Contagious Sets in Expanders

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Abstract

We consider the following activation process in undirected graphs: a vertex is active either if it belongs to a set of initially activated vertices or if at some point it has at least r active neighbors, where $r > 1$ is the activation threshold. A *contagious set* is a set whose activation results with the entire graph being active. Given a graph G, let $m(G,r)$ be the minimal size of a contagious set. It is known that for every d-regular or nearly d-regular graph on n vertices, $m(G,r) \leq O(\frac{nr}{d})$. We consider such graphs that additionally have expansion properties, parameterized by the spectral gap and/or the girth of the graphs.

The general flavor of our results is that sufficiently strong expansion properties imply that $m(G,2) \leq O(\frac{n}{d^2})$ (and more generally, $m(G,r) \leq O(\frac{n}{d^{r/(r-1)}})$). In addition, we demonstrate that rather weak assumptions on the girth and/or the spectral gap suffice in order to imply that $m(G,2) \leq O(\frac{n \log d}{d^2})$. For example, we show this for graphs of girth at least 7, and for graphs with $\lambda(G) < (1-\epsilon)d$, provided the graph has no 4-cycles.

Our results are algorithmic, entailing simple and efficient algorithms for selecting contagious sets.

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1 Introduction

Threshold models in graphs and networks have received much attention in diverse research fields. Typically in such models there is an undirected graph $G = (V, E)$ where every node $v \in V$ has a threshold function $t(v)$. In addition, it is assumed that every node can be in two states: either *active* or *inactive*. An initial set of nodes (termed seeds) is activated. An inactive vertex v becomes active once it has at least $t(v)$ active neighbors. In this work we focus on progressive models: once a vertex is active, it remains active forever.

Threshold models emerge in various settings such as brain modeling, diffusion of innovation, ideas, and trends in social networks as well as resilience to cascading failures in financial networks, power grids and communication networks [17, 34, 29, 39, 43]. Within computer science, the rising popularity of social media has resulted in much interest in various optimization problems related to cascading behavior in networks [25, 34, 38].

We shall focus on threshold models where every vertex has the same threshold r. Such activation rules, which are often referred to as *bootstrap percolation*, have been introduced in statistical physics settings [21] (a note regarding terminology. The term *bootstrap percolation* is sometimes used with the implicit assumption that the set of seeds is random. In this paper we use this term also when the set of seeds is selected deterministically rather than at random). Formally, in *r-neighbor bootstrap percolation* we are given an undirected graph $G = (V, E)$ and an integer $r > 1$. Every vertex is either *active* or *inactive*. A set of vertices composed entirely of active vertices is called active. Initially, a set of vertices A_0 is activated. These vertices are called *seeds*. A contagious process evolves in discrete steps where for $i > 0$,

$$A_i = A_{i-1} \cup \{v : |N(v) \cap A_{i-1}| \geq r\},$$

where $N(v)$ is the set of neighbors of v. In words, a vertex becomes active in a given step if it has at least r active neighbors. We refer to r as the *threshold*. Set

$$\langle A_0 \rangle = \bigcup_i A_i.$$

Definition 1.1 *Given $G = (V, E)$, a set $A_0 \subseteq V$ is called* contagious *if $\langle A_0 \rangle = V$. In words, activating A_0 results with the entire graph being activated. The minimal cardinality of a contagious set is denoted by $m(G, r)$. For a contagious set A_0, the number of generations is the minimal integer t with $\bigcup_{i \leq t} A_i = V$.*

Bootstrap percolation has been subjected to extensive research in computer science (see for example [2, 22, 41]) as well as in probabilistic and combinatorial settings [9, 14, 10, 12, 33]. It is known that in every d-regular graph $m(G, r) \leq \frac{rn}{d+1}$ [2, 42]. For certain families of graphs (a collection of disjoint cliques each of size $d + 1$), $m(G, r) = \frac{rn}{d+1}$.

1.1 Contagious sets in expander graphs: motivation

In this work we study how $m(G, r)$ depends on the *expansion* properties of G. Let G be a d-regular graph. We shall distinguish between two types of expansion properties, and associate one parameter with each type. One type is what we refer to as *global expansion*. The parameter that we associate with it is $\lambda(G)$, the second largest eigenvalue (in absolute value) of the adjacency matrix of G. We focus on *spectral expanders*, namely, graphs for which $\lambda(G) \leq \delta d$ for some $\delta < 1$ (observe that for every d-regular graph $\lambda(G) \leq d$). We refer to this class of graphs as (n, d, λ)-graphs, where n is the number of vertices. The other type is what we refer to as *local expansion*. The parameter that we associate with it is the girth g (the length of a shortest cycle in G). If $g \geq 2k+1$ this implies that every vertex has $d(d-1)^{k-1}$ distinct neighbors at distance k from it. We remark that large girth does not imply small λ (a graph might have high girth without even being connected, in which case $\lambda = d$), and $\lambda < \delta d$ need not imply high girth (a graph with $\lambda < \delta d$ may have triangles and four-cycles). We also remark that our results concerning high girth graphs can be extended to graphs that do have short cycles, provided that every small set of vertices has a sufficiently large neighborhood. Details of this are omitted from this manuscript.

Expanders are rich mathematical objects with diverse applications in algebra, combinatorics, probability and theoretical computer science [31]. Furthermore, expander graphs are used in designing fault tolerant networks, hence it makes sense to study various algorithmic problems on expanders and there are several works in this flavor [20, 3, 35]. Understanding optimization problems on expanders and random graphs may be useful in understanding these problems in worst-case settings (see for example [7]). The study of combinatorial optimization problems on graphs with high girth is quite natural as well.

Several works have demonstrated that expanders are resilient to random or adversarial faults in the sense that they keep a certain degree of connectivity in the presence of faulty edges or nodes [4, 5, 8]. As expanders are advocated as sparse graphs with fault tolerant against *static failures*, it is of interest to study their resilience to cascading failures that spread across the network topology.

1.2 Our results

For simplicity of the presentation, our results will be stated for the case $r = 2$, and we will only briefly mention extensions to larger values of r. These extensions do not involve new ideas, but rather a more complicated application of the ideas that work for $r = 2$.

It will be convenient for us to distinguish between three algorithms for selecting seeds.

Random-parallel. In this algorithm one fixes a parameter $p \in (0, 1)$ (that may depend on the input graph G), and initially activates each vertex independently with probability p. If the set of seeds (initially activated vertices) happens to be contagious the algorithm succeeds, and if not it fails. This is typically the algorithm implicitly associated with the term bootstrap percolation.

Random-sequential. This algorithm proceeds in rounds. In each round, the algorithm picks a new vertex at random to become a seed, but only among those vertices that have not been activated in previous rounds (neither by becoming seeds, nor by a cascade effect).

Greedy. This is a family of algorithms, parameterized by the greedy rule that is used. The algorithm proceeds in rounds. In each round the algorithm selects one vertex as a seed according to some greedy rule. A natural rule is to select the vertex whose activation will result in the largest cascade of newly activated vertices. In our work we shall consider other greedy rules as well.

Our first result concerns spectral expanders. To put the following theorem in context one should note that for every d-regular graph $\lambda \geq \Omega(\sqrt{d})$, and that for most d-regular graphs $\lambda \leq O(\sqrt{d})$ (see [31], for example).

Theorem 1 *Let G be an (n, d, λ)-graph. If $\lambda = O(\sqrt{d})$ then $m(G, 2) = O(\frac{n}{d^2})$. Moreover, a contagious set can be chosen by the random-parallel algorithm (with a value of $p = O(d^{-2})$). For the randomly constructed contagious set, the number of generations until complete activation is $O(\log_d \log n + \log \log d)$ with probability $1 - o(1)$.*

Our next result concerns high girth graphs. The random-parallel algorithm is inappropriate in this case (for example, when the graph is composed of many separate components, p might need to be very close to 1 to ensure that each component has at least two seeds), and hence we revert to the random-serial algorithm.

Theorem 2 *Let G be a d-regular graph of girth at least $2k + 1$. If $k \geq \log \log d$ then $m(G, 2) = O(\frac{n}{d^2})$.*

Proposition 1.1 below, shows that for constant d, the number of generations in Theorem 1 is best possible among all d-regular graphs (up to constant factors) as far as random parallel activation is concerned. We remark that Theorem 2 gives examples where random sequential activation leads to fewer generations than random parallel activation.

Proposition 1.1 *For every d-regular graph, if every vertex is initially activated independently with probability at most $1/4$, then with probability $1 - o(1)$ the number of generations until complete activation is at least $\log_d \log n$.*

Theorems 1 and 2 give nearly best possible bounds for $m(G, 2)$ when $\lambda \leq O(\sqrt{d})$ or the girth exceeds $2 \log \log d$.

Theorem 3 *For d large enough there are (n, d, λ)-graphs with $\lambda = O(\sqrt{d})$, girth $\Omega(\log \log d)$ and $m(G, 2) \geq \Omega(\frac{n}{d^2 \log d})$.*

The upper and lower bounds above extend to activation thresholds $2 < r \ll d$, with the adjustment that the terms d^2 need to be replaced by $d^{\frac{r}{r-1}}$ (for example, an upper bound of $m(G, 2) \leq O(\frac{n}{d^2})$ is replaced by $m(G, r) \leq O(\frac{n}{d^{\frac{r}{r-1}}})$). See Section 8 for precise statements of these results.

Theorem 1 does not address graphs for which $\lambda(d) \geq \Omega(d)$, and Theorem 2 does not address graphs of constant girth. One may conjecture that for every $\delta < 1$, an (n, d, λ)-graph with $\lambda < \delta d$ has $m(G, 2) \leq O(\frac{n}{d^2})$ (with the hidden constant in the O notation depending on δ). We do not know if this conjecture is true, but the following proposition gives partial progress towards this conjecture.

Proposition 1.2 *Let G be an (n, d, λ)-graph where $\lambda < \delta d$ where $\delta < 1$ is independent of d. Then there is a contagious set in G of size $O(\frac{n}{d^{\frac{3}{2}}})$. Moreover, the contagious set can be chosen by the random-parallel algorithm.*

Another conjecture is that for every d-regular graph with no 4-cycles, $m(G, 2) \leq O(\frac{n}{d^2})$. For graphs of girth 5, the proof of Theorem 2 establishes a bound of $m(G, 2) \leq O(\frac{n}{d^{3/2}})$. We can improve over this bound as follows.

Theorem 4 *Let G be a graph of minimum degree d and with no 4-cycles. Then $m(G, 2) \leq O(\frac{n}{d^{7/4}})$. Moreover, the contagious set can be chosen by the random-sequential algorithm.*

For graphs of girth at least 7 (in fact, absence of 4-cycles and 6-cycles suffices), we can nearly obtain the desired upper bound of $O(\frac{n}{d^2})$, with a significantly smaller girth than the girth required in Theorem 2. The algorithm used in the proof of Theorem 5 involves an interplay between random and greedy selection of seeds.

Theorem 5 *Let G be a d-regular graph of girth at least 7. Then $m(G, 2) \leq O(\frac{n \log d}{d^2})$.*

One can combine an even weaker girth requirement with a modest expansion requirement and nearly obtain the desired upper bound of $O(\frac{n}{d^2})$. Observe that in Theorem 6 we parameterize the spectral ratio $\lambda(G)/d$ by $1 - \epsilon$. Hence for smaller ϵ we get worst expansion, and our upper bounds on $m(G, 2)$ get *larger*.

Theorem 6 *For arbitrary $\epsilon \in (0, 1)$, let G be an (n, d, λ)-graph with $\lambda \leq (1-\epsilon)d$ and with no 4-cycles. Then $m(G, 2) \leq O(\frac{n \log d}{\epsilon^2 d^2})$. Moreover, the contagious set can be chosen by a greedy algorithm.*

The proof of Theorem 6 works without change when the condition $\lambda \leq (1 - \epsilon)d$ is replaced by the weaker condition $\lambda_2 \leq (1 - \epsilon)d$, where λ_2 is the second largest eigenvalue of the adjacency matrix. Moreover, the contagious set in Theorem 6 can also be chosen by the random-parallel algorithm, but the proof for this is more involved than the proof for the greedy algorithm, and is omitted.

