On the chromatic number of random graphs with a fixed degree sequence

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November 14, 2006

Abstract

Let $\mathbf{d} = 1 \leq d_1 \leq d_2 \leq \cdots \leq d_n$ be a non-decreasing sequence of n positive integers, whose sum is even. Let $\mathcal{G}_{n,\mathbf{d}}$ denote the set of graphs with vertex set $[n] = \{1, 2, \ldots, n\}$ in which the degree of vertex i is d_i. Let $G_{n,\mathbf{d}}$ be chosen uniformly at random from $\mathcal{G}_{n,\mathbf{d}}$. Let $d = (d_1 + d_2 + \cdots + d_n)/n$ be the average degree. We give a condition on $\mathbf{d}$ under which we can show that **whp** the chromatic number of $\mathcal{G}_{n,\mathbf{d}}$ is $\Theta(d/\ln d)$. This condition is satisfied by graphs with exponential tails as well those with power law tails.

1 Introduction

Let $\mathbf{d} = 1 \leq d_1 \leq d_2 \leq \cdots \leq d_n$ be a fixed non-decreasing sequence of n positive integers, whose sum is even. Let $\mathcal{G}_{n,\mathbf{d}}$ denote the set of graphs with vertex set $[n] = \{1, 2, \ldots, n\}$ in which the degree of vertex i is d_i. Let $G_{n,\mathbf{d}}$ be chosen uniformly at random from $\mathcal{G}_{n,\mathbf{d}}$. When $d_i = r$ for $i \in [n]$ then this models a random r-regular graph $G_{n,r}$ and there is a large literature on this subject. We refer the reader to the survey by Wormald [18] for an excellent summary. By now we know much about the structure of random regular graphs.

For general $\mathbf{d}$, less is known. In many, but not all, cases we can estimate $|\mathcal{G}_{n,\mathbf{d}}|$. See Bender and Canfield [5], McKay and Wormald [13, 14]. We have the *configuration model* to study them, