draw uniformly at random k colours from $\mathcal{S}$ to form a colour scheme $L(k,s)$. Denote by $p(n) = p(n,k,s)$ the probability that $K_{n,n}$ is $L(k,s)$-colourable from the random lists. For $k=1$ it is an easy exercise to check that the choosability property has a coarse threshold at $s(n) = n^2$. We prove the following:

Theorem 1.1 *For $k=2$ and any $\epsilon > 0$:*

$$p(n) = \begin{cases} o(1), & s(n) \leq (2-\epsilon)n, \\ 1-o(1), & s(n) \geq (2+\epsilon)n. \end{cases}$$

And for $k \geq 3$ and any $\epsilon > 0$:

$$p(n) = \begin{cases} o(1), & s(n) \leq \frac{(1-\epsilon)n}{2^{k-1}\ln 2}, \\ 1-o(1), & s(n) \geq \frac{(1+\epsilon)n}{2^{k-1}\ln 2-(k+1)\ln 2-1}. \end{cases}$$

We shall also use the standard asymptotic notation and assumptions. In particular we assume that the parameter n is large enough whenever necessary. For two functions $f(n)$ and $g(n)$, we write $f = o(g)$ if $\lim_{n\to\infty} f/g = 0$, and $f = \omega(g)$ if $g = o(f)$. Also, $f = O(g)$ if there exists an absolute constant $c > 0$ such that $f(n) < cg(n)$ for all large enough n; $f = \Theta(g)$ if both $f = O(g)$ and $g = O(f)$ hold; and $f \sim g$ if $\lim_{n\to\infty} f/g = 1$.

2 The general setting

We shall use the standard probability spaces of random hypergraphs. Denote by $H_k(n,p)$ the space of the k-uniform hypergraphs on n vertices where each of the $\binom{n}{k}$ edges is selected independently with probability p, and by $H_k(n,m)$ the space of all the k-uniform hypergraphs on n vertices and m edges uniformly distributed.

It is convenient to consider the following equivalent formulation: Given two independent k-uniform random hypergraphs, H_1 and H_2, distributed as $H_k(s(n),n)$ on the same vertex set, $[s(n)]$, then $p(n,k,s)$ is the probability that H_1 has a vertex cover (a set of vertices such that every edge contains at least one vertex from the set) disjoint from a vertex cover of H_2. To see the equivalence, simply let the vertex set be $\mathcal{S}$, the edges of H_1 be the lists of the left side of $K_{n,n}$ and similarly, the edges of H_2 be the lists of the right side of $K_{n,n}$. $K_{n,n}$ has a proper colouring from the lists iff $H_1 \cup H_2$ has such disjoint covers. We colour the graph by picking for each vertex on the left side a colour present in the corresponding edge in H_1 which is also a member of H_1's cover and similarly for vertices on the right side. This justifies the following definition:

Definition: A pair of hypergraphs on the same vertex set is said to admit *disjoint covers* if there exist two disjoint vertex covers, one for each hypergraph.

We shall often refer to edges of H_1 as *red*, to edges of H_2 as *blue* and to disjoint covers as a function $\sigma : [n] \to \{red, blue\}$ such that for every edge, $e \in H_1 \cup H_2$, there exists $v \in e$ such that $\sigma(v)$ is of the same colour as the e (note that there are many possible functions).

It is also worth noting that this problem is equivalent to the random k-SAT problem (see, e.g., [2] and its references) with the restriction that every clause must have its literals with the same

sign. To see the equivalence, simply let the variables be the set of vertices (originally $\mathcal{S}$), let every red edge be a clause with positive literals and let every blue edge be a clause with negative literals. Clearly disjoint covers exists iff the formula is satisfiable.
The following theorems clearly imply Theorem 1.1:

Theorem 2.1 *For every fixed $k \geq 2$, let H_1, H_2 be two random k-uniform random hypergraphs distributed independently as $H_k(n, m(n))$, denote their union by $H_{1,2}(n, m(n))$. Then the property of having disjoint covers has a sharp threshold. That is to say that for every $0 < C < D < 1$, and for every $\epsilon > 0$, if $e : \mathbb{N} \to \mathbb{N}$ is a function such that*

$$D > Pr[H_{1,2}(n, e(n)) \text{ admits disjoint covers}] > C,$$

then if $m(n) \leq (1 - \epsilon)e(n)$, $H_{1,2}(n, m(n))$ admits disjoint covers almost surely, and if $m(n) \geq (1 + \epsilon)e(n)$, $H_{1,2}(n, m(n))$ admits no disjoint covers almost surely.

Theorem 2.2 *Let G_1, G_2 be two random graphs on the vertex set $[n]$, distributed independently as $G(n, m(n))$. Then for any $\epsilon > 0$:*

$$Pr[G_1 \text{ and } G_2 \text{ admit disjoint covers}] = \begin{cases} 1 - o(1), & m(n) \leq (\frac{1}{2} - \epsilon)n, \\ o(1), & m(n) \geq (\frac{1}{2} + \epsilon)n. \end{cases}$$

Theorem 2.3 *For every fixed $k \geq 3$, let H_1, H_2 be two random k-uniform random hypergraphs distributed independently as $H_k(n, m(n) = rn)$. Then if $r > 2^{k-1} \ln 2$, H_1 and H_2 admit no disjoint covers almost surely.*
Furthermore, there exists some $C = C(k)$ such that $C > 0$ and if $r < 2^{k-1} \ln 2 - (k+1) \ln 2 - 1$ then:

$$Pr[H_1 \text{ and } H_2 \text{ admit disjoint covers}] > C.$$

An immediate corollary of Theorem 2.1 and Theorem 2.3 is:

Corollary 2.4 *For every $r < 2^{k-1} \ln 2 - (k+1) \ln 2 - 1$, if $m(n) \leq rn$, then H_1 and H_2 admit disjoint covers almost surely.*

The rest of the paper is organized as follows: In Section 3 we prove the sharpness of threshold result (Theorem 2.1). In section 4 we discuss the problem in the case $k = 2$ and prove Theorem 2.2, and in section 5 we prove Theorem 2.3.