We also obtain the following bounds on the size of $m(G,2)$ in the Binomial random graph $G(n,p)$:

Theorem 7 *Let $G \sim G(n,p)$ with $p := \frac{d}{n}$ and $w(n) < d < n^{\frac{1}{2}-\epsilon}$, where $\epsilon > 0$ is an arbitrary constant and $w(n)$ is an arbitrary function tending to infinity with n. Then with high probability*

$$\Omega\left(\frac{n}{d^2 \log d}\right) \le m(G,2) \le O\left(\frac{n \log^* d}{d^2 \log d}\right).$$

Some of our upper bounds on $m(G,2)$ are summarized in Table 1. They hold for every graph with the corresponding expansion property.

Let us comment on the range of d for which our results hold. All our upper bounds hold when d is a large enough constant, and furthermore, d can be a growing function of n. There are obvious limits on how quickly d can grow as a function of n for the results to make sense (e.g., in Theorem 2, one must have $d^{\log \log d} < n$ so as to satisfy the girth condition). We alert the reader that our proof of Theorem 1 builds on Theorem 2 and hence inherits the requirement that $d^{\log \log d} < n$. Our proof of the lower bound in Theorem 3 assumes that d is a constant independent of n, though the related lower bound in Theorem 7 does not make such an assumption. In general, we did not attempt to find the largest d as a function of n for which the Theorems in this work apply.

Graph Parameters	**Upper bound**
Girth larger than $2 \log \log d$	$O(\frac{n}{d^2})$
No 4-cycles	$O(nd^{-7/4})$
Girth at least 7	$O(\frac{n \log d}{d^2})$
$\lambda(G) \le O(\sqrt{d})$	$O(\frac{n}{d^2})$
No 4-cycles and $\lambda(G) \le (1-\epsilon)d$	$O(\frac{\log d}{\epsilon^2 d^2} n)$

Table 1: **Upper bounds on $m(G,2)$ as a function of graph parameters.** The results apply to d-regular graphs as a function of their girth and $\lambda(G)$, where $\lambda(G)$ is the second largest eigenvalue in absolute value.

Our current work is concerned with regular and nearly regular graphs. However, we remark here that the algorithmic question of finding a small contagious set in an irregular graph can be reduced to this question in regular graphs (though our reduction does not preserve expansion properties). See Section 9 for more details. We also note that insights from the study of contagious sets in expanding nearly regular graphs can be applied to expanding highly irregular graphs. See Section 10 for more details.

1.3 Overview of proof techniques

The following lemma simplifies the selection of contagious sets in spectral expanders (its proof is in Section 4). We remark that its proof works without change when the condition $\lambda \leq \delta d$ is replaced by the weaker condition $\lambda_2 \leq \delta d$.

Lemma 1.1 *Let G be an (n, d, λ)-graph such that $\lambda < \delta d$ with $\delta < 1$. Let the activation threshold of every vertex be $r = 2$. Then every set of size larger than $\frac{n}{(1-\delta)d}$ is contagious.*

Hence in spectral expanders it suffices to find a set that activates $\frac{n}{(1-\delta)d}$ vertices, and then the whole graph is activated by Lemma 1.1. A similar approach does not hold for graphs of large girth (which need not even be connected). For such graphs we shall use the random-sequential algorithm. We shall work in two stages, first finding a set of seeds that activates a large part of the graph, and then arguing that this suffices in order to activate the whole graph. However, now the second stage of the argument is more delicate and requires the selection of additional seeds.

Lemma 1.2 *Consider an arbitrary randomized algorithm RA for selecting seeds in a graph G with vertex set $[n]$. For every vertex i, let p_i denote the probability that vertex i is a seed, and let $q_i > 0$ denote the probability that vertex i is activated. (Observe that necessarily $q_i \geq p_i$). Then there is a distribution D over contagious sets such that for every vertex i, the probability that i is a seed in a random contagious set selected according to D is at most p_i/q_i.*

Proof: Consider a sequence of rounds, where in every round RA is applied on G with independent randomness. As $q_i > 0$ for every i, eventually every vertex is activated in at least one of the rounds. For every j, include vertex i in set S_j if and only if i was chosen as a seed in round j, and i has not been activated in any round prior to j. The set $S = \bigcup S_j$ is necessarily contagious. (One can show by induction on r that $\bigcup_{j=1}^{r} S_r$ activates all those vertices that are activated by round r.) Now:

$$Pr[i \in S] = \sum_{j=1}^{\infty} Pr[i \in S_j] = \sum_{j=1}^{\infty} p_i(1 - q_i)^{j-1} = p_i \sum_{j=0}^{\infty} (1 - q_j)^j = p_i/q_i$$

□

Corollary 1.1 *Let G be a graph on n vertices for which if every vertex is a seed independently with probability p, then for every vertex it holds that the probability that it is activated is at least $1/C$ $(C > 1)$. Then G has a contagious set of size at most Cpn.*

Proof: Applying Lemma 1.2 with $p_i = p$ and $q_i \geq 1/C$ we get for the random contagious set S:

$$E[|S|] = \sum_i Pr[i \in S] \leq \sum_i Cp = Cpn.$$

There must be at least one contagious set of size not larger than the expected size of contagious sets (taken from the distribution whose existence is implied by the proof). $\square$

We now explain how Theorem 2 (contagious sets in high girth graphs) is proved. As the girth of the graph is $2k+1$, every vertex v is a root of a d-regular tree of depth k. Suppose that every leaf (a vertex at distance k from v) is made a seed independently with probability p. Now we let a cascade of activations propagate from the leaves to the root, with the goal of inferring that the root is activated with constant probability. A simple calculation shows that once $p \geq \Omega(\frac{1}{d^2})$, we have "amplification" in the sense that the probability of a node being activated increases as we get closer to the root of the tree. Hence, the deeper the tree, the smaller p needs to be in order to ensure the root is activated with constant probability. Thereafter, an application of Corollary 1.1 proves Theorem 2.

Theorem 1 (contagious sets in spectral expanders) follows from a proof similar to that of Theorem 2, using a result of [11] that shows that every vertex of an (n, d, λ)-graph is a root of a sufficiently large tree (where $\lambda = O(\sqrt{d})$ implies that the degrees of nonleaf nodes in the tree are $\Omega(d)$). The resulting algorithm is random-parallel rather than random-sequential because there is no need to use Corollary 1.1 – we can use Lemma 1.1 instead. (Moreover, if one is not concerned with the number of generations until complete activation, it suffices to have the root of the tree activated with probability $\Omega(1/d)$ rather then constant, though this does not lead to substantial improvements in the bounds.) Let us comment that Theorem 1 can be extended to the case where $\lambda(G) \gg \sqrt{d}$, although the upper bounds in this case on $m(G, 2)$ are weaker – see the discussion after the proof of Theorem 1.

The lower bound argument (Theorem 3) is based on the observation that a "small" contagious set A entails a not much bigger set B ($A \subset B$) such that $G[B]$ (the induced subgraph on B) has average degree close to 4. This is because every newly activated vertex in B must be adjacent to two vertices causing it to become active. Hence it suffices to design (n, d, λ)-graphs with $\lambda = O(\sqrt{d})$ and large girth for which no set of $O(\frac{n}{d^2})$ vertices has average degree (at least) nearly $4 - \frac{2}{\log d}$. Such graphs can be constructed using the probabilistic method.

The proof of Proposition 1.2 follows quite easily from Lemma 1.1.

The proof of Theorem 4 (contagious sets in graphs with no 4-cycles) is based on considering all neighbors of a vertex v up to distance 3. However, as the girth is possibly smaller than 6, this neighborhood is no longer a tree, contrary to the case analyzed in Theorem 2. Hence analyzing the probability that this neighborhood activates v involves handling dependencies, making the analysis considerably more complicated than that of Theorem 2. The absence of 4-cycles gives some control over these dependencies, leading to essentially the same amplification effect that one would get had the neighborhood been a tree.

The proof of Theorem 5 (contagious sets in graphs with girth at least 7) involves selecting a random initial set A of $O(\frac{n \log d}{d^2})$ seeds, and considering the set B of vertices that have a neighboring seed. Girth considerations are used in order to show that the subgraph induced on B has large connected components. Thereafter, choosing one seed in each large connected component of B activates the whole component. This allows us to cheaply extend the set of activated vertices to include most of B, and hence reach a size of $\Omega(\frac{n \log d}{d})$. At this stage one would expect a typical vertex to have $\Omega(\log d)$ active neighbors, and hence it should not be difficult to activate the remaining vertices in the graph. Turning this intuition into a formal proof involves some extra work, including appealing to Lemma 1.2.

The proof of Theorem 6 (contagious sets in graphs with no 4-cycles and $\lambda = (1-\epsilon)d$) involves the following amplification effect. Consider $\log d$ rounds, where in each round n/d^2 seeds are selected at random. The property that we wish to maintain is that the number of active vertices doubles after every round (until we eventually apply Lemma 1.1). Hence after every round t we want there to be roughly $\frac{2^t n}{d^2}$ activated vertices (whereas there are only $\frac{tn}{d^2}$ seeds). For an inductive argument to apply, we would like the active vertices to have roughly $\frac{2^t n}{d}$ neighbors. These neighbors may be thought of as *excited* vertices, as they need only one additional active neighbor in order to become active. This makes it plausible that in the next round $\frac{2^t n}{d^2}$ new active vertices will be generated, because each new seed is likely to have 2^t neighbors that are already excited, and these excited neighbors will be activated. We show that such a delicate balance can be kept for $\log d$ rounds by a greedy choice of seeds. Initially, our greedy rule does not seek to select a seed that maximizes the number of newly activated vertices, but rather to maximize the number of newly excited vertices. Both spectral expansion and absence of 4-cycles are used in order to analyze this greedy rule. Only after the number of excited vertices reaches $n/2$, we switch to a greedy rule that maximizes the number of newly activated vertices.

1.4 Related work

As already noted, $m(G, r)$ has been determined for certain families of graphs. For example, if G is the k-dimensional grid $[n]^k$ then $m(G, r) = \Theta(n^{r-1})$ if $1 \le r \le k$ and $\Theta(n^k)$ otherwise [13]. If G is the n-dimensional hypercube on 2^n vertices it is known that $m(G, 2) = n$ [9]. To the best of our knowledge, the current work is the first to study how $m(G, r)$ depends on the girth of G and on $\lambda(G)$.

Random regular graphs are expected to have very good expansion properties, and hence results on $m(G, r)$ for random regular graphs can serve as a benchmark against which to compare results for expanders. Balogh and Pittel [14] proved an upper bound on $m(G,r)$ when G is chosen uniformly among all n-vertex d-regular graphs for $d > 2$. Using differential equations, they show that a random set of size smaller than $(p(G,r) - \epsilon_n)n$ will not be contagious with high probability[1]. On the other hand, a random set of size $(p(G,r) + \epsilon_n)n$ will be contagious with high probability, where $\lim_{n\to\infty} \epsilon_n = 0$ (for some explicitly defined function ϵ_n). The value of $p(G,r)$ is $1 - \inf_{y\in(0,1)} \frac{y}{R(y)}$ with $R(y) = \Pr(\mathrm{Bin}(d-1, 1-y) < r)$ where $\mathrm{Bin}(d-1, 1-y)$ is a binomial random variable with parameters $d-1$ and $1-y$. It can be shown that $p(G,2)$ tends to $\frac{1}{2d^2}$ as d grows [14]. We are not aware of a closed formula of $p(G,r)$, nor are we aware of asymptotic evaluations (as a function of d and r) of it for $2 < r < d-1$. The work of [14] on random d regular graph does not provide lower bounds on $m(G,r)$ – it only implies that with high probability (probability $1 - o(1)$) a *random* set of size $(\frac{1}{2d^2} - \epsilon)n$ is not contagious.

