On the asymptotic value of the choice number of complete multi-partite graphs

Nurit Gazit[*], Michael Krivelevich[†]

September 11, 2006

[*]School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel. E-mail: perfect@post.tau.ac.il.

[†]School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel. E-mail: krivelev@post.tau.ac.il. Research supported in part by USA-Israel BSF Grant 2002-133, and by grant 64/01 from the Israel Science Foundation.

Abstract

We calculate the asymptotic value of the choice number of complete multi-partite graphs, given certain limitations on the relation between the sizes of the different sides. In the bipartite case, we prove that if $n_0 \leq n_1$ and $\log n_0 \gg \log \log n_1$, then $ch(K_{n_0,n_1}) = (1+o(1))\frac{\log_2 n_1}{\log_2 x_0}$, where x_0 is the unique root of the equation $x - 1 - x^{\frac{k-1}{k}} = 0$ in the interval $[1, \infty)$ and $k = \frac{\log_2 n_1}{\log_2 n_0}$. In the multi-partite case, we prove that if $n_0 \leq n_1 ... \leq n_s$, and n_0 is not too small compared to n_s, then $ch(K_{n_0,\ldots,n_s}) = (1+o(1))\frac{\log_2 n_s}{\log_2 x_0}$. Here x_0 is the unique root of the equation $sx - 1 - \sum_{j=0}^{s-1} x^{\frac{k_j - 1}{k_j}} = 0$ in the interval $[1, \infty)$, and for every $0 \leq i \leq s-1$, $k_i = \frac{\log_2 n_s}{\log_2 n_i}$.

Key words: choice number.

1 Introduction

The *choice number* $ch(G)$ of a graph $G = (V, E)$ is the minimum number k such that for every assignment of a list $S(v)$ of at least k colors to each vertex $v \in V$, there is a proper vertex coloring of G assigning to each vertex v a color from its list $S(v)$. The concept of choosability was introduced by Vizing in 1976 [5] and independently by Erdős, Rubin and Taylor in 1979 [2]. It is also shown in [2] that the choice number of the complete bipartite graph $K_{n,n}$ satisfies $ch(K_{n,n}) = (1+o(1)) \log_2 n$. The choice number of the complete multi-partite graph has been investigated by several researchers. Among the results: Alon [1] proved that the choice number of a complete $r-$partite graph with parts of size m is $\Theta(r \log m)$, Kierstead [3] proved that the choice number of a complete $r-$partite graph with parts of size 3 is $[(4r-1)/3]$, and Reed and Sudakov [4] proved that if the number of parts r in the complete $r-$partite graph on n vertices is very large, i.e. $\frac{r}{n} = c$ for any constant $c > \frac{1}{2}$, then the choice number is r. In this paper we calculate the asymptotic value of the choice number of a general complete bipartite graph K_{n_0,n_1} and then expand the result to the case of a complete multi-partite graph. We begin by proving (note that throughout this paper all logs are binary):

Theorem 1 *Let $2 \leq n_0 \leq n_1$ be integers, and let $n_0 = (\log n_1)^{\omega(1)}$. Denote $k = \frac{\log n_1}{\log n_0}$. Let x_0 be the unique root of the equation $x - 1 - x^{\frac{k-1}{k}} = 0$ in the interval $[1, \infty)$. Then $ch(K_{n_0,n_1}) = (1 + o(1)) \frac{\log n_1}{\log x_0}$.*

As usual, $\omega(1)$ stands for a function tending to infinity arbitrarily slowly as its variable tends to infinity. Notice that for the case of equal parts (i.e., when $n_0 = n_1$), we have $k = 1$, $x_0 = 1$ and thus $ch(K_{n_0,n_0}) = (1+o(1)) \log n_0$, matching (naturally) the above mentioned result of Erdős, Rubin and Taylor [2].

We will prove the theorem in two parts, showing first the upper bound and then the lower bound. In the graph K_{n_0,n_1} we label the group of n_0

vertices by V_0 and the group of n_1 vertices by V_1.

2 The Upper Bound

Theorem 2 *Let $2 \leq n_0 \leq n_1$ be integers. Denote $k = \frac{\log n_1}{\log n_0}$. Let x_0 be the unique root of the equation $x - 1 - x^{\frac{k-1}{k}} = 0$ in the interval $[1, \infty)$. Then $ch(K_{n_0,n_1}) \leq \lceil \frac{\log n_1}{\log x_0} \rceil + 1$.*

Proof.

Lemma 2.1 *If there exists a $p, 0 \leq p \leq 1$, s.t. $n_0 p^r + n_1(1-p)^r \leq 1$ then $ch(K_{n_0,n_1}) \leq r$.*

Proof. We show that given, for each vertex $v \in V(K_{n_0,n_1})$, a set of colors $S(v)$ of size r, there is a proper vertex coloring of the graph, assigning to each vertex v a color from $S(v)$.

We partition the set of all available colors $S = \bigcup_{v \in V} S(v)$ into two subsets S_1 and S_0 in the following manner: each color $c \in S$ is chosen randomly and independently with probability p to be in S_1, and with probability $1-p$ to be in S_0. We will show that with positive probability the sets S_0 and S_1 chosen satisfy the condition: each vertex $v \in V_0$ has a color $c \in S(v)$ s.t. $c \in S_0$, and each vertex $v \in V_1$ has a color $c \in S(v)$ s.t. $c \in S_1$. Given such S_0 and S_1, we can color each vertex in V_0 with a color from S_0, and each vertex in V_1 with a color from S_1, and since $S_0 \cap S_1 = \emptyset$, we get a proper coloring.

For each $v \in V_1$ the probability that a bad event occurs, i.e. that all the colors in $S(v)$ are chosen to be in S_0, is $(1-p)^r$. For each $v \in V_0$ the probability that a bad event occurs, i.e. that all the colors in $S(v)$ are chosen to be in S_1, is p^r. Therefore the expectation of the number of bad events that occur is $n_0 p^r + n_1(1-p)^r \leq 1$. Since either $p > 0$ or $1 - p > 0$, we can assume w.l.o.g. that $1 - p > 0$. Then since, for example, the case in which all the colors in S are chosen to be in S_0 happens with probability

$(1-p)^{|S|} > 0$, and gives n_1 bad events, the case in which 0 events occur also happens with positive probability (otherwise the expectation would be greater than 1). Therefore we get the desirable partition. ■

Lemma 2.2 *Given r s.t. $(\frac{1}{n_0})^{\frac{1}{r-1}} + (\frac{1}{n_1})^{\frac{1}{r-1}} \geq 1$, let $p = \frac{(\frac{1}{n_0})^{\frac{1}{r-1}}}{(\frac{1}{n_0})^{\frac{1}{r-1}}+(\frac{1}{n_1})^{\frac{1}{r-1}}}$. Then $n_0 p^r + n_1(1-p)^r \leq 1$.*

Proof. If $p = \frac{(\frac{1}{n_0})^{\frac{1}{r-1}}}{(\frac{1}{n_0})^{\frac{1}{r-1}}+(\frac{1}{n_1})^{\frac{1}{r-1}}}$ then $(\frac{p}{1-p})^{r-1} = \frac{n_1}{n_0}$. Therefore

$$\begin{aligned}
n_0 p^r + n_1(1-p)^r &= n_0 p^r + n_1(\frac{n_0}{n_1})p^{r-1}(1-p) = n_0 p^{r-1} \\
&= n_0 \left(\frac{(\frac{1}{n_0})^{\frac{1}{r-1}}}{(\frac{1}{n_0})^{\frac{1}{r-1}} + (\frac{1}{n_1})^{\frac{1}{r-1}}} \right)^{r-1} = \left(\frac{1}{(\frac{1}{n_0})^{\frac{1}{r-1}} + (\frac{1}{n_1})^{\frac{1}{r-1}}} \right)^{r-1} \\
&\leq 1 \,.
\end{aligned}$$

■

All that remains now is to choose $r = r(n_0, n_1)$ satisfying the condition of Lemma 2.2. Let $r = \lceil \frac{\log n_1}{\log x_0} \rceil + 1$. Then $r - 1 \geq \frac{\log n_1}{\log x_0}$, and hence $x_0 \geq n_1^{\frac{1}{r-1}}$. Since the function $f_k(x) = x - 1 - x^{\frac{k-1}{k}}$, where $k \geq 1$, is a monotonely increasing function in the interval $[1, \infty)$, and since $f_k(x_0) = 0$, it follows that $n_1^{\frac{1}{r-1}} \leq 1 + n_1^{\frac{1}{r-1}\frac{k-1}{k}} = 1 + (\frac{n_1}{n_0})^{\frac{1}{r-1}}$ as required. ■

3 The Lower Bound

Theorem 3 *If $2 \leq n_0 \leq n_1$ are integers, and $n_0 = (\log n_1)^{\omega(1)}$, then $ch(K_{n_0,n_1}) \geq (1-o(1))\frac{\log n_1}{\log x_0}$, where x_0 is the unique root of the equation $x - 1 - x^{\frac{k-1}{k}} = 0$ in the interval $[1, \infty)$ and $k = \frac{\log n_1}{\log n_0}$.*

Proof.

A *cover* of a hypergraph H is a subset M of the vertices of the hypergraph such that every hyperedge of H contains at least one vertex of M. A minimum cover is a cover which has the least cardinality among all covers.

Let us generate the hypergraph H_0 created by the color lists of the vertices in V_0, i.e. the hypergraph whose vertices are the colors $\bigcup_{v\in V_0} S(v)$, and whose edges are the lists $S(v)$ for each $v \in V_0$. In the same way, we generate the hypergraph H_1 created by the color lists of the vertices in V_1.

For any r, if we wish to prove $ch(K_{n_0,n_1}) > r$, it is enough to show that there are parameters $t \geq r$ and $0 \leq l \leq t$ s.t. it is possible to choose for each vertex in K_{n_0,n_1} a list of r colors from $\{1, 2, ...t\}$, and the lists chosen satisfy:

1. The minimum cover of the hypergraph H_0 created by the color lists of the vertices in V_0 (i.e. the minimum size of a set L of colors s.t. for every $v \in V_0$, $S(v)$ contains at least one of the colors in L) is of cardinality at least l .

2. The minimum cover of the hypergraph H_1 created by the color lists of the vertices in V_1 is of cardinality at least $t - l + 1$.

If these conditions are satisfied, then when these color lists are assigned to the vertices of K_{n_0,n_1}, the graph cannot be properly colored. This is because at least l colors are needed to color one side, and at least $t-l+1$ to color the other. Since there are only t colors in all, at least one color will be chosen by both sides – i.e., at least two vertices on opposite sides must be given the same color, implying that a proper coloring is not possible. Therefore, the choice number of the graph is greater than r.

Lemma 3.1 *If there exist parameters t and l such that $t \geq r, 0 \leq l \leq t$ and*

$$2^t e^{-\frac{(l)_r}{(t)_r} n_1} + 2^t e^{-\frac{(t-l)_r}{(t)_r} n_0} \leq 1 \tag{1}$$

then $ch(K_{n_0,n_1}) > r$.

Proof. It is easy to see that at least l colors are required for a cover of the hypergraph H_0 created by the color lists of the vertices in V_0 if and only if for each subset C of size $t-l+1$ of $\{1, 2, ...t\}$ there is at least one $v \in V_0$ for which $S(v) \subset C$. In the same way, the minimum cover of the hypergraph H_1 created by the color lists of the vertices in V_1 is at least $t-l+1$ if and only if for each subset C of size l of $\{1, 2, ...t\}$ there is at least one $v \in V_1$ for which $S(v) \subset C$.

For each vertex v in K_{n_0,n_1}, let $S(v)$ be a random subset of cardinality r of $\{1, 2, ...t\}$, chosen uniformly and independently among all $\binom{t}{r}$ subsets of cardinality r of $\{1, 2, ...t\}$. We wish to find an r that guarantees that with positive probability:

1. For every subset C of size $t-l+1$ there is a vertex $v \in V_0$ s.t. $S(v) \subset C$, and

2. For every subset C of size l there is a vertex $v \in V_1$ s.t. $S(v) \subset C$.

To simplify the calculations, we will change Condition 1 above to the stronger condition that:

1. For every subset C of size $t-l$ there is a vertex $v \in V_0$ s.t. $S(v) \subset C$.

For each fixed subset C of cardinality l of $\{1, 2, ...t\}$ and each $v \in V_1$, the probability that $S(v) \not\subseteq C$ is $1 - \frac{l \cdot ... \cdot (l-r+1)}{t \cdot ... \cdot (t-r+1)} = 1 - \frac{(l)_r}{(t)_r}$. Since there are n_1 vertices in V_1 and $\binom{t}{l}$ subsets of cardinality l of $\{1, ...t\}$, and since the color groups of the vertices were chosen independently, the probability that there is a subset C of size l that does not contain $S(v)$ for any $v \in V_1$ is at most $\binom{t}{l}\left(1 - \frac{(l)_r}{(t)_r}\right)^{n_1} < 2^t e^{-\frac{(l)_r}{(t)_r} n_1}$. In a similar fashion, the probability that there is a subset C of size $t-l$ that does not contain $S(v)$ for any $v \in V_0$ is at most $\binom{t}{t-l}\left(1 - \frac{(t-l)_r}{(t)_r}\right)^{n_0} < 2^t e^{-\frac{(t-l)_r}{(t)_r} n_0}$.

We are looking for an r that guarantees that the probability that at least one of Conditions 1 and 2 does not hold is smaller than 1. Therefore it is enough to show the sum of these probabilities is smaller than 1, i.e., it is enough to show: $2^t e^{-\frac{(l)_r}{(t)_r} n_1} + 2^t e^{-\frac{(t-l)_r}{(t)_r} n_0} \leq 1$. ■

Before proceeding to find t and l that fit Lemma 3.1, we derive bounds on x_0 that will be useful at later stages of the proof.

Lemma 3.2 $2 \leq x_0(k) < \max(k, e+2)$

Proof. We begin by showing that if $k > e+1$, then $x_0(k) < k$. Since $f_k(x) = x - 1 - x^{\frac{k-1}{k}}$ is monotonely increasing, we need to show that $f_k(k) > 0$, or $k - k^{\frac{k-1}{k}} - 1 > 0$, or $(k-1)^{\frac{1}{k-1}} > k^{\frac{1}{k}}$. But the function $h(x) = x^{\frac{1}{x}}$ is monotonely decreasing for $x > e$. So if $k > e+1$ then $k-1 > e$ and therefore $(k-1)^{\frac{1}{k-1}} > k^{\frac{1}{k}}$.

It can easily be seen that x_0 increases monotonely as a function of k (i.e. if $k_2 \geq k_1$, $x_0(k_2) \geq x_0(k_1)$). Therefore if $k \leq e+2$, then $x_0(k) \leq x_0(e+2) < e+2$.

To prove the lower bound on x_0, observe that $f_k(2) = 2 - 1 - 2^{\frac{k-1}{k}} = 1 - 2^{\frac{k-1}{k}} \leq 0$ for every $k \geq 1$. ■

Lemma 3.3 *Let* $n_0 = (\log n_1)^{\omega(1)}$. *Define* $r_0 = \frac{\log n_1}{\log x_0}$, $u = \frac{4 \log\log n_1}{\log n_0} r_0$ *and* $r = r_0 - u$. *Then* $r = (1 - o(1)) r_0$, *and for* $t = (\frac{n_1}{n_0})^{\frac{1}{r}} r^2$ *and* $l = t \frac{1}{(\frac{n_1}{n_0})^{\frac{1}{r}} + 1}$, $2^t e^{-\frac{(l)_r}{(t)_r} n_1} + 2^t e^{-\frac{(t-l)_r}{(t)_r} n_0} \leq 1$.

Proof. If $n_0 = (\log n_1)^{\omega(1)}$ then $\log\log n_1 \ll \log n_0$, and therefore $u = o(r_0)$, and $r = (1 - o(1)) r_0$, as required. From the fact that $r = (1 - o(1)) r_0$, it also follows that $r = \omega(1)$. This is because $x_0 < \max(k, e+2)$, and therefore, if $k \leq e+2$ then $r_0 = \frac{\log n_1}{\log x_0} > \frac{\log n_1}{\log(e+2)} = \omega(1)$, and otherwise $r_0 = \frac{\log n_1}{\log x_0} > \frac{\log n_1}{\log k} = \frac{\log n_1}{\log \frac{\log n_1}{\log n_0}} = \frac{\log n_1}{\log\log n_1 - \log\log n_0} \geq \frac{\log n_1}{\log\log n_1} = \omega(1)$. Hence $r = (1 - o(1)) r_0 = \omega(1)$.

Let us denote $l_0 = l$ and $l_1 = t - l$. Then $t - l_i = t \frac{(\frac{n_1}{n_i})^{\frac{1}{r}}}{(\frac{n_1}{n_0})^{\frac{1}{r}} + 1}$, and $2^t e^{-\frac{(l)_r}{(t)_r} n_1} + 2^t e^{-\frac{(t-l)_r}{(t)_r} n_0} = \sum_{i=0}^{1} 2^t e^{-\frac{(t-l_i)_r}{(t)_r} n_i}$. In order for this sum to be not greater than 1, it is enough to show that $\frac{(t-l_i)_r}{(t)_r} n_i \gg t$ for $i = 0, 1$. We begin by estimating $\frac{(t-l_i)_r}{(t)_r} n_i$.

Claim 3.4 $\frac{(t-l_i)_r}{(t)_r} n_i > \frac{1}{2e^2} \frac{n_1}{\left((\frac{n_1}{n_0})^{\frac{1}{r}}+1\right)^r}$ *for* $i = 0, 1$.

Proof. $\frac{(t-l_i)_r}{(t)_r} > \left(\frac{t-l_i-r}{t-r}\right)^r = \left(\frac{t-l_i}{t}\right)^r \left(\frac{t(t-l_i-r)}{(t-l_i)(t-r)}\right)^r = \left(\frac{t-l_i}{t}\right)^r \left(1 - \frac{l_i r}{(t-l_i)(t-r)}\right)^r > \left(\frac{t-l_i}{t}\right)^r \left(1 - \frac{2l_i r}{(t-l_i)t}\right)^r$, where the last inequality is a result of $r < \frac{t}{2}$.