3 Sharpness of threshold

We follow the approach outlined in Friedgut's survey on the sharp threshold phenomenon (see [11]). We prove the sharpness result assuming H_1 and H_2 are distributed independently as $H_k(n, p)$. Our probability space is $\Omega = \{0, 1\}^N$, $N = 2\binom{n}{k}$, in which the first half of the coordinates represents the edges of H_1 and the second half represents the edges of H_2.

A subset A of $\{0, 1\}^N$ is called monotone increasing if whenever $x \in A, x' \in \{0, 1\}^N, x_i \leq x'_i$ for $i = 1, \ldots, N$ then $x' \in A$. For $0 \leq p \leq 1$, define μ_p, a product measure on $\{0, 1\}^N$ with weights $1 - p$ at 0 and p at 1. That is to say:

$$\mu(x) = p^{|x|}(1 - p)^{N - |x|} \text{ where } |x| = |1 \leq i \leq n : x_i = 1|.$$

Denote by $H(n,p)$ a random element of Ω equipped with μ_p, and let $A \subset \Omega$ be monotone increasing. For every n define $f_n(p) = Pr[H(n,p) \in A]$. For every n, f_n can easily be seen to be a monotone increasing polynomial in p. For every $\alpha \in (0,1)$ define the function $p_\alpha(n)$ as the unique value for which $f_n(p_\alpha(n)) = \alpha$.

In this context we say that a property, $A \subset \{0,1\}^N$, has a sharp threshold if for every $\alpha \in (0,1)$, every $\delta > 0$ and every $p = p(n)$,

$$\lim_{n\to\infty} Pr[H(n,p(n)) \in A] = \begin{cases} 1, & p(n) \geq (1+\delta)p_\alpha(n), \\ 0, & p(n) \leq (1-\delta)p_\alpha(n). \end{cases}$$

Since the property of not having disjoint covers is monotone, standard calculations (see, for example, [12]) show that this implies a sharp threshold for when the distribution is $H_k(n,m(n))$ thus implying Theorem 2.1.

The sharpness of threshold follows from the condition that for all $\alpha \in (0,1)$ and for all $C > 0$ exists n_0 such that for all $n > n_0$, $p_\alpha(n) f_n'(p_\alpha(n)) > C$. Indeed, for every $0 < \epsilon < 1-\alpha$, and for every $C > 0$, there exists $p^*(n)$ such that $p_\alpha(n) < p^*(n) < p_{1-\epsilon}(n)$ and,

$$1-\epsilon-\alpha = f_n(p_{1-\epsilon}(n)) - f_n(p_\alpha(n)) = f_n'(p^*(n))(p_{1-\epsilon}(n) - p_\alpha(n)) > C - \frac{Cp_\alpha(n)}{p_{1-\epsilon}(n)},$$

thus, $\frac{p_\alpha(n)}{p_{1-\epsilon}(n)} = 1 - o(1)$, hence for every $\delta > 0$ exists n_0 such for all $n > n_0$, $(1+\delta)p_\alpha(n) > p_{1-\epsilon}$, and thus $f_n((1+\delta)p_\alpha(n)) = 1 - o(1)$, as required. The second part of the requirement of a sharp threshold is proven similarly.

We will assume by contradiction that there exists $\alpha \in (0,1)$ such that $p_\alpha(n) f'(p_\alpha(n)) < C$ for some C. Our proof relies on the following result of Bourgain [10], which provides a sharp-threshold criteria for general monotone properties.

Theorem 3.1 (Bourgain) *Let $A \subset \{0,1\}^n$ be a monotone property, and $C > 0$ constant. Assume that there exists $\alpha \in (0,1)$ such that $\mu_p(A) = \alpha$, $p\frac{d\mu_p(A)}{dp} < C$ and $p = o(1)$, then there is $\delta = \delta(C) > 0$ such that either:*

$$\mu_p(x \in \{0,1\}^n : x \text{ contains } x' \in A \text{ of size } |x'| \leq 10C) > \delta,$$

or there exists $x' \notin A$ of size $|x'| \leq 10C$ such that the conditional probability satisfies:

$$\mu_p(x \in A | x' \subset x) > \alpha + \delta.$$

The idea of the proof of Theorem 2.1 is as follows. Assuming the threshold for disjoint covers is coarse (i.e., not sharp), we have by Theorem 3.1 that there exists a fixed set of edges, M (some red and some blue), whose addition changes the probability of disjoint covers by some positive constant, δ. Since the property of having disjoint covers is symmetric with respect to hypergraph automorphisms, it can easily be shown that adding a random copy of M has the same effect. On the other hand, the fact that the threshold is coarse implies that the addition of a large number of random edges has almost no effect on the existence of disjoint covers. We will use the Erdős-Simonovits Theorem [9] to show that these two conclusions contradict each other.

Proof of Theorem 2.1. Our probability space is $\Omega = \{0,1\}^{2\binom{n}{k}}$, equipped with the probability measure μ_p defined earlier. The first half of the coordinates represents H_1 and the second half represents H_2. The property $A \subset \Omega$ is all the pairs (H_1, H_2) with no disjoint covers.

Assume that the theorem does not hold, then by the above discussion, there exist constants α and C as in Theorem 3.1. Easy first moment calculations done in Section 5 show that if $p = p(n)$ is such that $\mu_p(A) = \alpha$ then $p(n) = O(n^{-k+1})$. Within this range of probabilities, it is an easy exercise to see that for every set of vertices, $B \subset [n]$ of size $|B| \leq 10kC$, the edges spanned on these vertices have disjoint covers almost surely (for instance, for $k = 2$ this can only be a union of trees and unicyclic components and for $k > 2$ check that almost surely for any subhypergraph exist a vertex of degree at most 1). Thus, the first option of Theorem 3.1 does not hold, therefore the second option must hold, i.e., there exists some $M \in \Omega - A$ of size $|M| \leq 10C$ such that $Pr[H(n,p) \in A | M \subset H(n,p)] > \alpha + \delta$. Since A is invariant under hypergraph vertex-automorphisms, the latter holds for any isomorphic copy M' of M. Because of this, if we first draw a random copy of M, from all possible ones on the vertex set $[n]$, and then draw $H \in \Omega$ their union will be in A with probability $> \alpha + \delta$ (note that the probability space here is the product space of Ω equipped with μ_p and the space of all isomorphic copies of M on the vertex set $[n]$, uniformly distributed). Now, since $p\frac{d\mu_p(A)}{dp} < C$, $\lim_{\epsilon \to 0} \frac{\mu_{p+\epsilon p}(A) - \mu_p(A)}{\epsilon} < C$, implying that there exists $\epsilon > 0$ such that $\mu_{p+\epsilon p}(A) < \alpha + \frac{\delta}{2}$. Furthermore, the usual double exposure routine shows that Ω equipped with the probability measure $\mu_{p+\epsilon p}$ is probabilistically isomorphic to the union of two independent copies of hypergraphs drawn from Ω: one with probability measure μ_p and another with $\mu_{\epsilon' p}$ for $\epsilon' = \epsilon'(\epsilon) > 0$ satisfying $(1 - (1+\epsilon)p) = (1-p)(1-\epsilon' p)$. Denote by M^* a uniformly random copy of M drawn from all possible copies on the vertex set $[n]$. To sum things up, we have:

$$Pr[H(n,p) \cup M^* \in A] > \alpha + \delta \tag{1}$$

$$Pr[H(n,p) \cup H(n,\epsilon' p) \in A] < \alpha + \frac{\delta}{2} \tag{2}$$

Now, from (1) and (2) we have that:

$$\sum_{H_0 \in \Omega} Pr[H(n,p) = H_0](Pr[H_0 \cup M^* \in A] - Pr[H_0 \cup H(n,\epsilon' p) \in A]) > \frac{\delta}{2}$$

Therefore, exists a fixed $H_0 \in \Omega$ such that:

$$Pr[H_0 \cup M^* \in A] - Pr[H_0 \cup H(n,\epsilon' p) \in A] > \frac{\delta}{2}. \tag{3}$$

From this obviously, H_0 admits disjoint covers. We show that this leads to a contradiction. Order the vertices of M arbitrarily. Denote by $t \leq 10kC$ the number of vertices spanned by the edges of M. Call an ordered t-tuple of vertices $(v_1, \ldots, v_t)$ in $[n]$ *bad* if the addition of an ordered copy of M on these vertices to H_0 results in a hypergraph with no disjoint covers. Clearly, by (3) at least $\frac{\delta}{2}$ fraction of the n^t ordered t-tuples of $[n]$ are bad. We now use a small variation on a theorem of Erdős and Simonovits [9], which differs only in that it deals with ordered t-tuples instead of sets of size t. Let T be a family of ordered t-tuples of distinct elements of $[n]$. An ordered t-tuple $(A_1, \ldots, A_t)$ of disjoint subsets of $[n]$ is called *T-complete* if for every choice of $v_i \in A_i$, $1 \leq i \leq t$, the resulting ordered t-tuple, $(v_1, \ldots, v_t)$, belongs to T.

Theorem 3.2 (Erdős, Simonovits) *For every positive integers k and t and $0 < \gamma \leq 1$ there exists $\gamma' > 0$ such that for sufficiently large n, if $T \subset [n]^t$ is such that $|T| > \gamma n^t$ then with probability at least γ' a random choice of t k-tuples from $[n]$ is T-complete.*

Now, we shall get a contradiction from (3). Draw uniformly at random t red edges (from H_1), $e_1, \ldots, e_t$, and then another t blue edges (from H_2), $e_{t+1}, \ldots, e_{2t}$. Define T to be the set of all bad t-tuples of $[n]$. By Theorem 3.2, applied with integers $2k$ and t, with probability $> \gamma'$ $(e_1 \cup e_{t+1}, e_2 \cup e_{t+2}, \ldots, e_t \cup e_{2t})$ is T-complete. Therefore, if we draw $w(n) \to \infty$ red and blue edges, we will get with probability tending to 1 at least one occurrence of such $2t$ edges such that $(e_1 \cup e_{t+1}, e_2 \cup e_{t+2}, \ldots, e_t \cup e_{2t})$ is T-complete. Also note that almost surely $H_{\epsilon' p}$ will indeed have $w(n) \to \infty$ red and blue edges drawn, and conditioning on this event, the edges are uniformly distributed.

Therefore, all that is left to show is that such a configuration implies the non-existence of disjoint covers. Assume otherwise, and let $\sigma : [n] \to \{red, blue\}$ be disjoint covers of $H_0 \cup e_1 \cup \ldots \cup e_{2t}$. Denote the vertices of M by $V(M) = \{m_1, \ldots, m_t\}$. M admits disjoint covers, $\chi : V(M) \to \{red, blue\}$. Note that for each $1 \leq i \leq t$ there exists a vertex $v_i \in e_i \cup e_{t+i}$ such that $\sigma(v_i) = \chi(m_i)$, thus adding an ordered copy of M on $(v_1, \ldots, v_t)$ to H_0 will leave the disjoint covers σ intact, contradicting the fact $(v_1, \ldots, v_t)$ is a bad t-tuple. Thus, it follows that if $Pr[H_0 \cup M^* \in A] > \frac{\delta}{2}$ then $Pr[H_0 \cup H_{\epsilon' p} \in A] = 1 - o(1)$, contradicting (3).

□

4 Case $k = 2$

In this section, we will denote by $G = G_1 \cup G_2$ our coloured random graph where the edges of G_1 are red and the edges of G_2 are blue. Our probability space is as usual, $\Omega = \{0,1\}^{2\binom{n}{2}}$, equipped with the probability measure μ_p defined earlier. The first half of the coordinates represents the red edges and the second half represents the blue edges. We also require the following definition:

Definition: An *even alternating cycle* in G is a sequence of an even number of vertices, $v_1, \ldots, v_{2t}$ such that $\{v_1, v_2\}$ is a red edge, $\{v_2, v_3\}$ is a blue edge and so on, $\{v_{2t}, v_1\}$ is a blue edge.
An *odd alternating cycle* in G is a sequence of an odd number of vertices $v_1, \ldots, v_{2t+1}$ such that $\{v_1, v_2\}$ is a red edge, $\{v_2, v_3\}$ is a blue edge and so on, $\{v_{2t}, v_{2t+1}\}$ is a blue edge and $\{v_{2t+1}, v_1\}$ is a red edge. An *alternating path* in G is a sequence of vertices $v_1, \ldots, v_p$ such that $\{v_1, v_2\}$ is a red (blue) edge, $\{v_2, v_3\}$ is a blue (red) edge and so on.

In the proof of Theorem 2.2 we will see that the appearance of large alternating cycles causes the disappearance of disjoint covers. In fact, we will find an exact structure which determines the existence of disjoint covers in this case.

Proof of Theorem 2.2. We will prove the theorem assuming G_1 and G_2 are distributed $G(n,p)$ where $p(n) = \frac{m(n)}{\binom{n}{2}}$. Since the property of having disjoint covers is monotone, standard calculations (see [12]) show that this implies the theorem. The theorem consists of two parts. We first prove that if $p(n) = \frac{c}{n}$, such that c is any fixed constant satisfying $c < 1$, then disjoint covers exist almost surely. We will present two simple proofs for this fact, an indirect proof using Theorem 2.1, and a direct proof (which also leads to a polynomial algorithm for finding such disjoint covers). We begin with the indirect proof.

An important observation is that if G has no alternating cycles, then every subgraph of G has a vertex, v, such that all edges containing v have the same colour. Thus we can build disjoint covers using a greedy approach: at each step choose such v and colour it with the colour of the edges containing it. Clearly this procedure results in disjoint covers. We will show that with probability larger than some constant, $C > 0$, G has no alternating cycles and thus, by Theorem 2.1 we will get Theorem 2.2. One approach is to show that the number of vertices that participate in alternating cycles is distributed asymptotically Poisson with constant expectation, $\mu > 0$. Thus, the probability that there exist no alternating cycles is approximately $e^{-\mu}$. We choose to use the FKG-inequality (see, e.g., [4]) which we quote, phrasing it according to our needs.

Theorem 4.1 *Let $\mathcal{A} \subset \Omega$ and $\mathcal{B} \subset \Omega$ be two monotone subsets of Ω. Then:*

$$Pr[\mathcal{A} \cap \mathcal{B}] \geq Pr[\mathcal{A}]Pr[\mathcal{B}].$$

First, for any l denote by X_l the random variable counting the number of alternating cycles of length l in G. For any odd l, denote by $C_1, \ldots, C_m \in \Omega$ all the possible odd alternating cycles of length l. For every $1 \leq i \leq m$ define $A_i = \{G \in \Omega : C_i \not\subset G\}$, i.e., $A_i \subset \Omega$ is the set of all graphs not containing the cycle C_i. Easily, since for any l vertices there are $\frac{1}{2}(l-1)!$ possible cycle orderings, and for each $2l$ possible edge colourings to make it an odd alternating cycle,

$$m = \binom{n}{l} l!.$$

Furthermore, observe that $\{X_l = 0\} = \bigcap_{1 \leq i \leq m} A_i$ and that A_i is a decreasing monotone subset of Ω for each i. Thus, using Theorem 4.1 m times gives:

$$Pr[X_l = 0] \geq \left[1 - \left(\frac{c}{n}\right)^l\right]^m.$$

We use the easily verifiable facts that for any $x \in [0, 1/2]$, $1 - x \geq e^{-4x}$, for any natural $n > l$ $\binom{n}{l} \leq \frac{n^l}{l!}$ to get:

$$Pr[X_l = 0] \geq e^{-4\binom{n}{l} l! \left(\frac{c}{n}\right)^l} \geq e^{-4c^l}.$$

Also, a similar calculation shows that also $Pr[X_l = 0] \geq e^{-4c^l}$ for any even l.

Now, observe that the subset $\{X_l = 0\}$ is a monotone decreasing subset of Ω for every l. Thus using Theorem 4.1 n times,

$$\begin{aligned} Pr[\text{G admits disjoint covers}] &\geq Pr\left[\bigcap_{l=1}^{n} \{X_l = 0\}\right] \geq \prod_{l=1}^{n} Pr[X_l = 0] \\ &\geq e^{-\sum_{l=1}^{n} 4c^l} \geq e^{-\frac{4}{1-c}} > 0, \end{aligned}$$

which concludes the existential proof.

For the direct proof we find an exact structure in G which determines the existence of disjoint covers. This will imply a polynomial algorithm for finding disjoint covers for $k = 2$ of which we omit the details (for $k > 2$, deciding whether two hypergraphs have disjoint covers is NP-hard). In [5] the authors give a necessary and sufficient condition for a 2-SAT formula to be satisfiable. Since, as we mentioned in Section 2, our problem is equivalent to a certain random 2-SAT problem, we are looking for the translation of this condition to our problem.