A different proof of the result of [14] building on cores in random graphs was given by Janson [32]. Interestingly, $p(G,r)$ is identical to the critical threshold for complete activation of the infinite d-regular tree [12]. Our bounds for expander graphs are partly based on analyzing the spread of activation from the leafs of a d-regular tree to its root, and this part of the analysis involves a recursive approach similar to those employed in previous work (though we do so in a setting in which the depth of the tree is finite rather than infinite).

The critical size of a *random* set needed for full activation (with high probability) of the binomial random graph $G(n,p)$ was studied in [33] where the critical size of a random set required for complete activation of $G(n,p)$ for arbitrary constant threshold r is determined in great detail of precision. We shall apply the following Theorem (focusing on the case $r = 2$) from [33] (which follows from Theorem 3.1, page 1996 in [33]:

Theorem 8 *Let ϵ, δ be arbitrary (small) positive constants. Suppose that $p < n^{-1/2-\epsilon}$, with $pn = w(n)$ being some function tending to infinity with n. Let A be an arbitrary set of vertices that are activated as seeds. Then with high probability over the choice of random graph from $G(n,p)$ the following holds:*

1. *If $|A| \geq \frac{(1+\delta)}{2np^2}$ then at least $n - (n^2p)e^{-pn}(1 + O(1))$ vertices will be activated.*
2. *if $|A| \leq \frac{(1-\delta)}{2np^2}$ then at most $o(n)$ vertices will be activated.*

In particular, Theorem 8 implies that when $G \sim G(n,p)$ with p as above, then with high probability $m(G,2) \leq \frac{1+\delta}{2np^2}$. (Observe that for $d = (n-1)p$, $(n^2p)e^{-pn} = o(n/d^2)$ for the range of p in Theorem 8, and hence the set of vertices not activated by A is small and can be added to the set of seeds with only negligible effect on the total number of seeds.) The asymptotic behavior of

[1]That is, with probability tending to 1 as the number of vertices n tends to infinity.

$m(G,2)$ in $G(n,p)$ with p as in Theorem 8 was recently shown to be $\Theta(\frac{n}{d^2 \log d})$ in [26] improving upon an earlier result appearing in [24].

The time (number of generations) until complete activation in bootstrap percolation is the topic of several recent works such as [19]. For $G(n,p)$, Janson et al., [33] studied the number of generations until complete activation for various parameters (e.g., Theorem 3.10, pp. 2000). In particular, for $r = 2$, they show that when $p = n^{-\alpha}$ where $1/2 < \alpha < 1$ and for a fixed set of size $\frac{1+\delta}{np^2}$ (namely., a set of cardinality twice as large than the critical cardinality needed for complete activation), the number of generations is with high probability $\log\log(np) + O(1)$.

The optimization problem, where given $G = (V,E)$ with threshold r, we seek to activate a set of minimum cardinality (that is, of cardinality $m(G,r)$) so that the whole of G is activated, is called the *Target Set Selection* problem [22]. Calculating $m(G,r)$ exactly is NP-hard and obtaining an approximation better than $O(2^{\log^{1-\epsilon} n})$ (n is the number of vertices) is intractable, unless $NP \subseteq DTIME(n^{poly(\log n)})$ [22]. These hardness results hold even when $r = 2$ and G has maximal degree d, where d is a constant not depending on the size of G [22]. For recent results demonstrating the tractability of target set selection in graphs with certain structural properties such as bounded treewidth see [16, 23]. To the best of our knowledge, no approximation algorithm with approximation ratio significantly better than the trivial n approximation is known for the target set selection problem. The results of [2, 42] are algorithmic and they imply for a fixed threshold r a polynomial time $O(n/d)$ approximation algorithm for $m(G,r)$. We are not aware of an approximation algorithm achieving better approximation ratio as a function of d for $m(G,r)$ in d-regular graphs. Approximation and hardness of other propagation problems that are similar to target set selection was considered in [1].

2 Preliminaries and notation

Unless explicitly stated, we will always deal with d-regular, undirected graphs on n vertices. A graph G has *girth* g if the shortest cycle in G is of length g. For clarity reasons, floor and ceiling signs are omitted. For a natural number l, we denote the set $\{1,...,l\}$ by $[l]$. log refers to the logarithm in base 2. We denote by $\mathrm{Bin}(k,p)$ the binomial distribution with k independent trials, each with success probability p. Given a d-regular graph $G = (V,E)$ in the bootstrap percolation model with threshold r, we shall often be interested in the case where every vertex is chosen to belong to A_0 independently with probability $p_0 \in [0,1]$. We denote by $p_c(G,r)$ the minimal p_0 such that a set A_0 whose elements are chosen independently with probability p_0 is contagious with probability $\frac{1}{2}$.

$$p_c(G,r) = \inf_p[\Pr(\langle A_0\rangle = V) = \frac{1}{2}],$$

where every vertex is chosen independently to A_0 with probability p. Observe that we always have that $m(G,r) \leq p_C(G,r) \cdot n$. In general $m(G,r)$ may be much smaller than $p_c(G,r) \cdot n$. For example, for the hypercube over 2^n vertices, $m(G,2) = n$ whereas $p_c(G,2) = \Theta(\frac{2^{-2\sqrt{n}}}{n^2})$ [9].

Given a vertex v and a set S, the number of neighbors of v in S is denoted by $deg_S(v)$. For two sets of vertices A and B let $e(A,B)$ be the number of ordered pairs of vertices (u,v) with $u \in A$, $v \in B$ and (u,v) in E (A, B, need not be disjoint). We denote by $e(A)$ the set of all edges whose two endpoints belong to A. For a subset A of vertices, we denote by $\partial(A)$ the set of all vertices in $V \setminus A$ having a neighbor in A and by $N(A)$ the set of all vertices in V having a neighbor in A. The adjacency matrix of an n–vertex graph G, A_G, is symmetric hence it has n real eigenvalues. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of A_G. It is known that $\lambda_1 = d$ and for every $i > 1$, $|\lambda_i| \leq d$ (see for example [36]). Let $\lambda(G) = \max\{|\lambda_2|, |\lambda_n|\}$. We say that G is an (n,d,λ)-graph if G is d-regular and $\lambda(G) \leq \lambda$. We will focus on the case that λ is smaller than δd where $\delta < 1$.

The following Lemma relates edge expansion to λ_2, the second largest positive eigenvalue of G. The proof can be found in [6].

Lemma 2.1 *Let $G = (V,E)$ be a d-regular graph. Then, for every partition B,C of V, $e(B,C) \geq \frac{(d-\lambda_2)|B||C|}{n}$.*

A graph is called an *expander graph* if for every set of vertices W of size at most $n/2$, the set $\partial(W)$ is of size at least $c|W|$ with $c > 0$ independent of n. It can be verified that if G is a (n,d,λ) graph with $\lambda \leq \delta d$ ($\delta < 1$) then G is an expander with c being at least $\frac{d-\lambda}{2d}$ (see [6], Corollary 9.2.2).

We shall use Azuma's inequality to prove concentration results.

Lemma 2.2 *Let $X_0, ..., X_n$ be a martingale such that for every $1 \leq k < n$ it holds that $|X_k - X_{k-1}| \leq c_k$. Then for every nonnegative integer t and real $B > 0$*

$$\Pr(|X_t - X_0| \geq B) \leq 2\exp\left(\frac{-B^2}{\sum_{i=1}^{t} c_i^2}\right).$$

We shall sometimes use the term *infected* to describe an activated vertex that is not one of the *seeds*, but rather became activated by having at least r active neighbors. When $r = 2$, the term *excited* describes a non-active vertex that has one active neighbor.

3 Contagious sets in graphs with large girth

In this section we focus on the case where the threshold r of every vertex equals 2. We derive upper bounds on $m(G,2)$ as a function of the girth of G. We do this by using bounds on bootstrap percolation on d-regular trees. It is known and easy to see that if one considers an infinite rooted tree in which every vertex has d children, the following holds. Let $p_0 = p$ denote the initial activation probability, let p_i denote the probability that the root becomes activated by generation at most i of the bootstrap percolation process, and let $q_i = 1 - p_i$. Then for $i \geq 1$, $q_i = q_0((q_{i-1})^d + dp_{i-1}(q_{i-1})^{d-1})$. Using this recursive relation it is not difficult to show that for $p = c/d^2$ (for a sufficiently large value of c) we have $p_k = \Omega(1)$ already for some $k = \log\log d + O(1)$, and $p_k = 1 - o(1/n^2)$ already for some $k = O(\log_d \log n) + \log\log d + O(1)$. The following lemma provides a short proof of these statements in which no attempt was made to optimize the constants involved. For simplicity, given a finite tree, the lemma only uses the assumption that the leaves are initially activated with probability p, ignoring the fact that also internal vertices may be initially activated.

Lemma 3.1 *Let $T_{d,k}$ be the complete d-regular tree (e.g., the root being of degree d and all other nonleaf vertices are of degree $d+1$) of depth k, with d being sufficiently large.*

1. *Suppose every leaf of the tree is activated independently with probability $p = \frac{g(k)}{d^2}$ with $g(k) = 10d^{\frac{1}{2^{k-1}}}$. Then the probability the root is activated once we apply the bootstrap percolation process is at least $\frac{1}{2e}$. As a special case, if $k > \log\log d + 1$ then a value of $p = O(\frac{1}{d^2})$ suffices in order to activate the root with probability at least $\frac{1}{2e}$.*
2. *If $k = C\log_d \log n + \log\log d + O(1)$ (for a sufficiently large absolute constant C) then a value of $p = O(\frac{1}{d^2})$ suffices in order to activate the root with probability at least $1 - \left(\frac{1}{n}\right)^2$.*

Proof: A vertex in $T_{d,k}$ is said to be in level ℓ with $0 \leq \ell \leq k$ if its distance from the root is ℓ. Hence the root is in level 0 whereas the leaves are in level k. Let p_i $(0 \leq i \leq k)$ be the probability that a vertex in level $k-i$ gets activated. Hence $p_0 = p$ and p_k is the probability of the root being activated in the bootstrap percolation process. We shall write $p_i = \frac{g_i}{d^2}$ with $g_0 = g(k)$ as defined in the lemma. An internal vertex w of the tree becomes activated if it has at least two active children. Hence for $j < k$, $p_{j+1} \geq \Pr(\mathrm{Bin}(d, p_j) \geq 2) \geq \binom{d}{2}{p_j}^2(1-p_j)^{d-2}$ where we used the fact that $\mathrm{Bin}(k,p)$ is an increasing function of p. Hence $g_{j+1} \geq \frac{1}{2}(g_j)^2(1-p_j)^{d-2}$. As long as $p_j \leq \frac{1}{d}$ then we have that $g_{j+1} \geq \frac{1}{3}(g_j)^2\frac{1}{e} \geq \frac{1}{10}(g_j)^2$, and by induction we have that

$$p_i \geq 10(\frac{g_0}{10})^{2^i} d^{-2} = 10d^{\frac{1}{2^{k-1-i}}} d^{-2}.$$

Substituting $i = k-1$, we have that every child of the root is activated with probability at least $\frac{1}{d}$ independently of all the other children of the root implying that $p_k \geq \frac{1}{2e}$, proving item 1 of the lemma.

We now prove item 2 of the lemma. By item 1, every vertex $v \in T$ in level $k - \log\log d - 1$ gets activated with probability at least $\frac{1}{2e}$. We now use the inequality $p_{j+1} > 1-(1-p_j)^d - p_j d(1-p_j)^{d-1}$ that holds for every j. Let $q_j = 1-p_j$. Then $q_{j+1} \le q_j^{d-1}(d+1) \le q_j^{d/2}$, as for $j > \log\log d + 1$ it holds that $q_j < \frac{1}{2e}$ and assuming d is large enough it holds that $q_j^{d/2-1} < \frac{1}{d+1}$. We get by induction that $q_i \le e^{-(\frac{d}{2})^i}$. Now we consider two cases. If $d < \log n$, then when $i = \log\log d + C\log_d \log n$ and C is sufficiently large the probability the root is *not* infected is at most $\left(\frac{1}{n}\right)^2$. If $d \ge \log n$, the same consequence is obtained by taking $i = \log\log d + O(1)$. $\square$

We can now present a proof of Theorem 2:

Proof: Observe that as the girth of G is $2k+1$, every vertex is the root of a $(d-1)$-regular tree of depth k. Activate independently every vertex with probability $p = \frac{g(k)}{d^2}$ with $g(k) = 10d^{\frac{1}{2^k-1}}$. Lemma 3.1 implies that every vertex is activated with probability at least $\frac{1}{2e}$. Applying Corollary 1.1 concludes the proof of the Theorem. $\square$

We remark that when G is a d-regular graph of order n and girth $\Omega(\log\log n)$ then Lemma 3.1 implies that $p_c(G,2) = O(\frac{1}{d^2})$. In other words, in such graphs the random parallel algorithm will infect all vertices of G with high probability.

4 Bounds for $m(G,2)$ in spectral expanders

In this section we concentrate on (n,d,λ)-graphs. Our main goal is to derive upper bounds on $m(G,2)$ in terms of $\lambda(G)$. We start by proving Lemma 1.1.

Proof: Consider a set S of size $|S|$ that is not contagious. We can assume without loss of generality that S is inclusion-maximal with respect to being active (namely, *every* vertex *not* belonging to S is not active). For every $u \in V \setminus S$ it holds that $deg_S(u) \le 1$. Thus $e(S, V\setminus S) \le |V \setminus S| = n - |S|$. On the other hand, by Lemma 2.1

$$e(S, V\setminus S) \ge \frac{(1-\delta)d|S|(n-|S|)}{n}.$$

Combining these inequalities we have that

$$\frac{(1-\delta)d|S|(n-|S|)}{n} \le n - |S|.$$

Hence $|S| \le \frac{n}{(1-\delta)d}$. As required. $\square$

Using Lemma 1.1 we first prove Proposition 1.2.

Proof: Activate independently every vertex with probability p (where p will be chosen later). Let A_1 denote the set of non-seed vertices that have at least two seed neighbors (and hence become active), and let p_1 denote the probability that a vertex belongs to A_1. Then

$$p_1 \ge (1-p)\binom{d}{2}p^2(1-p)^{d-2}.$$

Assuming d is sufficiently large and p is smaller than $\frac{1}{d}$ we get that $p_1 \geq \frac{(dp)^2}{4}$. By Lemma 1.1, every set of size $\frac{cn}{d}$, where $c > \frac{1}{1-\delta}$, is contagious. If $p > \frac{4\sqrt{c}}{d^{\frac{3}{2}}}$ we get that the *expected* number of vertices in A_1 is at least $\frac{2cn}{d}$. We proceed and show that w.h.p. $|A_1| > \frac{cn}{d}$ vertices. Define the familiar Doob exposure martingale, e.g., exposing the set of seeds according to some predetermined order and considering the expected number of vertices in A_1. Observe that whether an exposed vertex is a seed or not can effect at most d neighboring vertices. We get using Lemma 2.2 (Azuma's inequality) that for such p with high probability $|A_1| \geq \frac{cn}{d}$. The Lemma follows. $\square$

We now turn to prove Theorem 1. The proof of Theorem 2 can be generalized to the case where every vertex is contained in a regular tree of degree $\Omega(l)$ and sufficiently large depth (even if the tree is not induced). There is a long line of research concerned with embedding trees in expanders, starting with the works of Pósa [40] and Friedman and Pippenger [28]. We will use the recent result of Balogh, Csaba, Pei and Samotij [11], building on the work of Haxell [30].

Theorem 9 (Theorem 5 in [11]) *Let $l \geq 2$ and $\epsilon \in (0, \frac{1}{2})$. If $\lambda < \frac{\epsilon d}{\sqrt{8l}}$ then every (n, d, λ)-graph contains every tree of order at most $(1-\epsilon)n$ and maximum degree l. Furthermore, for every vertex $v \in G$, fixing a (rooted) tree T satisfying these conditions, T can be embedded into G with v being the root of T.*

We now prove Theorem 1.

Proof: By Theorem 9 every vertex is the root of a regular tree of degree $\Omega(d)$ of depth $k = \Omega(\log_d \log n + \log \log d)$. The proof of Lemma 3.1 then implies that if every vertex in G is activated independently with probability $p \geq \Omega(\frac{1}{d^2})$, then for every vertex v in G the probability v is not activated by the bootstrap percolation process is $O(\frac{1}{n^2})$. Hence the entire graph is activated with high probability by taking union bounds over all vertices. Furthermore, it is immediate that the number of generations until complete activation is $O(\log_d \log n + \log \log d)$. $\square$

Note: By applying Theorem 9, Theorem 1 can be generalized without much difficulty to the case where $\lambda < O(\frac{d}{\sqrt{l}})$ when $l < \sqrt{d}$. In this case, $m(G, 2) \leq O(\frac{n}{l^2})$ and when vertices are activated independently with probability $p = \Omega(\frac{1}{l^2})$, the number of generations until complete activation is $O(\log_l \log n + \log \log d)$.

The proof of Proposition 1.1 is based on elementary probabilistic arguments.

Proof: Consider an arbitrary d-regular graph. For a fixed vertex v there are at most $d(d-1)^{\log_d \log n - 1} < \log n$ vertices of distance $\log_d \log n$ from v. Vertex v is activated within $\log_d \log n$ generations only if at least one vertex (possibly v itself) within its $\log_d \log n$ neighborhood is initially activated. A simple greedy argument shows that there is a set U of at least $n/(\log n)^2$ vertices in G such that the distance between any two vertices of U is at least $2 \log_d \log n$. Hence for every two vertices in U, the events that they are activated within $\log_d \log n$

generations are independent. It follows that if every vertex is initially activated independently with probability $1/4$, the probability that all vertices of U are activated in $\log_d \log n$ generations is at most:

$$\left(1 - (\frac{3}{4})^{\log n}\right)^{n/(\log n)^2} = o(1).$$

□

We now turn to Theorem 3, exhibiting d-regular expanders for which $m(G, 2) = \Omega(\frac{n}{d^2 \log d})$. Our lower bound on $m(G, 2)$ is based on the following lemma.

Lemma 4.1 *Suppose an n-vertex graph $G = (V, E)$ has a contagious set of size t_0. Then for every t such that $t_0 \leq t \leq n$ there is a subgraph of G induced by t vertices, spanning at least $2(t - t_0)$ edges.*

Proof: Let A_0 be a contagious set of size t_0. Then there exists an ordering of the vertices of $V \setminus A_0$, $v_1, ..., v_{n-t}$ such that $\forall i, 1 \leq i \leq n - t_0$, v_i is connected to at least two vertices in $A_0 \cup \{v_1, ..., v_{i-1}\}$. Given $t_0 \leq t$, let B_t be $A_0 \cup \{v_1, ..., v_{t-t_0}\}$. Then $2(t - t_0) \leq |E(B_t)|$. As required. □

Lemma 4.1 implies that in order to prove lower bounds on $m(G, 2)$ it suffices to exhibit graphs that do not have small subgraphs of average degree nearly 4. To exhibit expander graphs that do not have small subgraphs of average degree nearly 4 we apply the probabilistic method. For the expansion property, we shall use the following theorem of Friedman [27].

Theorem 10 (Friedman [27]) *For arbitrary $\delta > 0$, a random d-regular graph G has probability $1 - o(1)$ (the $o(1)$ term tends to 0 as n grows) of satisfying $\lambda(G) \leq 2\sqrt{d-1} + \delta$.*

We remark that the bound in Theorem 10 matches (up to low order terms) the lower bound on λ for arbitrary d-regular graphs (see for example, [6]).

We now find it convenient to temporarily switch to the configuration model $G^*(n, d)$ of random d-regular multigraphs (see for example [44]). Let nd be even, the vertex set of the sampled graph be $[n]$, and let d be a constant independent of n. Let $W = [n] \times [d]$. Elements of W are called cells. For $i \in [n]$ we define W_i, as the set $\{i\} \times [d]$. Now we generate G by choosing a uniform perfect matching over all matchings of all cells in W. Suppose a cell from W_i is matched to a cell in W_j: in this case we add an edge between two vertices $i, j \in [n]$. Observe that the resulting graph need not be simple and may contain multiple edges and self loops. However, we shall use the following known theorem (see for example [44]).

Theorem 11 *A graph G sampled from $G^*(n, d)$ is simple (has no parallel edges and no self loops) with probability tending to $e^{-(d^2-1)/4}$ (which is bounded away* *from 0 for a constant d) as n tends to infinity. Conditioned on being simple, G is distributed as $G(n,d)$. Namely, G is a uniform sample of a d-regular n-vertex graph.*

As edges in $G^*(n,d)$ are not independent, we shall use the following known lemma:

Lemma 4.2 *Let $G=(V,E)$ be a graph sampled from $G^*(n,d)$. Let E_0 be a set of k distinct unordered pairs $e_1,..,e_k$ where each pair consists of two distinct vertices in V where $k < \frac{nd-1}{4}$. Then the probability that $e_1,...,e_k$ simultaneously belong to E is bounded by $(\frac{2d}{n})^k$.*

Proof: In the configuration model, fix $\widetilde{e}_1,..,\widetilde{e}_k$ with $\widetilde{e}_i$ being an edge connecting a *fixed* cell in W_r to a *fixed* cell in W_s where it is assumed that e_i is between the vertices r and s $(r,s \in V)$. Then the probability that $\widetilde{e}_1,..,\widetilde{e}_k$ all exist in the configuration model is exactly $\frac{1}{nd-1}\cdot\frac{1}{nd-3}\cdot\ldots\cdot\frac{1}{nd-2k+1}$ which is bounded by $(\frac{2}{nd})^k$. The lemma follows as for each $i \le k$, conditioned on $e_1,..,e_{i-1}$ chosen there are at most d^2 choices for cells realizing e_i. $\square$

In our analysis, we shall include two parameters α and β that can simultaneously be optimized to give the best possible lower bound provable with our current approach. For simplicity of the presentation, rather than optimizing α and β, we shall fix $\alpha = 6$ and $\beta = 2 - \frac{1}{\log d}$, where log is in base 2.

Let G be a random graph sampled from $G^*(n,d)$. Let $t = \frac{n}{\alpha d^2}$, though note that this equality will be used as $n = \alpha d^2 t$. We assume that d is bounded from below by some sufficiently large constant (that can be computed explicitly from the proof of Lemma 4.3), and bounded from above by $o(\sqrt{n})$.