Now since $\frac{l_0 r}{(t-l_0)t} = \frac{lr}{(t-l)t} = \frac{t\frac{1}{(\frac{n_1}{n_0})^{\frac{1}{r}}+1} r}{t^2 \frac{(\frac{n_1}{n_0})^{\frac{1}{r}}}{(\frac{n_1}{n_0})^{\frac{1}{r}}+1}} = \frac{r}{t(\frac{n_1}{n_0})^{\frac{1}{r}}} \leq \frac{r(\frac{n_1}{n_0})^{\frac{1}{r}}}{t} = \frac{1}{r} = o(1)$, and $\frac{l_1 r}{(t-l_1)t} = \frac{(t-l)r}{lt} = \frac{r(\frac{n_1}{n_0})^{\frac{1}{r}}}{t} = \frac{1}{r} = o(1)$ we get (recalling that $1 - x \geq e^{-x}/2$ for $0 \leq x \leq 1/2$) $\frac{(t-l_i)_r}{(t)_r} > \left(\frac{t-l_i}{t}\right)^r \frac{1}{2e^2}$. Therefore $\frac{(t-l_i)_r}{(t)_r} n_i > \left(\frac{t-l_i}{t}\right)^r n_i \frac{1}{2e^2} = \left(\frac{(\frac{n_1}{n_i})^{\frac{1}{r}}}{(\frac{n_1}{n_0})^{\frac{1}{r}}+1}\right)^r n_i \frac{1}{2e^2} = \frac{1}{2e^2} \frac{n_1}{\left((\frac{n_1}{n_0})^{\frac{1}{r}}+1\right)^r}$. ∎

Hence in order to prove that (1) holds it is now enough to prove that $\frac{n_1}{\left((\frac{n_1}{n_0})^{\frac{1}{r}}+1\right)^r} \gg t$.

Claim 3.5 $\frac{n_1}{\left((\frac{n_1}{n_0})^{\frac{1}{r}}+1\right)^r} \gg t$.

Proof. $\frac{n_1}{\left((\frac{n_1}{n_0})^{\frac{1}{r}}+1\right)^r} = \left(\frac{n_1^{\frac{1}{r}}}{(\frac{n_1}{n_0})^{\frac{1}{r}}+1}\right)^r = \left[\frac{n_1^{\frac{1}{r_0}}}{(\frac{n_1}{n_0})^{\frac{1}{r_0}}+1} \frac{n_1^{\frac{1}{r}-\frac{1}{r_0}}}{(\frac{n_1}{n_0})^{\frac{1}{r}}+1}\left((\frac{n_1}{n_0})^{\frac{1}{r_0}} + 1\right)\right]^r$.

Since $\frac{n_1^{\frac{1}{r_0}}}{\left((\frac{n_1}{n_0})^{\frac{1}{r_0}}+1\right)} = \frac{n_1^{\frac{\log x_0}{\log n_1}}}{(\frac{n_1}{n_0})^{\frac{\log x_0}{\log n_1}}+1} = \frac{x_0}{\frac{x_0}{x_0^{\frac{\log n_0}{\log n_1}}}+1} = \frac{x_0}{x_0^{\frac{k-1}{k}}+1} = 1$, we get $\frac{n_1}{\left((\frac{n_1}{n_0})^{\frac{1}{r}}+1\right)^r} = \left(n_1^{\frac{1}{r}-\frac{1}{r_0}} \frac{(\frac{n_1}{n_0})^{\frac{1}{r_0}}+1}{(\frac{n_1}{n_0})^{\frac{1}{r}}+1}\right)^r > \left(n_1^{\frac{1}{r}-\frac{1}{r_0}} \frac{(\frac{n_1}{n_0})^{\frac{1}{r_0}}}{(\frac{n_1}{n_0})^{\frac{1}{r}}}\right)^r$, where the last inequality follows from $r < r_0$. So $\frac{n_1}{((\frac{n_1}{n_0})^{\frac{1}{r}}+1)^r} > \left(n_1^{\frac{1}{r}-\frac{1}{r_0}} \frac{(\frac{n_1}{n_0})^{\frac{1}{r_0}}}{(\frac{n_1}{n_0})^{\frac{1}{r}}}\right)^r = n_0^{(\frac{1}{r}-\frac{1}{r_0})r} = n_0^{1-\frac{r}{r_0}} = n_0^{\frac{u}{r_0}} = n_0^{\frac{4\log\log n_1}{\log n_0}} = \log^4 n_1$.

Let us now estimate $t = (\frac{n_1}{n_0})^{\frac{1}{r}} r^2$. Observe that $r^2 < {r_0}^2 = \left(\frac{\log n_1}{\log x_0}\right)^2 \leq \log^2 n_1$. Also,

$$\left(\frac{n_1}{n_0}\right)^{\frac{1}{r}} = 2^{\frac{\log n_1 - \log n_0}{r}} = 2^{\frac{\log n_1 - \log n_0}{\left(1-\frac{4\log\log n_1}{\log n_0}\right)\frac{\log n_1}{\log x_0}}} = x_0^{\frac{\frac{\log n_1 - \log n_0}{\log n_1}}{1-\frac{4\log\log n_1}{\log n_0}}} \leq x_0^{1+o(1)},$$

where the last inequality stems from the assumption that $n_0 = (\log n_1)^{\omega(1)}$. Since $x_0 = O(k)$, $(\frac{n_1}{n_0})^{\frac{1}{r}} \leq x_0^{1+o(1)} = (O(k))^{1+o(1)} = O((\log n_1)^{1+o(1)})$. Therefore $t = (\frac{n_1}{n_0})^{\frac{1}{r}} r^2 = O((\log n_1)^{3+o(1)}) \ll \log^4 n_1$. ■

This also ends the proof of Lemma 3.3, and therefore of the lower bound and of Theorem 1.