Definition: An *odd bicycle* in G consists of two disjoint odd alternating cycles, $v_1, \ldots, v_p$ and $u_1, \ldots, u_q$ (q, p odd) such that the edges $\{v_1, v_2\}$, $\{v_{p+1}, v_1\}$ are in the same colour, and the edges $\{u_1, u_2\}$, $\{u_q, u_1\}$ are in the same colour, and one alternating path $v_1, w_1, \ldots, w_r, u_1$ such that the edges $\{v_1, v_2\}$, $\{v_1, w_1\}$ are of distinct colours, and the edges $\{u_1, u_2\}$, $\{w_r, u_1\}$ are of distinct colours as well (the path's edges are not necessarily disjoint from the cycles' edges).

Proposition 4.2 *G admits disjoint covers if and only if G does not contain an odd bicycle.*

Proof.
It is easy to check that an odd bicycle admits no disjoint covers. It is left to prove that if G admits no disjoint covers then it has an odd bicycle. Let G be a minimal counterexample with respect to containment. Pick an arbitrary vertex $v \in V(G)$, and let $\sigma : V \to \{red, blue\}$ be disjoint covers of $E(G - v)$.

A simple path P in G starting at v is called a *Red Alternating Path*, or RAP for brevity, if: 1) the colour of the edges of P alternates starting from a red edge containing v; 2) the colour of the vertices along $P - v$ given by σ alternates starting from a blue neighbour of v. Let V_1 be a set of vertices of $G - v$ reachable from v by a RAP. We define a new colouring σ' by colouring v in blue and flipping the colours of the vertices of V_1. By our assumption on G, σ' does not define disjoint covers. Therefore there exists an edge $e = (u_1, u_2)$ missed by its corresponding colour in σ'. Obviously, at least one of the endpoints of e is in V_1, say, it is u_1. Let P_1 be a RAP from v to u_1. Observe that the colour of e should coincide with $\sigma(u_1)$. Then, unless $u_2 \in P_1$, we can extend P_1 to a RAP ending at u_2 by adding e. This shows that there is a RAP P containing both u_1 and u_2, implying in particular $u_1, u_2 \in V_1$. Since e is uncovered under σ', the colours of u_1, u_2 in σ should be identical – and coincide with that of e. But then it follows that a union of P and e contains an odd alternating cycle with an alternating path connecting it to v, where the edge of the path containing v is red.

The same argument, with red and blue interchanged, shows that G should contain an alternating cycle connected to v by an alternating path whose last edge is blue. We thus reached a contradiction. □

All that is left now is to show that when $c < 1$ an odd bicycle does not exist almost surely. We use easy first moment calculations. Denote by Y the r.v. counting the number of odd bicycles. Then,

$$E[Y] \leq \sum_{q>2, p>2, r\geq 0} \binom{n}{q}\binom{n}{p}\binom{n}{r} p!q!r!2\left(\frac{c}{n}\right)^{q+p+r+1},$$

where p, q indicate the numbers of vertices in the odd alternating cycles, and r indicates the size the alternating path connecting them. Note that after choosing the cycles' ordering, the path's ordering and for each cycle the unique vertex which is contained in two edges of the same colour, there are exactly two edge colourings available. Evaluating this sum gives,

$$
\begin{aligned}
E[Y] &\le \frac{2c}{n}\left(\sum_{i=0}^{n}\binom{n}{i} i! \left(\frac{c}{n}\right)^i\right)^3 \le \frac{2c}{n}\left(\sum_{i=0}^{n} n^i \left(\frac{c}{n}\right)^i\right)^3 \\
&\le \frac{2c}{n}\left(\frac{1}{1-c}\right)^3 = o(1).
\end{aligned}
$$

Thus, when $c < 1$ almost surely there exists no odd bicycles, and therefore, disjoint covers exists almost surely.

We now prove the second part of Theorem 2.2. The crux of the proof is to show that when $c > 1$ almost surely there exists an alternating cycle (not necessarily a simple cycle) of length $\omega(\sqrt{n})$. The theorem follows from the observation that by the double exposure routine, this is the same as drawing first G_1 and G_2 distributed $G(n, \frac{c'}{n})$ where $1 < c' < c$ and then adding red and blue random edges distributed $G(n, \frac{\epsilon}{n})$ for some $\epsilon > 0$ small enough. Now, after the first draw we have almost surely an alternating cycle of length $\omega(\sqrt{n})$. The second draw ensures us $\omega(1)$ random chords on that cycle. It is also easy to observe that $\omega(1)$ random chords on an alternating cycle results almost surely in an odd bicycle, and thus no disjoint covers exist.

To reach an optimal result, perhaps it is best to follow the classical paper of Ajtai, Komlós and Szemerédi [3] in which, using branching process techniques, they analyze the DFS algorithm on the random graph $G(n, p = \frac{c}{n})$ where $c > 1$. This analysis leads to the existence of a cycle of linear size. It can be shown that a slight change in their proof, in particular drawing red/blue edges alternately, results in an alternating cycle also of linear size. However, for the sake of brevity, we will show by easier arguments the existence of an alternating cycle of size $\Omega(\frac{n}{\ln^8 n})$ using a theorem of McDiarmid [14]. The following theorem will conclude the proof:

Theorem 4.3 *If $c > 1$, G contains almost surely an alternating cycle of size $\Omega(\frac{n}{\ln^8 n})$.*

Proof.

We associate with our $G = G_1 \cup G_2$ a new auxiliary directed bipartite graph $H(V, E(H))$ where $V = A \uplus B$, $|A| = |B| = n$, $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. Furthermore, $(a_i, b_j) \in E(H)$ iff $\{i, j\}$ is a red edge in G and $(b_i, a_j) \in E(H)$ iff $\{i, j\}$ is a blue edge in G. Observe that a directed cycle in H corresponds to an even alternating cycle in G (again, not necessarily a simple cycle. In fact, we construct a cycle in which the number of edges is $\Omega(\frac{n}{\ln^8 n})$, but it is a simple large deviations exercise to see that almost surely vertex sets of size t span at most $O(t)$ edges in G. Thus the obtained cycle spans $\Omega(\frac{n}{\ln^8 n})$ vertices and has therefore enough potential chords to be hit at the second stage of the double exposure procedure.). Also, note that every ordered pair (a_i, b_j) or (b_i, a_j), for $i \neq j$, is drawn independently with probability p.

Denote by $B(n, n, p)$ the space of random bipartite graphs, with n vertices in each side, such that each edge appears with probability p independently of other edges. By $\vec{B}(n, n, p)$ denote the corresponding space of random directed bipartite graphs such that each **directed** edge appears with probability p independently of other edges. We also denote by $\{v_1, \ldots, v_n\}$ and $\{u_1, \ldots, u_n\}$ the corresponding sides.

Now, clearly, if we condition on the event that no edges of the form (v_i, u_i) or (u_i, v_i), $1 \le i \le n$ exist in $\vec{B}(n, n, p)$ we get a graph distributed precisely as $H(V, E(H))$. It is also easy to show that the probability of such an event tends to e^{-2c}, thus, it is enough to show that $\vec{B}(n, n, p)$ has a directed cycle of length $\Omega(\frac{n}{\ln^8 n})$ almost surely. We will in fact show that $B(n, n, p)$ has an **undirected** cycle of that length almost surely and then use a theorem of McDiarmid [14] to attain the same for $\vec{B}(n,n,p)$. We restate McDiarmid's result, phrasing it in according to our needs (the theorem is far more general).

Theorem 4.4 (McDiarmid) *Let Q be a set of directed bipartite graphs with n vertices in each side that satisfies the following two conditions:*

1. *Q is an anti-chain, i.e., for any $D, D' \in Q$, $D \not\subset D'$ and $D' \not\subset D$.*

2. *None of the graphs in Q contains opposing edges, i.e. (u,v) and (v,u).*

Then,

$$Pr[\vec{B}(n,n,p) \text{ contains a subgraph from } Q] \geq Pr[B(n,n,p) \text{ contains a subgraph from } Q].$$

It is easy to observe that the set $Q = \{\text{all cycles of length} > \Omega(\frac{n}{\ln^8 n})\}$ satisfies the conditions of Theorem 4.4, and therefore, we are left to prove that $B(n,n,p)$ has an undirected cycle of length $\Omega(\frac{n}{\ln^8 n})$.

Proposition 4.5 *Let $G = (V,E)$ be a connected graph on m vertices with maximum degree at most k. Then for every natural l, there exist disjoint vertex sets $V_1, \ldots, V_t \subset V$, with the following properties:*

1. *$lk \leq |V_i| \leq lk^2$ for every $1 \leq i \leq t$.*

2. *$\sum_{i=1}^{t} |V_i| \geq m - lk$.*

3. *$G[V_i]$ is connected for every $1 \leq i \leq t$.*

Proof. By induction on m. For $lk \leq m \leq lk^2$ we take the whole graph to be V_1. For $m > lk^2$, we choose an arbitrary vertex v and build from it a BFS tree in which v is the root. For every vertex w, denote by $D(w)$ the set of w's descendants in the BFS tree. We claim that there exists w such that $lk \leq |D(w)| \leq lk^2$. Otherwise, choose w such that $|D(w)| \geq lk$ and is minimal, it follows by our assumption that $|D(w)| > lk^2$. Since the maximum degree is no more than k, w has no more than k direct children. Thus, since $|D(w)| > lk^2$, one of w's children, w', must have $|D(w')| \geq lk$ and of course $|D(w')| < |D(w)|$, thus contradicting the minimality of $|D(w)|$. Now, we take a w such that $lk \leq |D(w)| \leq lk^2$ and take $V_1 = D(w)$, then V_1 is clearly connected . We remove $D(w)$ from the graph and since the remaining graph is still connected, we apply the induction hypothesis to it. $\square$

It is a well known fact that if $p = \frac{c}{n}$, where $c > 1$, then $B(n,n,p)$ almost surely has a connected component of size $> Dn$, (often referred to as the *giant component*) where $D = D(c) > 0$ is constant. It is also very easy verify that for such p, the maximum degree of $B(n,n,p)$ is almost surely no more than $\ln n$. Let $H \in B(n,n,p)$ be such a graph, denote its sides by A, B ($|A| = |B| = n$) and apply Proposition 4.5 to its giant component with $k = \ln n$ and $l = ln^6 n$. Thus we have $V_1, \ldots, V_t$, such that $\ln^7 n \leq V_i \leq \ln^8 n$, $H[V_i]$ is connected for every $1 \leq i \leq t$ and $\sum_{i=1}^{t} |V_i| \geq Dn - \ln^7 n$. Also, $|V_i \cap A| > \ln^5 n$ and $|V_i \cap B| > \ln^5 n$ since otherwise, recalling that the maximum degree is at most $\ln n$, we would have $|V_i| \leq \ln^5 n + \ln^6 n < \ln^7 n$, for large enough n.

We use now the usual double exposure routine. We first draw $B(n,n,p = \frac{c'}{n})$ where $1 < c' < c$, then $B(n,n,p = \frac{\epsilon}{n})$ for $\epsilon = \epsilon(c,c') > 0$ satisfying $(1 - \frac{c}{n}) = (1 - \frac{c'}{n})(1 - \frac{\epsilon}{n})$. Their union is distributed precisely $B(n,n,p = \frac{c}{n})$. After the first draw, we get almost surely such $V_1, \ldots, V_t$ as before. After the second draw, define a new auxiliary graph $\tilde{H}$ with $V(\tilde{H}) = \{V_1, \ldots, V_t\}$ and an edge $\{V_i, V_j\} \in E(\tilde{H})$ for $1 \le i \ne j \le t$ iff in the second draw we have drawn an edge between a vertex in V_i and a vertex in V_j. Clearly, $\tilde{H}$ is a random graph on no more than $\frac{Dn}{\ln^7 n}$ vertices, and since each set V_i has $\ln^5 n$ vertices on each side, the edge probability, $\tilde{p}$, satisfies $1-\tilde{p} = \left(1 - \frac{\epsilon}{n}\right)^{\ln^{10} n}$ implying $\tilde{p} > \frac{\epsilon \ln^{10} n}{4n} \gg \frac{\ln V(\tilde{H})}{V(\tilde{H})} \sim \frac{\ln^9 n}{n}$. But, within this probability range, $\tilde{H}$ has a Hamilton cycle almost surely (see, e.g., [6]). Since $\tilde{H}$ has at least $\frac{Dn - \ln^7 n}{\ln^8 n}$ vertices, and since $H[V_i]$ is connected for every $1 \le i \le t$ this translates to a cycle in the original graph, H, of length at least $\frac{Dn - \ln^7 n}{\ln^8 n}$, thus concluding our proof. $\square$

5 Case $k \ge 3$

In [2], Achlioptas and Peres use a clever refinement of the second moment method to estimate the probability that there exists a satisfying assignment to a random k-CNF formula.

We imitate Achlioptas and Peres' proof to prove Theorem 2.3. In fact, although our proof is only slightly different, we have not found a way to use Achlioptas and Peres' result to obtain Theorem 2.3. For the sake of completeness, we repeat the proof here.

Proof of Theorem 2.3.

To prove the first part of Theorem 2.3, let X be the r.v. counting the number of disjoint covers of $H = H_1 \cup H_2$, each drawn from $H_k(n, m = rn)$ for some constant $r > 0$. For a fixed partition $A \uplus B = [n]$, where $|A| = a$ and $|B| = b = n - a$, the probability that these are disjoint covers is clearly:

$$\left(1 - \frac{\binom{a}{k}}{\binom{n}{k}}\right)^{rn} \left(1 - \frac{\binom{b}{k}}{\binom{n}{k}}\right)^{rn}.$$

As $\binom{y}{k}$ is convex, this expression is maximized when $a = b = \lfloor \frac{n}{2} \rfloor$, which is then approximately, $\left(1 - \frac{1}{2^k}\right)^{2rn}$. Thus,

$$E[X] \le \sum_{A \uplus B = [n]} Pr[A \uplus B \text{ are disjoint covers}] \le \left[2\left(1 - \frac{1}{2^k}\right)^{2r}\right]^n;$$

So, if $2\left(1 - \frac{1}{2^k}\right)^{2r} < 1$ then $E[X] = o(1)$ and almost surely, no disjoint covers exist. This implies that for $r > 2^{k-1} \ln 2$, no disjoint covers exist almost surely.

Now we prove the second part of the Theorem. We will assume during the proof that n is a large enough even integer. To get the theorem for all sufficiently large n, observe that if for r^* and even n, $H_1, H_2 \sim H_k(n, r^* n)$ have almost surely disjoint covers, then for any $r < r^*$ and odd n, $H_1, H_2 \sim H_k(n, rn)$ have almost surely disjoint covers. This is because if we first draw $H_1, H_2 \sim H_k(n+1, r^*(n+1))$ and then delete an arbitrary vertex we get a hypergraph on n vertices, having almost surely $r^* n - o(n)$ random edges that almost surely admits disjoint covers.

A partition $A \uplus B = [n]$ is called *balanced* if $|A| = |B| = n/2$. For a hypergraph $H = H_1 \cup H_2$ denote by $D(H)$ the set of all balanced disjoint covers of H. Similarly, for a blue (red) edge e, denote by $D(e)$ the set of all balanced partitions which colour some vertex of e with blue (red). For a blue edge, e, and a partition, σ, define $W(\sigma,e)$ to be the number of blue vertices in e minus the number of red vertices in e, with respect to σ. Similarly, define $W(\sigma,e)$ for a red edge, e, as the number of red vertices in e minus the number of blue vertices in e, with respect to σ. For a blue (red) edge, e and a vertex $v \in e$, define $W(\sigma,e,v)$ to be 1 if v is coloured blue (red) by σ and -1 otherwise. Note that $W(\sigma,e)$ is an integer between $-k$ and k, and if $e=(v_1,\ldots,v_k)$, then $W(\sigma,e)=\sum_{i=1}^k W(\sigma,e,v_i)$. For constant $0<\gamma\le 1$ to be determined later define the following random variable:

$$X=\sum_{\sigma:A\uplus B=[n]}\left(\prod_{e\text{ red}}\gamma^{W(\sigma,e)}\right)\left(\prod_{e\text{ blue}}\gamma^{W(\sigma,e)}\right)\mathbf{1}_{\{\sigma\in D(H)\}};$$

Note that $X>0$ implies $S(D)\neq\emptyset$, i.e. disjoint covers exist. We will use the following inequality, which easily follows from Chebyschev's inequality: for any non-negative random variable X,

$$Pr[X>0]\ge\frac{E[X]^2}{E[X^2]}.$$

Thus, to prove Theorem 2.3 it is enough to prove that $\frac{E[X]^2}{E[X^2]}>C$ for some fixed C. To simplify the analysis, we draw an edge by drawing uniformly at random k vertices with replacements (it is easy to see that the number of "defect" edges, i.e., edges of size smaller than k is $o(n)$ and hence the same argument used for dealing with odd n will work here as well). We begin by evaluating the first moment.