Lemma 4.3 *For the setting above, w.h.p. G does not have a subgraph with t vertices and βt edges.*

Proof: There are $\binom{n}{t} \simeq (e\alpha d^2)^t$ possible choices of a set T of t vertices in G. There are $\binom{\binom{t}{2}}{\beta t} \simeq (\frac{et}{2\beta})^{\beta t}$ ways of choosing βm edge locations in T. By Lemma 4.2, the probability that all these locations are indeed edges is at most $(\frac{2d}{n})^{\beta t} = (\frac{2}{\alpha dt})^{\beta t}$. Hence the probability that G has a subgraph with t vertices and βt edges is upper bounded by roughly:

$$(e\alpha d^2)^t(\frac{et}{2\beta})^{\beta t}(\frac{2}{\alpha dt})^{\beta t} = \left(\frac{e^{\beta+1}d^{2-\beta}}{\alpha^{\beta-1}\beta^{\beta}}\right)^t$$

Now in the exponent for d substitute $\beta = 2 - \frac{1}{\log d}$, obtaining $d^{2-\beta} = 2$. For the other terms we can substitute an approximation $\beta \simeq 2$, because for sufficiently large d, the error introduced by this is offset by our choice of α that is larger than needed for the proof. The expression $\frac{e^{\beta+1}d^{2-\beta}}{\alpha^{\beta-1}\beta^{\beta}}$ is then roughly $\frac{2e^3}{4\alpha}$ and is strictly smaller than 1 for $\alpha = 6$. Raising to the power of t, the probability that G has a subgraph with t vertices and βt edges, tends to 0 as n grows. As desired. □

Corollary 4.1 *For the parameters as above,* $m(G,2) \geq \frac{n}{12d^2 \log d}$ *w.h.p.*

Proof: Suppose otherwise. Then for $t = \frac{n}{6d^2}$, the set of $t_0 = \frac{n}{12d^2 \log d}$ seeds and first $t - t_0$ infected vertices induces a subgraph with t vertices and $2(t - t_0) = (2 - \frac{1}{\log d})t$ edges, contradicting Lemma 4.3. □

We can proceed and prove Theorem 3:

Proof: Sample at random a graph G from $G^*(n,d)$. By Theorem 4.1 we have that $m(G,2) \geq \Omega(\frac{n}{d^2 \log d})$ with probability $1 - o(1)$. By Theorem 11, G is simple with probability bounded away from 0. Hence conditioned on G being simple, the probability that it fails to have $m(G,2) \geq \Omega(\frac{n}{d^2 \log d})$ is still $o(1)$. Conditioned on being simple, Theorem 10 implies that G fails to have $\lambda(G) = O(\sqrt{d})$ with probability $o(1)$. For a fixed integer k it is known, that with probability $p(d,k) > 0$ (where $p(d,k)$ depends only on d,k but not on n) a random d-regular graph has girth at least k (see for example, [44]). Hence there is positive probability that G is simultaneously simple, of girth at least k, has $\lambda(G) = O(\sqrt{d})$, and moreover, $m(G,2) \geq \Omega(\frac{n}{d^2 \log d})$. Plugging $k = P(\log \log d)$ proves Theorem 3. □

5 Contagious sets in graphs with no 4-cycles

We have seen that for d-regular graphs $m(G,2)$ may be at least $\frac{2n}{d+1}$. It is not hard to construct triangle free graphs with $m(G,2)$ at least $\frac{n}{d}$ (take $\frac{n}{2d}$ disjoint copies of complete bipartite d-regular graphs). In this section we show that situation is different for graphs without 4-cycles, proving Theorem 4.

Given a graph G of minimum degree at least $d+1$ (for notational reasons, we find it easier in this section to work with degree $d+1$ as opposed to d), a vertex v and and a parameter $k \geq 0$, a (d,k)-tree rooted at v is a d-ary tree of depth k that can be defined by induction on k as follows. A $(d,0)$-tree is v itself. A $(d,1)$ tree has v as its root, and d distinct neighbors of v as its leaves. Thereafter, a $(d,k+1)$-tree is obtained from a (d,k)-tree as follows: every leaf of the (d,k)-tree gets d of its neighbors in G (excluding its parent node in the tree) as children in the $(d,k+1)$-tree. Hence for every node in a (d,k)-tree, all its tree neighbors are distinct vertices of G. However, the same node of G may appear multiple times in the (d,k)-tree.

For a vertex v and $k \geq 0$, a k-witness is a $(2,k)$-tree rooted at v in which all its leaves are seeds. A k-witness implies that v is activated, by propagating activations from the leaves to the root. Observe that we do not require the leaves to represent distinct vertices of G, or to represent vertices different from internal nodes of the tree. Observe also that v might be activated without there being any k-witness to its activation (for example, by having one neighbor of v as a seed and another neighbor of v activated by two of its seed neighbors).

Proposition 5.1 *Consider a (d,k)-tree T rooted at v. Then the number of $(2,k)$-trees rooted at v that T contains is $\binom{d}{2}^{2^k-1}$.*

Proof: A $(2,k)$-tree has $2^k - 1$ non-leaf nodes. Every non-leaf node has $\binom{d}{2}$ ways of choosing two children different from its parent node. $\Box$

Proposition 5.2 *Let v be the root of a (d,k)-tree T in G. Suppose we activate every vertex in G independently with probability p. Then the expected number of k-witnesses for v in T is at least $\binom{d}{2}^{2^k-1} p^{2^k}$.*

Proof: By Proposition 5.1 the number of $(2,k)$-trees rooted at v that T contains is $\binom{d}{2}^{2^k-1}$. Each one of them has 2^k leaves, and all its leaves are seeds with probability p^{2^k} if these leaves are distinct, and higher probability otherwise. $\Box$

To show that a vertex v is likely to be activated, we shall view it as a root of a (d,k)-tree, and show that this tree is likely to contain a $(2,k)$-witness for v. A necessary condition for this is that the expected number of $(2,k)$-witnesses will exceed 1. By Proposition 5.2, this will happen when $p > d^{\frac{2}{2^k}-2}$. To make this into a sufficient condition, we develop tools for bounding the variance of this random variable.

Definition 5.1 *A (d,k)-tree T in a graph G is* proper *if all its nodes correspond to distinct vertices of V. Equivalently, the subgraph of G induced by the edges of T does not contain a cycle. The tree T is t-proper if the subgraph of G induced by the edges of T does not contain a t-cycle in G. (Edges of T that correspond to the same edge in G are counted only once.)*

Proposition 5.3 *Let G be a graph with no 4-cycles. Then every (d,k)-tree in G is 4-proper.*

Proof: By definition. $\Box$

Lemma 5.1 *Let v be the root of a 4-proper $(d,2)$-tree T, and let $\omega(d^{-2}) \leq p \leq o(d^{-3/2})$. Then the probability that v has a 2-witness in T is at least $(1-o(1))\binom{d}{2}^3 p^4$.*

Proof: All d^2 leaves in T are distinct, because T is 2-proper. Let W_i denote the indicator random variable for the event that the ith $(2,2)$-tree in T is a 2-witness for v. Then $Pr[W_i = 1] = p^4$. Let $W = \sum W_i$ be a random variable that counts the number of 2-witnesses in T for v. Then $E[W] = \binom{d}{2}^3 p^4$ (which is the same as substituting $k = 2$ in Proposition 5.2).

Consider an arbitrary $(2,2)$-tree in T, and suppose that it happens to be a witness. W.l.o.g we can assume T is the ith tree, that is, $W_i = 1$ (all its leaves are seeds). We compute an upper bound on $E[W|W_i = 1]$. Hence conditioned on $W_i = 1$, we only know of four leaves that are seeds. The number of $(2,2)$-trees that share three leaves with T is $4(d-2)$ (each of the four leaves of W_i can be replaced by $d-2$ alternative leaves). The number of $(2,2)$-trees that share two leaves with T is at most $2(d-2)\binom{d}{2} + 4(d-2)^2$ (either one of the two children of v in W_i is replaced by a different child with two leaves, or each of the children of v has one of its leaves replaced). The number of $(2,2)$-trees that share one leaf with T is at most $4(d-2)^2\binom{d}{2}$ (one child of v replaces a leaf, and another child of v is placed completely). Hence

$$E[W|W_i = 1] \leq 1 + 4dp + (d^3 + 4d^2)p^2 + 2d^4p^3 + \binom{d}{2}^3 p^4 \leq 1 + O(d^3p^2),$$

where the last inequality used $\omega(d^{-2}) \leq p \leq o(d^{-3/2})$. It follows that

$$E[W^2] = \sum_i Pr[W_i]E[W|W_i = 1] \leq (1+O(d^3p^2))\sum_i Pr[W_i = 1] = (1+O(d^3p^2))E(W).$$

Observe that by definition $1 = \sum_{i=0}^{\infty} Pr[W = i]$, that $E[W] = \sum_{i=0}^{\infty} iPr[W = i]$, and that $E[W^2] = \sum_{i=0}^{\infty} i^2 Pr[W = i]$. Hence (see [18], Theorem 1.16)

$$Pr[W = 0] \leq 1 - 2E[W] + E[W^2] \leq 1 - (1 - O(d^3p^2))E(W),$$

implying that $Pr[W > 0] \geq (1 - O(d^3p^2))E(W) = (1 - o(1))\binom{d}{2}^3 p^4 \geq \Omega(d^6p^4)$. $\square$

Lemma 5.2 *Let v be the root of a 4-proper $(d,3)$-tree T. Then v has probability at least $1/2$ of being activated when $p = 4d^{-7/4}$. (The leading constant 4 was chosen for concreteness. A smaller constant suffices.)*

Proof: Let $v_1, \ldots, v_d$ denote the neighbors of v in T. Let X_i be an indicator random variable for the event that v_i has a 2-witness in the subtree of T rooted at v_i. Lemma 5.1 implies that $Pr[X_i = 1] = (1-o(1))\binom{d}{2}^3 p^4$. Let $X = \sum X_i$. Then $E[X] = (\frac{1}{8} - o(1))d^7p^4 \simeq 32$. Observe that when $X \geq 2$ at least two neighbors of v are activated, and then v is activated as well. Hence if X behaves similar to its expectation, we expect v to be activated. To show that X is concentrated around its expectation, we compute $E[X^2]$.

Let us compute $Pr[X_i \wedge X_j]$ for $i \neq j$. The fact that T is 4-proper implies the following useful facts:

1. All d^2 leaves in the subtree of T rooted at v_i are distinct. All d^2 leaves in the subtree of T rooted at v_j are distinct.

2. All children of v_i in T are distinct from all children of v_j in T.

3. No child of v_i in T has two common children with a child of v_j in T.

The probability $Pr[X_i \wedge X_j]$ depends on the pattern of common grandchildren that the vertices v_i and v_j has. The above facts show that every child of v_i and every child of v_j have at most one common neighbor. We consider two cases.

In the first case every child of v_i and every child of v_j have *exactly* one common neighbor. This case can be visualized as a d by d table M of distinct grandchildren. The rows are indexed by the children of v_i and the columns are indexed by the children of v_j. Every child of v_i is a neighbor of those grandchildren in its respective row, and every child of v_j is a neighbor of those grandchildren in its respective column. Each entry of the table is a seed with probability p and not a seed otherwise. For the event $X_i \wedge X_j$ we need two rows to have two seed entries, and two columns to have two seed entries. This requires between four to eight seed entries, depending on where the seeds are located within the table. We compute the number of possibilities for each case separately.

1. Four seed entries. One needs to choose the two rows and two columns that contain them, giving $\binom{d}{2}^2$ possibilities.

2. Five seed entries. There are $\Theta(d^6)$ possibilities. (Details omitted.)

3. Six seed entries. There are $\Theta(d^8)$ possibilities. (Details omitted.)

4. Seven seed entries. There are $\Theta(d^{10})$ possibilities. (Details omitted.)

5. Eight seed entries. One needs to choose two rows and two locations within these rows, and likewise for the columns. This gives at most $\binom{d}{2}^6$ possibilities.

As $d^2p \gg 1$, the dominating term is $\binom{d}{2}^6 p^8$, giving $Pr[X_i \wedge X_j] = (1 + o(1))Pr[X_i]Pr[X_j]$. It follows that

$$E[X^2] = \sum_i \sum_j Pr[X_i \wedge X_j] \leq \sum_i Pr[X_i](1+(1+o(1))E[X]) = E[X]+(1+o(1))(E[X])^2,$$

Hence $var[X] = E[(X - E[X])^2] = E[X^2] - (E[X])^2 = E[X] + o((E[X])^2)$.

Now Chebyschev's inequality implies that $Pr[X \geq 2] > 1/2$.

The remaining case to consider is the one in which some pairs of children, one child of v_i and one child of v_j, have no common neighbors at all. In this case, some entries of the table M referred to above are empty, and instead the vertices representing the corresponding rows and columns have additional children not accounted for in M (and not shared by other vertices). Imitating the analysis performed for the first case, the number of possibilities for eight seed entries remains at most $\binom{d}{2}^6$, and $\binom{d}{2}^6 p^8$ remains the dominating term (the nondominating terms can easily be seen not to increase by more than a constant factor). Hence the bounds proven for the first case above apply also in the current case. □

We can now prove Theorem 4.

Proof: As the minimum degree of G is $d+1$, every vertex v in G is a root of a $(d,3)$-tree. By Proposition 5.3 this $(d,3)$-tree is proper. By Lemma 5.2, if $p = 4d^{-7/4}$ then v is activated with probability at least $1/2$. By Corollary 1.1, there is a contagious set of size $2pn$. □

6 Contagious sets in graphs of girth at least 7

Before proving Theorem 5, let us present a lemma that summarizes the only property of d-regular graphs of girth at least 7 that will be used in the proof. Given a graph $G(V,E)$, for a set S of vertices, recall that $N(S)$ denote the set of those vertices that are neighbors of some vertex in S, and let $N^2(S)$ denote the set of those vertices that are at distance exactly 2 from some vertex in S. Observe that we do not require the sets S, $N(S)$ and $N^2(S)$ to be disjoint.