4 Generalization - Multi-Partite Graphs

We wish to estimate the choice number of a general $(s+1)$-partite graph $K_{n_0,n_1,\ldots,n_s}$. In the graph $K_{n_0,n_1,\ldots,n_s}$ we label the group of n_i vertices by V_i, for each $0 \leq i \leq s$. Using a proof similar to that of the bipartite case, we will prove:

Theorem 4 *Let $s \geq 1$ be a fixed integer. Let $2 \leq n_0 \leq n_1 ... \leq n_s$, and assume that $n_0 = (\log n_s)^{\alpha}$, where $\alpha \geq 2\sqrt{\frac{\log n_s}{\log\log n_s}}$. For every $0 \leq i \leq s-1$ denote $k_i = \frac{\log n_s}{\log n_i}$. Let x_0 be the unique root of the equation $sx - 1 - \sum_{j=0}^{s-1} x^{\frac{k_j - 1}{k_j}} = 0$ in the interval $[1, \infty)$. Then $ch(K_{n_0,\ldots,n_s}) = (1+o(1))\frac{\log n_s}{\log x_0}$.*

Observe that in the most basic case of equally sized parts (i.e. whenever $n_0 = \ldots = n_s$), we have $x_0 = (s+1)/s$, and thus $ch(K_{n_0,n_0,\ldots,n_0}) = (1 + o(1)) \log n_0 / \log((s+1)/s)$. Since $\log((s+1)/s) = \Theta(1/s)$, we recover the result of Alon [1] mentioned in the introduction.

Again we divide the proof into two parts – the upper bound and the lower bound.

5 The Upper Bound for Multi-Partite Graphs

Theorem 5 *Let $2 \leq n_0 \leq ... \leq n_s$ be integers, and let $0 < \epsilon < 1$ be a constant. For every $0 \leq i \leq s-1$ denote $k_i = \frac{\log n_s}{\log n_i}$. Let x_0 be the unique* *root of the equation* $(s+\epsilon)\cdot x - 1 - \sum_{j=0}^{s-1} x^{\frac{k_j-1}{k_j}} = 0$ *in the interval* $[1,\infty)$. *Define* $r = \lceil \frac{\log n_s}{\log x_0} \rceil + 1$. *Then* $ch(K_{n_0,\ldots,n_s}) \leq r$, *for* n_s *large enough.*

Proof.

Lemma 5.1 *If there exist* $p_0, \ldots p_s$ *such that* $0 \leq p_i \leq 1$ *for every* $0 \leq i \leq s$, $\sum_{i=0}^{s} p_i = 1$ *and* $\sum_{i=0}^{s} n_i (1-p_i)^r \leq 1$, *then* $ch(K_{n_0,n_1,\ldots,n_s}) \leq r$.

Proof. The proof is identical to that of the bipartite case (Lemma 2.1), only this time we partition the set of all available colors into $s+1$ sets, using the probabilities p_i. A bad event for a vertex $v \in V_i$ is one in which all the colors in $S(v)$ are chosen to be in color groups other than S_i, and it happens with probability $(1-p_i)^r$. ■

Lemma 5.2 *Given* r *s.t.* $\sum_{i=0}^{s} n_i^{-\frac{1}{r-1}} \geq s^{\frac{r}{r-1}}$, *let* $p_i = 1 - \frac{s n_i^{-\frac{1}{r-1}}}{\sum_{j=0}^{s} n_j^{-\frac{1}{r-1}}}$ *for* $0 \leq i \leq s$. *Then* $0 \leq p_i \leq 1$ *for each* $0 \leq i \leq s$, $\sum_{i=0}^{s} p_i = 1$, *and* $\sum_{i=0}^{s} n_i (1-p_i)^r \leq 1$.

Proof. In order for p_i to be non-negative, we must demand that for every $0 \leq i \leq s$, $\frac{s n_i^{-\frac{1}{r-1}}}{\sum_{j=0}^{s} n_j^{-\frac{1}{r-1}}} \leq 1$, or $s \leq \sum_{j=0}^{s} (\frac{n_i}{n_j})^{\frac{1}{r-1}}$. But if $s^{\frac{r}{r-1}} \leq \sum_{j=0}^{s} n_j^{-\frac{1}{r-1}}$, then for every $0 \leq i \leq s$, $s < s^{\frac{r}{r-1}} \leq \sum_{j=0}^{s} n_j^{-\frac{1}{r-1}} \leq \sum_{j=0}^{s} (\frac{n_i}{n_j})^{\frac{1}{r-1}}$. Also,

$$\sum_{i=0}^{s} p_i = s+1 - \sum_{i=0}^{s}(1-p_i) = s+1 - \sum_{i=0}^{s} \frac{s(n_i^{-\frac{1}{r-1}})}{\sum_{j=0}^{s} n_j^{-\frac{1}{r-1}}} = s+1-s = 1\,.$$

If $1-p_i = \frac{s n_i^{-\frac{1}{r-1}}}{\sum_{j=0}^{s} n_j^{-\frac{1}{r-1}}}$ then $(\frac{1-p_i}{1-p_j})^{r-1} = \frac{n_j}{n_i}$. Therefore, for any i,

$$\begin{aligned}
\sum_{j=0}^{s} n_j (1-p_j)^r &= n_i (1-p_i)^{r-1} \sum_{j=0}^{s} (1-p_j) = s \cdot n_i (1-p_i)^{r-1} \\
&= s \cdot n_i \left(\frac{s n_i^{-\frac{1}{r-1}}}{\sum_{j=0}^{s} n_j^{-\frac{1}{r-1}}} \right)^{r-1} = \left(\frac{s^{\frac{r}{r-1}}}{\sum_{j=0}^{s} n_j^{-\frac{1}{r-1}}} \right)^{r-1} \\
&\leq 1\,.
\end{aligned}$$

■

Let $r = \lceil \frac{\log n_s}{\log x_0} \rceil + 1$. Then $r-1 \geq \frac{\log n_s}{\log x_0}$, and thus $x_0 \geq n_s^{\frac{1}{r-1}}$.