For any random edge e (either blue or red) and fixed balanced partition σ,

$$E\left[\gamma^{W(\sigma,e)}\mathbf{1}_{\{\sigma\in D(e)\}}\right]=E\left[\gamma^{W(\sigma,e)}\right]-\gamma^{-k}Pr[\mathbf{1}_{\{\sigma\notin D(e)\}}]=\left(\frac{\gamma+\gamma^{-1}}{2}\right)^k-(2\gamma)^{-k}.$$

Preserving the notation in [2], define $\psi(\gamma):=\left(\frac{\gamma+\gamma^{-1}}{2}\right)^k-(2\gamma)^{-k}$. Since edges are drawn independently, it follows that:

$$E[X]=\sum_{\sigma:A\uplus B=[n]}E\left[\left(\prod_{e\text{ red}}\gamma^{W(\sigma,e)}\right)\left(\prod_{e\text{ blue}}\gamma^{W(\sigma,e)}\right)\mathbf{1}_{\{\sigma\in D(H)\}}\right]=2\binom{n}{n/2}\psi(\gamma)^{2rn}.$$

We now evaluate the second moment. Let σ,τ be two fixed, balanced partitions which have in common precisely αn vertices in the first part, and let $e=(v_1,\ldots,v_k)$ be a blue edge. Note that since σ,τ are balanced, both also have in common precisely αn vertices in the second part and $(1-\alpha)n$ vertices on which they do not "agree" upon. Then,

$$E\left[\gamma^{W(\sigma,e)+W(\tau,e)}\right]=\prod_{i=1}^k E\left[\gamma^{W(\sigma,e,v_i)+W(\tau,e,v_i)}\right]=\left[1-2\alpha+\alpha(\gamma^2+\gamma^{-2})\right]^k,$$

$$E\left[\gamma^{W(\sigma,e)+W(\tau,e)}\mathbf{1}_{\{\sigma\notin D(e)\}}\right]=\prod_{i=1}^k E\left[\gamma^{W(\sigma,e,v_i)+W(\tau,e,v_i)}\mathbf{1}_{\{\sigma\text{ colours }v_i\text{ red}\}}\right]=\left[\alpha\gamma^{-2}+\frac{1-2\alpha}{2}\right]^k,$$

$$E\left[\gamma^{W(\sigma,e)+W(\tau,e)}\mathbf{1}_{\{\sigma,\tau\notin D(e)\}}\right]=\prod_{i=1}^k E\left[\gamma^{W(\sigma,e,v_i)+W(\tau,e,v_i)}\mathbf{1}_{\{\sigma,\tau\text{colour }v_i\text{ red}\}}\right]=\alpha^k\gamma^{-2k}.$$

So,

$$\begin{aligned} E\left[\gamma^{W(\sigma,e)+W(\tau,e)}\mathbf{1}_{\{\sigma,\tau\in D(e)\}}\right] &= E\left[\gamma^{W(\sigma,e)+W(\tau,e)}(1-\mathbf{1}_{\{\sigma\notin D(e)\}}-\mathbf{1}_{\{\tau\notin D(e)\}}+\mathbf{1}_{\{\sigma,\tau\notin D(e)\}})\right] \\ &= \left[1-2\alpha+\alpha(\gamma^2+\gamma^{-2})\right]^k - 2\left[\alpha\gamma^{-2}+\frac{1-2\alpha}{2}\right]^k + \alpha^k\gamma^{-2k}. \end{aligned}$$

Denote $f(\alpha) = \left[1-2\alpha+\alpha(\gamma^2+\gamma^{-2})\right]^k - 2\left[\alpha\gamma^{-2}+\frac{1-2\alpha}{2}\right]^k + \alpha^k\gamma^{-2k}$, then,

$$\begin{aligned} E[X^2] &= \sum_{\sigma,\tau} E\left[\left(\prod_{e \text{ red}}\gamma^{W(\sigma,e)+W(\tau,e)}\mathbf{1}_{\{\sigma,\tau\in D(e)\}}\right)\left(\prod_{e \text{ blue}}\gamma^{W(\sigma,e)+W(\tau,e)}\mathbf{1}_{\{\sigma,\tau\in D(e)\}}\right)\right] \\ &= 2\binom{n}{n/2}\sum_{z=0}^{n/2}\binom{n/2}{z}^2 f(z/n)^{2rn}. \end{aligned}$$

To estimate this sum we use a lemma proved in [1] which is based on the Laplace method for asymptotic integrals [7].

Lemma 5.1 *Let ϕ be a positive, twice differentiable function on $[0,1]$ and let $q \geq 1$ be a fixed integer. Let,*

$$S_n = \sum_{z=0}^{n/q}\binom{n/q}{z}^q \phi(zq/n)^n.$$

Letting $0^0 := 1$, define g on $[0,1]$ as

$$g(\alpha) = \frac{\phi(\alpha)}{\alpha^\alpha(1-\alpha)^{1-\alpha}}.$$

If there exists $\alpha_{max} \in (0,1)$ such that $g(\alpha_{max}) := g_{max} > g(\alpha)$ for all $\alpha \neq \alpha_{max}$, and $g''(\alpha_{max}) < 0$, then there exists a constant $C = C(q, g_{max}, g''(\alpha_{max}), \alpha_{max}) > 0$ such that

$$S_n < Cn^{-(q-1)/2}g_{max}^n.$$

We apply Lemma 5.1 with $q = 2$, $\phi(\alpha) = f(\alpha/2)^{2r}$. A considerable piece of [2] is devoted to prove that for all $k \geq 3$, if $r < 2^{k-1}\ln 2 - (\ln 2)(k+1) - \frac{1}{2} - \frac{3}{2k}$, there exists γ such that $\phi(\alpha)$ satisfies the conditions of Lemma 5.1 with $\alpha_{max} = 1/2$ (see Lemma 3 of [2]). Note that since $f(1/4) = \psi(\gamma)^2$, it follows by Lemma 5.1 that,

$$\frac{E[X^2]}{E[X]^2} \leq \frac{2\binom{n}{n/2}Cn^{-1/2}2^n\psi(\gamma)^{4rn}}{4\binom{n}{n/2}^2\psi(\gamma)^{4rn}} \to \frac{C}{2}\sqrt{\frac{\pi}{2}},$$

thus concluding our proof. □

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