Lemma 6.1 *Let G be a d-regular graph of girth at least 7. Then for every $1 \le k < d$ and every set S of k vertices it holds that $|N^2(S)| \ge \frac{kd^2}{2}$.*

Proof: Given a d-regular graph $G(V,E)$ of girth at least 7, consider an arbitrary set S of k vertices. For every vertex $v \in S$ we have that $|N^2(v)| = d(d-1)$, because otherwise G has a cycle of length at most 4. Hence $\sum_{v\in S} |N^2(v)| = kd(d-1)$. To provide a lower bound on $|N^2(S)|$, we use the first two terms of the inclusion exclusion formula. Namely:

$$|N^2(S)| \ge kd(d-1) - \sum_{u,v\in S} |N^2(u) \cap N^2(v)|$$

We now claim that for every $u, v \in V$ it holds that $|N^2(u) \cap N^2(v)| \le d$. Suppose otherwise that $|N^2(u) \cap N^2(v)| > d$. Then by the pigeon-hole principle, and least one vertex $x \in N(u)$ has at least two neighbors x_1, x_2 in $N^2(v)$. Suppose first that $x \notin N(v)$. Then x_1 and x_2 cannot have a common neighbor y in $N(v)$, because then x, x_1, y, x_2 would form a 4-cycle. Hence there are two vertex disjoint paths from x to v (one through x_1, the other through x_2). This forms a 6-cycle, which contradicts the girth assumption.

The other case to consider is that $x \in (N(u) \cap N(v))$. (Note that it cannot be that $x = v$ because in that case neighbors of x will not be in $N^2(u) \cap N^2(v)$.) Observe that then there cannot be any other vertex y that is in $N(u) \cap N(v)$, because x, u, y, v would form a 4-cycle. Observe also that $|N^2(u) \cap N^2(v)| > d$ implies that there is a vertex $z \notin N(x)$ that is in $N^2(u) \cap N^2(v)$. This z has two vertex disjoint paths of length 3 to x, one through u and the other through v. This forms a 6-cycle, contradicting the girth assumption.

If follows (using also $k < d$) that:

$$|N^2(S)| \geq kd(d-1) - d\binom{k}{2} = kd(d-1-\frac{k-1}{2}) \geq \frac{kd^2}{2}$$

□

Remark. The proof of Lemma 6.1 only requires the graph not to have 4-cycles and 6-cycles. Having arbitrarily short odd cycles does not matter, up to some minimal changes in the parameters, such as the allowed range of k, or the leading term of $\frac{1}{2}$ for the expression kd^2. Consequently, the proof of Theorem 5 only uses the absence of 4-cycles and 6-cycles, and not the full requirement of girth at least 7. More generally, existence of odd cycles can have only limited effect on upper bounds on $m(G, 2)$, as long as these upper bounds are expressed as function of the degree and do not require the graph being exactly regular. This can be seen by recalling that every d-regular graph has a maximal cut in which every vertex has between $d/2$ and d edges crossing the cut. Removing all edges except for cut edges leaves us with a bipartite graph G', which has no odd cycles. Furthermore, all degrees are between $d/2$ and d. Upper bounds on $m(G', 2)$ trivially apply to G as well. Finally, observe that Lemma 6.1 is no longer true if we only require the graph to have no four-cycles (or girth 5) as there are d regular graphs with girth 5 and $O(d^2)$ vertices.

We now prove Theorem 5.

Proof: We present an algorithm that is partly random and partly greedy for selecting a contagious set in $G(V, E)$. Let $p = \frac{4 \ln d}{d^2}$. Let A be an initial set of seeds, where every vertex of G in included in A independently at random with probability p. Given A, consider the following sets of vertices.

1. Set A of seeds.
2. Set B of excited vertices: vertices in $V \setminus A$ that have at least one neighbor in A. Observe that under our definition of B, a vertex in B may have two or more neighbors in A and hence be activated, but we still refer to it as excited. Consider the subgraph $G(B)$ of G induced on the vertices of B. Call a connected component in $G(B)$ *large* if it contains at least d vertices, and *small* otherwise. Based on this distinction, we partition B into two disjoint subsets.
   (a) The set B_L of vertices that are in large connected components in $G(B)$.
   (b) The set B_S of vertices that are in small connected components in $G(B)$.
3. Set C of those vertices in $V \setminus (A \cup B)$ that have at least one neighbor in B_L.

As a memory aid, one may think of A as representing *activated*, B as representing *boundary*, and C as representing *close*.

Consider an arbitrary vertex $v \in V$. We analyze the probability of the event that $v \in C$. This event can be broken into several other events that all need to happen simultaneously.

Event $\bar{A}_v$, which holds if $v \notin A$. This happens with probability $1-p$.

Event $\bar{B}_v$, which holds if $v \notin B$. This happens with probability at least $1-dp$, because v has d neighbors.

Event NB_v, which holds if v has at least one neighbor in B. Consider the vertices at distance 2 from v. As G has no 4-cycles, these are $d(d-1)$ distinct vertices. The expected number of these vertices that are in A is $pd(d-1) \simeq 4\ln d$. Hence the probability that at least one of them is in A is roughly $1-e^{-4\ln d} > 1-\frac{1}{d}$. Let $w \in A$ be a vertex at distance 2 from v, and let u be the common neighbor of v and w. If u is not in A (which happens with probability $1-p$) then u is in B. Hence event NB_v holds with probability at least $1-\frac{1}{d}-p \geq 1-\frac{2}{d}$.

Event $\overline{NS}_v$, which holds if v has no neighbor in B_S.

Lemma 6.2 *The Event $\overline{NS}_v$ holds with probability $1-O(1/d)$.*

Proof: Consider an arbitrary vertex $u \in N(v)$, and for $k < d$, let K be a connected set of k vertices that contains u. Consider the event $\bar{K}$ that K forms one of the connected components in B. This event involves two requirements: one is that $K \subset B$ and the other is that no vertex in $\partial(K)$ is in B. Observe that by considering all possible connected K that contain u, exactly one of the events $\bar{K}$ needs to happen in order for k to be the size of the connected component of u in $G(B)$. Given that G is of degree d and that $u \in K$, there are at most $\binom{(k-1)d}{k-1} \simeq (ed)^{k-1}$ ways of choosing the k vertices of K.

Given K, we now upper bound the probability of event $\bar{K}$. For this, it suffices to upper bound the probability that no vertex in $\partial(K)$ is in B (while ignoring the requirement that $K \subset B$). This event fails if a vertex z at distance 2 from a vertex of $x \in K$ is *violating*, namely, $z \in A$, and there is a vertex $y \in N(x) \cap N(z)$ such that $y \notin (A \cup K)$. This y is in B and can be used to enlarge K. Lemma 6.1 implies that $N^2(K) \geq \frac{kd^2}{2}$. Using this, we now estimate the probability that no violating vertices exist.

For every vertex $z \in N^2(K)$, designate one vertex in $N(z) \cap N(K)$ to be the *link* $l(z)$ to K. Observe that every vertex in $N(K)$ can serve as a link to at most d vertices in $N^2(K)$ (because the graph has degree d). At most k of the links are in K ($N(K)$ may not be disjoint from K). Ignore those vertices in $N^2(K)$ whose link is in K. This still leaves at least $\frac{kd^2}{2} - kd$ vertices in $N^2(K)$ whose link is not in K. With each link l that is not in K, associate a 0/1 random variable y_l whose value is 1 if and only if the following two conditions hold: $z \in A$ for at least one $z \in N^2(K)$ for which $l(z) = l$, and $l \notin A$. Let $d_l \leq d$ denote the number of $z \in N^2(K)$ for which $l(z) = l$. We get that $Pr[y_l = 1] \geq (1-p)d_l p(1-p)^{d_l-1} \simeq pd_l$ (where the near equality holds because for our choice of p and d, $(1-p)^d \simeq 1$). If $y_l = 1$ then there is a violating vertex. Let $Y = \sum_l y_l$. There is no violating vertex only if $Y = 0$. Note that the expectation of Y is roughly $\sum pd_l \geq p(\frac{kd^2}{2} - kd) \simeq 2k \ln d$. Observe that the random variables y_l are independent, and each of them is a 0/1 variable, hence standard concentration results imply that $Pr[Y = 0] \leq e^{-2k \ln d} \simeq d^{-2k}$.

Taking a union bound over all choices of K, it follows that the size of the connected component of u in $G(B)$ is exactly k with probability at most $d^{-2k}(ed)^{k-1} \leq (\frac{e}{d})^{k+1}$. Summing over all values of $1 \leq k < d$, the probability that $u \in B_S$ is $O(1/d^2)$. Taking a union bound over all neighbors of v, we get that $Pr[\overline{NS}_v] = 1 - O(1/d)$. $\square$

For a given vertex v, if all four events listed above hold simultaneously then $v \in C$ (observe that the combination of NB_v and $\overline{NS}_v$ imply that v has a neighbor in B_L). Hence $v \in C$ with probability at least $1-p-pd-O(1/d) > 3/4$ (for our choice of p and sufficiently large d).

Within every large component (in B_L), chose at random one vertex to be a seed. Observe that the probability that v becomes a seed by this is at most p (probability of pd for being in B, times probability at most $1/d$ of being selected as seed in his large component). Observe also that this activates the whole large component. Hence by now every vertex of C has at least one active neighbor.

Let us repeat the above experiment of selecting a random A twice, each time with fresh randomness. Call a vertex *lucky* if it is in C in both experiments. Hence the probability that a vertex v is lucky is at least $(\frac{3}{4})^2 = \frac{9}{16}$. If the two active neighbors of v are distinct, then v is infected as well. What is the probability that these two active neighbors are not distinct? For this, v would have to have a neighbor that is in B in both experiments. This happens with probability at most $d(pd)^2 \leq \frac{1}{16}$ (for our choice of parameters). Hence v has probability at least $1/2$ of becoming infected. Note also that v had probability at most $4p$ of becoming a seed in at least one of the experiments. Hence Lemma 1.2 implies that G has a contagious set of size $8pn = O(\frac{n \log d}{d^2})$. $\square$

7 Contagious sets in expanders with no 4-cycles

In this section we prove Theorem 6.

Our strategy in building a small contagious set for expanders with no 4-cycles will be to choose the seeds (the vertices we activate) one by one in rounds in a greedy manner, where for a given round t, s_t will denote the seed chosen in round t, and S_t will denote the set of all t seeds chosen up to and including round t. Given a set S_t of seeds, an activation cascade may activate additional vertices. We let A_t denote the set of all activated vertices after round t, with $S_t \subset A_t$. We shall be concerned also with neighbors of vertices in A_t, and denote $B_t = A_t \cup \partial(A_t)$. The set of remaining vertices in $V \setminus B_t$ will be denoted by R_t. Initially, S_0, A_0 and B_0 are empty, and $R_t = V$.

Our greedy algorithm has two phases, each employing a different greedy rule. It switches between phases once B_t becomes the majority of the graph. Specifically, at round $t \geq 1$, if $A_{t-1} \neq V$, the greedy algorithm proceeds as follows:

1. If $|B_{t-1}| < n/2$, select as seed s_t a vertex $v \in (V \setminus A_{t-1})$ such that $S_t = S_{t-1} \bigcup \{v\}$ maximizes $|B_t|$ (after applying the activation cascade).

2. If $|B_{t-1}| \geq n/2$, select as seed s_t a vertex $v \in (V \setminus A_{t-1})$ such that $S_t = S_{t-1} \bigcup \{v\}$ maximizes $|A_t|$ (after applying the activation cascade).

We let T denote the total number of rounds until $A_T = V$. We now establish that $T = O(\frac{n \log d}{\epsilon^2 d^2})$. The following lemma does not require any expansion properties.