Since the function $g_{k_0,\ldots k_{s-1},\epsilon}(x) = (s+\epsilon) \cdot x - 1 - \sum_{j=0}^{s-1} x^{\frac{k_j-1}{k_j}}$, where $k_j \geq 1$ for each j, is a monotonely increasing function in the interval $[1,\infty)$, and since $g_{k_0,\ldots k_{s-1},\epsilon}(x_0) = 0$, it follows that for r large enough, or for n_s large enough (see Lemma 6.2 below, and the beginning of the proof of Lemma 3.3), $s^{\frac{r}{r-1}} n_s^{\frac{1}{r-1}} \leq (s+\epsilon) n_s^{\frac{1}{r-1}} \leq 1 + \sum_{j=0}^{s-1} n_s^{\frac{1}{r-1}\frac{k_j-1}{k_j}} = 1 + \sum_{i=0}^{s-1} (\frac{n_s}{n_i})^{\frac{1}{r-1}}$ as required.

■

6 The Lower Bound for Multi-Partite Graphs

Theorem 6 *Let $2 \leq n_0 \ldots \leq n_s$ be integers, and let $n_0 = (\log n_s)^{\alpha}$, where $\alpha \geq 2\sqrt{\frac{\log n_s}{\log\log n_s}}$. For every $0 \leq i \leq s-1$ denote $k_i = \frac{\log n_s}{\log n_i}$. Let x_0 be the unique root of the equation $s \cdot x - 1 - \sum_{j=0}^{s-1} x^{\frac{k_j-1}{k_j}} = 0$ in the interval $[1,\infty)$. Then $ch(K_{n_0,\ldots,n_s}) \geq (1-o(1))\frac{\log n_s}{\log x_0}$.*

Proof. Similarly to the bipartite case, in order to prove $ch(K_{n_0,\ldots,n_s}) > r$, it is enough to show that there are a $t \geq r$ and a sequence of $0 \leq l_i \leq t$ for which $\sum_{i=0}^{s} l_i = t$, s.t. it is possible to choose for each vertex in $K_{n_0,\ldots,n_s}$ a list of r of colors from $\{1,2,\ldots t\}$, and the lists chosen satisfy the following s conditions: For each $0 \leq i \leq s-1$ the minimum cover of the hypergraph created by the color lists of the vertices in V_i is of cardinality at least l_i, and the additional condition: the minimum cover of the hypergraph created by the color lists of the vertices in V_s is of cardinality at least $l_s + 1$.

As in the bipartite case, if these conditions are satisfied, then by the pigeonhole principle at least 2 vertices in different groups must be given the same color, so the choice number is greater than r.

Lemma 6.1 *If there exist a parameter $t \geq r$ and a sequence of $0 \leq l_i \leq t$ for which $\sum_{i=0}^{s} l_i = t$ and*

$$\sum_{i=0}^{s} 2^t e^{-\frac{(t-l_i)_r}{(t)_r} n_i} \leq 1 \tag{2}$$

then $ch(K_{n_0,\ldots,n_s}) > r$.

Proof. Similar to the bipartite case. ∎

As in the bipartite case, we calculate bounds on x_0 that will help us later on.

Lemma 6.2 $\frac{s+1}{s} \leq x_0 < \max(k_0, e+2)$

Proof. Since for every $0 \leq i \leq s$, $n_0 \leq n_i$, it follows that $k_0 = \frac{\log n_s}{\log n_0} \geq \frac{\log n_s}{\log n_i} = k_i$. Therefore, for a given x in the range $[1, \infty)$, $x^{\frac{k_0-1}{k_0}} \geq x^{\frac{k_i-1}{k_i}}$ for all i, and $f_{k_0,\ldots k_{s-1}}(x) = sx - 1 - \sum_{i=0}^{s-1} x^{\frac{k_i-1}{k_i}} \geq sx - 1 - sx^{\frac{k_0-1}{k_0}} = s(x - x^{\frac{k_0-1}{k_0}}) - 1 \geq x - x^{\frac{k_0-1}{k_0}} - 1$ (note all these functions increase monotonely as functions of x). Therefore the root x_0 in the range $[1, \infty)$ of the first equation $sx - 1 - \sum_{i=0}^{s-1} x^{\frac{k_i-1}{k_i}} = 0$, which is our equation, is not greater than the root x_1 of the equation $x - x^{\frac{k_0-1}{k_0}} - 1 = 0$.

But the last equation is $f_{k_0}(x) = 0$, and we already know from the bipartite case that its root is smaller than $\max(k_0, e+2)$.

To prove the lower bound observe that $f_{k_0,\ldots,k_{s-1}}(\frac{s+1}{s}) = s + 1 - 1 - \sum_{j=0}^{s-1} \left(\frac{s+1}{s}\right)^{\frac{k_j-1}{k_j}} \leq s - s = 0$, and thus by monotonicity $x_0 \geq \frac{s+1}{s}$. ∎

Lemma 6.3 *Let $n_0 = (\log n_s)^{\alpha}$, where $\alpha \geq 2\sqrt{\frac{\log n_s}{\log\log n_s}}$. Define $r_0 = \frac{\log n_s}{\log x_0}$, $u = \frac{4 \log\log n_s}{\log n_0} r_0$ and $r = r_0 - u$. Then $r = (1 - o(1)) r_0$, and for $t = (\frac{1}{s} \sum_{j=0}^{s} (\frac{n_s}{n_j})^{\frac{1}{r}} - 1) r^2$ and $t - l_i = t \frac{s(\frac{n_s}{n_i})^{\frac{1}{r}}}{\sum_{j=0}^{s} (\frac{n_s}{n_j})^{\frac{1}{r}}}$, one has: $0 \leq l_i \leq t$, $\sum_{i=0}^{s} l_i = t$, and $\sum_{i=0}^{s} 2^t e^{-\frac{(t-l_i)_r}{(t)_r} n_i} \leq 1$, i.e., the assumptions of Lemma 6.1 are satisfied.*

Proof. Since $n_0 = (\log n_s)^{\omega(1)}$, it follows that $r = (1-o(1))r_0$, as in the bipartite case. Also, again as in the bipartite case, from $x_0 < \max(k_0, e+2)$ it follows that $r_0 = \omega(1)$, and therefore $r = \omega(1)$.