Lemma 7.1 *Let $G(V, E)$ be an arbitrary d-regular graph. Then for $t \leq \frac{n}{d}$ the above greedy algorithm can maintain $|B_t| \geq \frac{d}{2}t$.*

Proof: By induction on t. For $t = 1$ we have $A_1 = \{s_1\}$ and hence $|A_1| = 1$, $\partial(A_1) = d$, and $|B_1| = d + 1 \geq d/2$. Assume now that the lemma holds for $t < \frac{n}{d}$ and prove for $t+1$. If $|B_t| \geq \frac{d}{2}(t+1)$ there is nothing to prove. Hence we may assume that $|B_t| < \frac{d}{2}(t+1) \leq n/2$, implying that $|R_t| \geq n/2$. Therefore $\sum_{v \in V} deg_{R_t}(v) \geq \frac{nd}{2}$, and a random vertex has in expectation at least $d/2$ neighbors in R_t. Hence there is at least one vertex v with at least $d/2$ neighbors in R_t. It cannot be that $v \in A_t$ because vertices in A_t have no neighbors in R_t. Hence taking this vertex v as s_t we have $|B_{t+1} \setminus B_t| \geq d/2$, proving the inductive step. $\square$

The weakness of Lemma 7.1 is that the rate of growth of B_t is limited to $O(dt)$. To reach B_T linear in n will require $T \geq \Omega(n/d)$, which we cannot afford. Hence we shall want to establish that B_t grows at a rate significantly larger than d per round. This is clearly not true in the first set of rounds (in particular, $|B_1| = d+1$), but we shall show that it becomes true after t exceeds n/d^2. Our next lemma does use expansion properties of G.

Lemma 7.2 *For $0 < \epsilon < 1$, let $G(V, E)$ be an (n, d, λ)-graph with $\lambda \leq (1-\epsilon)d$ and without 4-cycles. Let $\frac{4}{\epsilon^2 d} \leq c \leq \frac{d}{2}$. Let A be an arbitrary set of activated vertices in G, let $B = A \cup \partial(A)$ and let $R = V \setminus B$. If $|B| = \frac{cn}{d}$ then there is a vertex $u \in R$ such that $|R \cap N(B \cap N(u))| \geq c\epsilon^2 d/2$.*

Proof: Three vertices $u, v \in R$ and $w \in B$ will be called a *triplet* if $(u, w) \in E$ and $(v, w) \in E$. Let f denote the number of triplets in G. For $w \in B$, let $d_R(w) = |N(w) \cap R|$. Then $f = \sum_{w \in B} \binom{d_R(w)}{2}$. Using Lemma 2.1, $\sum_{w \in B} d_R(w) = e(B, R) \geq \frac{\epsilon d |B||R|}{n}$. Hence the average value of $d_R(w)$ is at least $\frac{\epsilon d |R|}{n}$, implying by convexity that

$$f \geq |B|\binom{\epsilon d|R|/n}{2} \simeq \frac{|R|^2|B|\epsilon^2 d^2}{2n^2}.$$

Every triplet involves two vertices from R. Hence on average, a vertex from R is involved in $2f/|R|$ triplets. This together with the lower bound on f implies that there is some $u \in R$ involved in at least $\frac{|R||B|\epsilon^2 d^2}{n^2}$ triplets. In any two such triplets, (u, w_1, v_1) and (u, w_2, v_2) $(v_1, v_2 \in R)$, the vertices v_1 and v_2 must be distinct, because G has no 4-cycles. This implies that $|R \cap N(B \cap N(u))| \geq \frac{|R||B|\epsilon^2 d^2}{n^2}$. Substituting $|B| = cn/d$ and noting that $|R| \geq n/2$, the lemma follows. □

We now proceed to prove Theorem 6:

Proof: Lemma 7.1 implies that for $t = \frac{8n}{\epsilon^2 d^2}$ the greedy algorithm reaches $|B_t| \geq \frac{4n}{\epsilon^2 d}$. Thereafter, in every $O(\frac{n}{\epsilon^2 d^2})$ iterations of the algorithm, Lemma 7.2 implies that B_t grows by a multiplicative factor of 2 (in every iteration choose the vertex u whose existence is guaranteed by Lemma 7.2). It follows that for $T \leq O(\frac{n \log d}{\epsilon^2 d^2})$ the greedy algorithm manages to achieve $|B_T| \geq \frac{n}{2}$, and the first phase of the greedy algorithm ends.

We now analyze the second phase of the greedy algorithm. We may assume that $|A_t| \leq \frac{n}{\epsilon d}$, because otherwise the whole graph is activated, by Lemma 1.1. Moreover, we may assume that $d > \frac{10}{3\epsilon}$, as otherwise the statement of Theorem 6 only requires $m(G, 2) \leq O(n \log d)$ which is trivially true. For this range of parameters, $|\partial(A_t)| = |B_t| - |A_t| \geq \frac{n}{2} - \frac{n}{\epsilon d} > \frac{2n}{5}$. Each vertex in $\partial(A_t)$ has exactly one neighbor in A_t, and hence $e(\partial(A_t), V \setminus A_t) \geq (d-1)\frac{2n}{5} \geq \frac{dn}{3}$. This implies that there is some vertex in $V \setminus A_t$ whose activation will activate at least $d/3$ new vertices. Hence the greedy algorithm activates at least $d/3$ vertices in each step of the second round, implying that in $O(\frac{n}{\epsilon d^2})$ rounds of the second phase $|A_t|$ exceeds $\frac{n}{\epsilon d}$. Lemma 1.1 then implies that the whole graph is activated. □

8 Bounds for $m(G, r)$: $r > 2$

In this section we give upper bounds for $m(G, r)$ where r is a small constant (e.g., 3,4) not depending on d. The ideas are similar to Section 4, hence our proofs are less detailed.

Lemma 8.1 *Let G be an (n, d, λ)-graph such that $\lambda < \delta d$ and $\delta < 1$. Suppose that the activation threshold of every vertex is r which is independent of d. Then every set of size larger than $\frac{(r-1)n}{(1-\delta)d}$ is contagious.*

Proof: Consider a set S of size $|S|$ that is not contagious. We can assume without loss of generality that S is inclusion-maximal with respect to being active (namely, *every* vertex *not* belonging to S is not active). For every $u \in$

$V \setminus S$ it holds that $deg_S(u) \leq r-1$. Thus $e(S, V \setminus S) \leq (r-1)(n-|S|)$. On the other hand, by Lemma 2.1

$$e(S, V \setminus S) \geq \frac{(1-\delta)d|S|(n-|S|)}{n}.$$

Combining these inequalities we have that

$$\frac{(1-\delta)d|S|(n-|S|)}{n} \leq (r-1)(n-|S|)$$

Hence $|S| \leq \frac{(r-1)n}{(1-\delta)d}$. $\square$

Theorem 12 *Let G be a d-regular graph with girth $\Omega(\log\log d)$. Then there is a contagious set of size $C(r)nd^{-\frac{r}{r-1}}$ where $C(r)$ is a constant depending only on r.*

Proof: The proof is similar to the proof of Lemma 3.1. Again, we consider $T_{d,k}$ the complete d-regular tree of depth k. Recall that a vertex in $T_{d,k}$ is said to be at level ℓ with $0 \leq \ell \leq k$ if its distance from the root is ℓ. Activate all the leafs of $T_{d,k}$ independently with probability $h(k)d^{-\frac{r}{r-1}}$ where $h(k) = (2e \cdot r!)^{\frac{2}{r}} d^{\frac{1}{r^{k-1}}}$ Let p_i $(0 \leq i \leq k)$ be the probability that a vertex in level $k-i$ gets activated. Hence $p_0 = p$ and p_k is the probability of the root being activated in the bootstrap percolation process. We shall write $p_i = h_i d^{-\frac{r}{r-1}}$ with $h_0 = h(k)$. An internal vertex w of the tree becomes activated if it has at least r active children. Hence for $j < k$, using the Poisson approximation $\Pr(\text{Bin}(d,q) = r) \sim e^{-qd}(qd)^r/r!$ we get

$$p_{j+1} \geq \Pr(\text{Bin}(d,p_j) \geq r) \sim e^{-p_j d}(p_j d)^r/r!.$$

As long as $p_j \leq \frac{1}{d}$ then we have that $h_{j+1} \geq \frac{1}{2 \cdot e \cdot r!}(h_j)^r$, and by induction we have that

$$p_i \geq (2e \cdot r!)^{\frac{2}{r}} \left(\frac{h_0}{(2e \cdot r!)^{\frac{2}{r}}}\right)^{r^i} d^{-\frac{r}{r-1}} = (2e \cdot r!)^{\frac{2}{r}} d^{\frac{1}{r^{k-1-i}}} d^{-\frac{r}{r-1}}.$$

Substituting $i = k-1$, the children of the root have probability at least $\frac{1}{d}$ to become active, implying that $p_k \geq B$ where $B > 0$ is a constant independent of d. The theorem now follows from Corollary 1.1. $\square$

Theorem 13 *Given an integer l, let G be an (n,d,λ) graph such $\lambda \leq \frac{1}{\sqrt{l}}d$, and l is sufficiently large. Then $m(G,r) = O(\frac{n}{l^{\frac{r}{r-1}}})$. In particular if $\lambda = O(\sqrt{d})$ then $m(G,r) = O(\frac{n}{d^{\frac{r}{r-1}}})$.*

Proof: Follows from Theorem 12, the proof of Theorem 9, and Lemma 8.1. □

Let us comment that similar ideas to the $r = 2$ case, can be used to show our upper bounds are nearly best possible. Namely the ideas in [26] imply that for a fixed $\epsilon > 0$, there exist d_0 such that for every $d > d_0$, if G is a random d-regular graph then w.h.p.

$$m(G, r) \geq nd^{-(\frac{r}{r-1}+\epsilon)}.$$

9 Hardness of target set selection in regular graphs

We set the activation threshold r to be 2 throughout this section. Recall that it is known that $m(G, 2)$, the size of the smallest contagious set, is hard to approximate within any constant factor (and even for factors that depend on n) [22]. The following theorem implies that approximating $m(G, 2)$ in regular graphs is roughly as hard as doing so in arbitrary graphs.

Theorem 14 *There is a polynomial time reduction that for every n and every $2 \leq \Delta \leq n-1$, given an arbitrary graph G with n vertices and maximum degree Δ, transforms G into a Δ-regular graph H on $O(n\Delta^2)$ vertices, such that*

$$m(G, 2) \leq m(H, 2) \leq 6m(G, 2)$$

Proof: Given Δ, we introduce a certain graph that we call a Δ-*regularizer*, which will be used as a gadget in our reduction. The Δ-regularizer is a complete graph on $\Delta+1$ vertices, but with three of its edges removed. The removed edges are picked in such a way that they form a triangle. Hence three vertices, that we call *connector vertices*, have degree $\Delta-2$, and the remaining vertices have degree Δ. Observe that if the three connector vertices are activated, this activates the remaining vertices in the Δ-regularizer. (In fact, when $\Delta \geq 4$, any two vertices are a contagious set for the Δ-regularizer, but this fact is not needed for our proof.)

Given a graph $G(V, E)$ on n vertices and with maximum degree Δ, our reduction works as follows. Make six independent copies of G (with no edges between different copies). Hence now every vertex $v \in V$ has six copies, $v_1, \ldots, v_6$. Let d_v denote the degree of v in G. If $d_v < \Delta$, we wish to raise the degrees of each of the vertices of $v_1, \ldots, v_6$ to Δ. To do this we introduce $\Delta - d_v$ fresh copies of the the Δ-regularizer gadget. For every copy of these Δ-regularizers, we introduce edges between its three connector vertices and the six copies of v, such that each copy of v gets one new edge, and each connector vertex gets two new edges. Hence all vertices of the Δ-regularizer become of degree Δ, and every copy of v gets $\Delta - d_v$ new edges, making it of degree Δ as well. Repeating this process for every vertex $u \in V$ (each time with fresh copies of Δ-regularizers) completes the description of the Δ-regular graph H.

To see that $m(H,2) \leq 6m(G,2)$, consider an arbitrary contagious set in G, and observe that taking six copies of this set, one in each copy of G, will also activate all of H.

To see that $m(G,2) \leq m(H,2)$, consider an arbitrary contagious set S in H, and observe that the following set S' is contagious in G: include vertex v in S' if and only if at least one of its six copies or at least one of the vertices in its Δ-regularizers is in S.

Further details are omitted from the proof. □

In the statement and proof of Theorem 14 we preferred simplicity, and hence made no attempt to minimize the size of H or to tighten the relation between $m(G,2)$ and $m(H,2)$.

10 Contagious sets in non-regular expanding graphs

Our work in this manuscript is concerned with contagious sets in regular graphs, and in nearly regular random graphs. In this section we discuss how insights obtained from these results extend to graphs that are not regular. Rather than attempt to formally define expansion in non-regular graphs (there are several alternative definitions that one may consider), we shall limit our discussion to random graphs (under various models), which would qualify as very good expanders under any reasonable definition of expansion.

Let us set the activation threshold r to be 2 throughout this section. A natural model for random irregular graphs is as follows. Given the number of vertices n, one first fixes a degree sequence $d_1 \leq d_2 \ldots, \leq d_n \leq n-1$, where $\sum_i d_i$ is even. We shall assume that $d_1 \geq 2$, because the activation threshold is 2. Thereafter one draws a multigraph at random using the *configuration model* with this degree sequence. Namely, a vertex i corresponds to d_i endpoints of edges, and the multi-graph is generated by selecting a random matching between all endpoints. Thereafter, self loops are removed, and among parallel edges, only one edge is maintained. For degree sequences that will interest us, self loops and parallel edges will be rare and their removal will not significantly change the degree sequence.

Rather than study the configuration model directly, it would be simpler to consider an alternative process for generating a random non-regular graph, which we illustrate by the following example. Let d be roughly $n^{1/4}$ for concreteness. Generate a random graph G of average degree roughly d using the Erdos-Renyi random graph model $G_{n,p}$ with $p = \frac{d}{n-1}$. By the results of [33], a random subset of $\frac{(1+\delta)n}{2d^2}$ vertices is almost surely contagious. By our Theorem 7, the smallest contagious set is of size $\Omega\left(\frac{n}{d^2 \log np}\right)$. Now modify G to become a non-regular expander G' as follows: pick at random two disjoint sets of vertices A and B, each of size $k = \frac{n}{d^2}$, and within every set, unite all vertices of the set to get a single vertex, thus obtaining vertices a and b. Removing parallel edges and self loops that might be generated by this process, each of the vertices a and b has degree roughly $\frac{n}{d}$, whereas the degrees of the remaining vertices remain roughly d. In G', the set $\{a, b\}$ is almost surely contagious. (Had we not removed parallel edges, each of a or b by itself would be contagious, and the fact that we take both a and b compensates for the removal of parallel edges. Details are omitted.) Moreover, a and b have multiple common neighbors, and any set of two such common neighbors is contagious as well (because it activates a and b).

Returning to the configuration model, the above argument shows that for a degree sequence that has $n-2$ vertices of degree roughly $n^{1/4}$ and two vertices of degree roughly $n^{3/4}$, the size of the smallest contagious set in the corresponding random graph is almost sure the minimum possible, namely, two. Moreover, the contagious set need not contain the high degree vertices. Observe that the average degree $\bar{d}$ of G' is roughly $n^{1/4}$, and hence though an upper bound of $O\left(n/(\bar{d})^2\right)$ on the size of the contagious set holds, this upper bound is very far from being tight.

Let us now modify the degree sequence by scaling all degrees by a factor of $1/\log n$. Namely, there are $n-2$ vertices of degree roughly $n^{1/4}/\log n$ and two vertices of degree roughly $n^{3/4}/\log n$. Observe that for the original nearly regular graph G, such a scaling would increase the size of the smallest contagious set by a modest polylogarithmic factor. However, this has a dramatic effect regarding G'. The vertices a and b no longer correspond to sets that are sufficiently large to be contagious, and hence the size of the smallest contagious set jumps to at least $\Omega\left(\frac{n}{d^2 \log n}\right) = \Omega(\sqrt{n}\log n)$.

The example above was presented so as to convey two messages.

- Understanding contagious sets in regular graphs leads us a long way towards understanding contagious sets in irregular graphs. Specifically, in the example above, the non-regular graph G' could be analyzed as a graph derived from a nearly regular graph G.

- Results regarding irregular graphs are much more sensitive to a change in the underlying parameters than the results for regular graphs. Multiplying the degree sequence by a small factor has only a small effect on the size of contagious sets in regular graphs, but a dramatic effect in non-regular graphs. Hence for non-regular graphs, even for random ones, we should not expect to have a single simple parameter (such as average degree) that roughly characterizes the size of contagious sets. This is unlike the case of random nearly regular graphs for which the average degree provides a rough characterization.

Further discussion of contagious sets in irregular graphs is beyond the scope of the current paper.

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References

[1] A. Aazami, M.D. Stilp. Approximation algorithms and hardness for domination with propagation. *SIAM Journal on Discrete Mathematics*, 23: 1382-1399, 2009.

[2] E. Ackerman, O. Ben-Zwi, and G. Wolfovitz. Combinatorial model and bounds for target set selection. *Theoretical Computer Science*, 411: 4017–4022, 2010.

[3] N. Alon, and M. R. Capalbo. Finding disjoint paths in expanders deterministically and online. In *FOCS*, 518–524, 2007.

[4] N. Alon and F. R. K. Chung. Explicit construction of linear sized tolerant networks. *Discrete Mathematics*, 306: 1068–1071, 2006.

[5] N. Alon, H. Kaplan, M. Krivelevich, D. Malkhi, and J. P. Stern. Scalable secure storage when half the system is faulty. *Information and Computation*, 174: 203–213, 2002.

[6] N. Alon and J. Spencer. *The Probablistic Method.* Wiley, 2008.

[7] S. Arora, S. Khot, A. Kolla, D. Steurer, M. Tulsiani, and N. K. Vishnoi. Unique games on expanding constraint graphs are easy. In *STOC*, 21–28, 2008.

[8] A.Bagchi, A. Bhargava, A. Chaudhary, D. Eppstein, and C. Scheideler. The effect of faults on network expansion. *Theory Comput. Syst*, 39: 903–928, 2006.

[9] J. Balogh and B. Bollobás. Bootstrap percolation on the hypercube, in *Probabilty Theory and Related Fields*, 134: 624–648, 2006.

[10] J. Balogh, B. Bollobás, H. Duminil-Copin and R. Morris. The sharp threshold for bootstrap percolation in all dimensions, in *Trans. Amer. Math. Soc.*, to appear.

[11] J. Balogh, B. Csaba, M. Pei and W. Samotij. Large bounded degree trees in expanding graphs. *Electronic Journal of Combinatorics*, 17: 2010, Publication R6.

[12] J. Balogh, Y. Peres and G. Pete. Bootstrap percolation on infinite trees and non-amenable groups. *Combinatorics, Probability and Computing*, 15: 715–730, 2006.

[13] J. Balogh and G. Pete. Random disease on the square grid. *Random Structures and Algorithms*, 13: 409–422, 1998.

[14] J. Balogh and B. Pittel. Bootstrap percolation on the random regular graph. *Random Structures and Algorithms*, 30: 257–286, 2007.

[15] A. L. Barabasi and R. Albert. Emergence of scaling in random networks. *Science*, 286: 509-512, 1999.

[16] O. Ben-Zwi, D. Hermelin, D. Lokshtanov, and I. Newman. Treewidth governs the complexity of target set selection. *Discrete Optimization*, 8: 87–96, 2011.

[17] L. E. Blume, D. A. Easley, J. M. Kleinberg, R. Kleinberg and E. Tardos. Which networks are least susceptible to cascading failures? In *FOCS*, 393–402, 2011.

[18] B. Bollobás. *Random graphs*. Cambridge Stud. Adv. Math. 73, Cambridge University Press, Cambridge, 2001.

[19] B. Bollobás, C. Holmgren, P.J. Smith, and A.J. Uzzell. The time of bootstrap percolation with dense initial sets. *Annals of Probability*, to appear.

[20] A. Z. Broder, A. M. Frieze, and E. Upfal. Existence and construction of edge disjoint paths on expander graphs. In *STOC*, 140–149, 1992.

[21] J. Chalupa, P. L. Leath and G. R. Reich. Bootstrap percolation on a Bethe latice. *J. Phys. C: Solid State Phys.*, 12, p L31, 1979.

[22] N. Chen. On the approximability of influence in social networks. *Siam Journal of Discrete Math*, 23: 1400–1415, 2009.

[23] M. Chopin, A. Nichterlein, R. Niedermeier and M. Weller. Constant thresholds can make target set selection tractable. In *MedAlg*, 120–133, 2012.

[24] A. Coja-Oghlan, U. Feige, M. Krivelevich and D. Reichman. Contagious sets in expanders. arXiv prepreint, Febuary 2014, arXiv:1306.2465.

[25] P. Domingos and M. Richardson. Mining the network value of customers, In *KDD*, 57-66, 2001.

[26] U. Feige, M. Krivelevich and D. Reichman. Contagious sets in random graphs. *Manuscript*.

[27] J. Friedman. A proof of Alon's second eigenvalue conjecture and related problems. *Memoirs of the AMS*, 910: 720–724, 2008.

[28] J. Friedman and N. Pippenger. Expanding graphs contain all small trees. *Combinatorica*, 7: 71-76, 1987.

[29] M. Granovetter Threshold models of collective behavior. *American Journal of Sociology*, 83: 1420–1443, 1978.

[30] P. Haxell. Tree embeddings. *Journal of Graph Theory*, 36: 121–130, 2001.

[31] S. Hoory, N. Linial, and A. Widgerson. Expander graphs and their applications. *Bulletin of the American Mathematical Society*, 43: 439–561, 2006.

[32] S. Janson. On percolation in random graphs with given degree sequence. *Electronic Journal of Probability*, 14: 86–118, 2009.

[33] S. Janson, T. Łuczak, T. Turova and T. Vallier. Bootstrap percolation on the random graph $G_{n,p}$. *Annals of Appied Probability*, 22: 1989–2047, 2012.

[34] D. Kempe, J. Kleinberg and E. Tardos. Maximizing the spread of influence through a social network. In *KDD*, 137–146, 2003.

[35] J. Kleinberg and R. Rubinfeld. Short paths in expander graphs. In *FOCS*, 86–95, 1996.

[36] M. Krivelevich and B. Sudakov. Pseudo-random graphs. In *More Sets, Graphs and Numbers, Bolyai Society Mathematical Studies 15, Springer*, 199–262, 2006.

[37] M. Krivelevich and B. Sudakov. Minors in expanding graphs. *Geometric and Functional Analysis*, 19, 294–331, 2009.

[38] E. Mossel and S. Roch. Submodularity of influence in social networks: From local to global. *SIAM Journal of Computing*, 39: 2176–2188, 2010.

[39] D. Peleg. Size bounds for dynamic monopolies. *Discrete Applied Mathematics*, 86: 263–273, 1998.

[40] L. Pósa. Hamiltonian circuits in random graphs. *Discrete Mathematics*, 14: 359-364, 1976.

[41] I. Rapaport, K. Suchan, I. Todinca, J. Verstraéte. On dissemination thresholds in regular and irregular graph classes. *Algorithmica*, 59: 16–34, 2011.

[42] D. Reichman. New bounds for contagious sets. *Discrete Mathematics*, 312: 1812–1814, 2012.

[43] T. Tlusty and J. P. Eckmann. Remarks on bootstrap percolation in metric networks. *J. Phys. A*, 205004, 2009.

[44] N. C. Wormald. Models of random regular graphs. In J. Lamb and D. Preece, editors, Surveys in Combinatorics, volume 276 of London Mathematical Society Lecture Note Series, pages 239-298. Cambridge University Press, 1999.