We need to show that for every i, $0 \le l_i \le t$, or $0 \le t - l_i \le t$. Since $t - l_i$ is obviously non-negative, we need to prove that $t - l_i \le t$, or $\frac{s(\frac{n_s}{n_i})^{\frac{1}{r}}}{\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r}}} \le 1$, or $s \le \sum_{j=0}^{s}(\frac{n_i}{n_j})^{\frac{1}{r}}$. Since $n_0 \le n_i$ for every i, it is enough to show $s \le \sum_{j=0}^{s}(\frac{n_0}{n_j})^{\frac{1}{r}}$.

Since $r_0 = \frac{\log n_s}{\log x_0}$, we have: $x_0 = n_s^{\frac{1}{r_0}}$, and so $sn_s^{\frac{1}{r_0}} = 1 + \sum_{j=0}^{s-1} n_s^{\frac{1}{r_0}\frac{k_j-1}{k_j}} = \sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r_0}}$, or $s = \sum_{j=0}^{s}(\frac{1}{n_j})^{\frac{1}{r_0}}$. But

$$\sum_{j=0}^{s}\left(\frac{n_0}{n_j}\right)^{\frac{1}{r}} = \sum_{j=0}^{s}\left(\frac{1}{n_j}\right)^{\frac{1}{r_0}} \frac{n_0^{\frac{1}{r}}}{n_j^{\frac{1}{r}-\frac{1}{r_0}}} \ge \frac{n_0^{\frac{1}{r}}}{n_s^{\frac{1}{r}-\frac{1}{r_0}}} \sum_{j=0}^{s}\left(\frac{1}{n_j}\right)^{\frac{1}{r_0}} = s\frac{n_0^{\frac{1}{r}}}{n_s^{\frac{1}{r}-\frac{1}{r_0}}},$$

so it is enough to show $\frac{n_0^{\frac{1}{r}}}{n_s^{\frac{1}{r}-\frac{1}{r_0}}} \ge 1$. But $\frac{1}{r} - \frac{1}{r_0} = \frac{1}{r}\frac{u}{r_0}$, so $\frac{1}{n_s^{\frac{1}{r}-\frac{1}{r_0}}} = 2^{-\frac{1}{r}\log n_s \frac{u}{r_0}} = 2^{-\frac{1}{r}\log n_s \frac{4\log\log n_s}{\log n_0}} = 2^{-\frac{1}{r}\log n_s \frac{4}{\alpha}}$. Also $n_0^{\frac{1}{r}} = (\log n_s)^{\alpha\frac{1}{r}} = 2^{\frac{1}{r}\alpha\log\log n_s}$. Therefore

$$\frac{n_0^{\frac{1}{r}}}{n_s^{\frac{1}{r}-\frac{1}{r_0}}} = (2^{\alpha\log\log n_s - \log n_s\frac{4}{\alpha}})^{\frac{1}{r}} \ge 1,$$

where the last inequality stems from the condition on α. Also,

$$\sum_{i=0}^{s} l_i = (s+1)t - \sum_{i=0}^{s}(t - l_i) = (s+1)t - \sum_{i=0}^{s} t\frac{s(\frac{n_s}{n_i})^{\frac{1}{r}}}{\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r}}} = st + t - st = t.$$

All that is left for us to verify is that Condition (2) is fulfilled. The proof is, again, similar to the bipartite case.

Claim 6.4 $\frac{(t-l_i)_r}{(t)_r} n_i > \frac{s^r n_s}{\left(\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r}}\right)^r}\frac{1}{2e^2}$ *for* $0 \le i \le s$.

Proof. We have: $\frac{(t-l_i)_r}{(t)_r} > \left(\frac{t-l_i-r}{t-r}\right)^r = \left(\frac{t-l_i}{t}\right)^r\left(1 - \frac{l_i r}{(t-l_i)(t-r)}\right)^r > \left(\frac{t-l_i}{t}\right)^r\left(1 - \frac{2l_i r}{(t-l_i)t}\right)^r$ where the last inequality is a result of $r < \frac{t}{2}$. By definition $t - l_i = t\frac{s(\frac{n_s}{n_i})^{\frac{1}{r}}}{\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r}}}$, so $l_i = \frac{t\left(\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r}} - s(\frac{n_s}{n_i})^{\frac{1}{r}}\right)}{\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r}}}$, and $\frac{l_i}{t-l_i} = \frac{\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r}} - s(\frac{n_s}{n_i})^{\frac{1}{r}}}{s(\frac{n_s}{n_i})^{\frac{1}{r}}} = \frac{1}{s}\sum_{j=0}^{s}(\frac{n_i}{n_j})^{\frac{1}{r}} - 1$.

Now since $\frac{l_i r}{(t-l_i)t} = (\frac{1}{s}\sum_{j=0}^{s}(\frac{n_i}{n_j})^{\frac{1}{r}} - 1)\frac{r}{t} \leq (\frac{1}{s}\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r}} - 1)\frac{r}{t} = \frac{1}{r} = o(1)$, we get $\frac{(t-l_i)_r}{(t)_r} > (\frac{t-l_i}{t})^r \frac{1}{2e^2}$.

Hence $\frac{(t-l_i)_r}{(t)_r} n_i > (\frac{t-l_i}{t})^r n_i \frac{1}{2e^2} = \left(\frac{s(\frac{n_s}{n_i})^{\frac{1}{r}}}{\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r}}}\right)^r n_i \frac{1}{2e^2} = \frac{s^r n_s}{\left(\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r}}\right)^r}\frac{1}{2e^2}$.

■

Therefore in order to prove that (2) holds it is now enough to prove that $\frac{s^r n_s}{\left(\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r}}\right)^r} \gg t$ (assuming s is constant).

Claim 6.5 $\frac{s^r n_s}{\left(\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r}}\right)^r} \gg t$.

Proof. We have:

$$\frac{s^r n_s}{\left(\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r}}\right)^r} = \left(\frac{s n_s^{\frac{1}{r}}}{\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r}}}\right)^r = \left[\frac{s n_s^{\frac{1}{r_0}}}{\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r_0}}}\frac{n_s^{\frac{1}{r}-\frac{1}{r_0}}}{\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r}}}\sum_{j=0}^{s}\left(\frac{n_s}{n_j}\right)^{\frac{1}{r_0}}\right]^r .$$

Since $\frac{s n_s^{\frac{1}{r_0}}}{\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r_0}}} = \frac{s n_s^{\frac{\log x_0}{\log n_s}}}{\sum_{j=0}^{s-1}(\frac{n_s}{n_j})^{\frac{\log x_0}{\log n_s}}+1} = \frac{s x_0}{\sum_{j=0}^{s-1} x_0^{\frac{k_j-1}{k_j}}+1} = 1$, we get $\frac{s^r n_s}{\left(\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r}}\right)^r} = \left(n_s^{\frac{1}{r}-\frac{1}{r_0}}\frac{\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r_0}}}{\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r}}}\right)^r = \left(\frac{\sum_{j=0}^{s}(\frac{1}{n_j})^{\frac{1}{r_0}}}{\sum_{j=0}^{s}(\frac{1}{n_j})^{\frac{1}{r}}}\right)^r$. Now,

$$\frac{\sum_{j=0}^{s}\left(\frac{1}{n_j}\right)^{\frac{1}{r_0}}}{\sum_{j=0}^{s}\left(\frac{1}{n_j}\right)^{\frac{1}{r}}} = \frac{\sum_{j=0}^{s}\left(\frac{1}{n_j}\right)^{\frac{1}{r}}\left(\frac{1}{n_j}\right)^{\frac{1}{r_0}-\frac{1}{r}}}{\sum_{j=0}^{s}\left(\frac{1}{n_j}\right)^{\frac{1}{r}}} = \frac{\sum_{j=0}^{s}\left(\frac{1}{n_j}\right)^{\frac{1}{r}} n_j^{\frac{1}{r}-\frac{1}{r_0}}}{\sum_{j=0}^{s}\left(\frac{1}{n_j}\right)^{\frac{1}{r}}} \geq \frac{\sum_{j=0}^{s}\left(\frac{1}{n_j}\right)^{\frac{1}{r}} n_0^{\frac{1}{r}-\frac{1}{r_0}}}{\sum_{j=0}^{s}\left(\frac{1}{n_j}\right)^{\frac{1}{r}}},$$

where the last inequality is a result of $n_i \geq n_0$ for all $1 \leq i \leq s$ and of $r < r_0$. So $\frac{\sum_{j=0}^{s}(\frac{1}{n_j})^{\frac{1}{r_0}}}{\sum_{j=0}^{s}(\frac{1}{n_j})^{\frac{1}{r}}} \geq n_0^{\frac{1}{r}-\frac{1}{r_0}}$, and $\frac{s^r n_s}{\left(\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r}}\right)^r} \geq \left(n_0^{\frac{1}{r}-\frac{1}{r_0}}\right)^r = n_0^{1-\frac{r}{r_0}} = n_0^{\frac{u}{r_0}} = n_0^{\frac{4\log\log n_s}{\log n_0}} = \log^4 n_s$.

Let us now estimate $t = (\frac{1}{s}\sum_{j=0}^{s}(\frac{n_s}{n_j})^{\frac{1}{r}} - 1)r^2$. First, $r^2 < {r_0}^2 = (\frac{\log n_s}{\log x_0})^2 \leq (\frac{\log n_s}{\log\frac{s+1}{s}})^2 = C\log^2 n_s$ where $C = C(s)$ is a constant. Second,

$$\left(\frac{n_s}{n_0}\right)^{\frac{1}{r}} = 2^{\frac{\log n_s - \log n_0}{r}} = 2^{\frac{\log n_s - \log n_0}{\left(1-\frac{4\log\log n_s}{\log n_0}\right)\frac{\log n_s}{\log x_0}}} = x_0^{\frac{\frac{\log n_s - \log n_0}{\log n_s}}{1-\frac{4\log\log n_s}{\log n_0}}} \leq x_0^{1+o(1)},$$

where the last inequality stems from the assumption that $n_0 = (\log n_s)^{\omega(1)}$. Since $x_0 = O(k_0)$, we get: $(\frac{n_s}{n_0})^{\frac{1}{r}} \leq x_0^{1+o(1)} = (O(k_0))^{1+o(1)} = O((\log n_s)^{1+o(1)})$.

Therefore

$$\begin{aligned} t &= \left(\frac{1}{s}\sum_{j=0}^{s}\left(\frac{n_s}{n_j}\right)^{\frac{1}{r}} - 1\right)r^2 = \left(\frac{1}{s}\sum_{j=0}^{s-1}\left(\frac{n_s}{n_j}\right)^{\frac{1}{r}} - \frac{s-1}{s}\right)r^2 \\ &\leq \left(\frac{1}{s}\sum_{j=0}^{s-1}\left(\frac{n_s}{n_j}\right)^{\frac{1}{r}}\right)r^2 \leq \frac{1}{s}s\left(\frac{n_s}{n_0}\right)^{\frac{1}{r}}r^2 = \left(\frac{n_s}{n_0}\right)^{\frac{1}{r}}r^2 = O((\log n_s)^{3+o(1)}) \\ &\ll \log^4 n_s\,. \end{aligned}$$

■

This also ends the proof of Lemma 6.3, and therefore of the lower bound of the multi-partite case and of Theorem 4.

References

[1] Alon, N., *Choice numbers of graphs: a probabilistic approach*, Combin. Probab. Comput. 1, no. 2 (1992), 107-114.

[2] Erdős P., Rubin A. L. and Taylor H., *Choosability in Graphs*, Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium XXVI, 1979, 125-157.

[3] Kierstead H., *On the choosability of complete multipartite graphs with part size 3*, Discrete Math. 211 (2000), 255-259.

[4] Reed B. and Sudakov B. *List colouring of graphs with at most* $(2-o(1))\chi$ *vertices*, Proc. Intern. Congress Math., Vol. III (Beijing, 2002), 587-603.

[5] Vizing V. G., *Coloring the vertices of a graph in prescribed colors* (in Russian), Diskret. Analiz. No. 29, Metody Diskret. Anal. v. Teorii Kodov i Shem 101 (1976), 3-10.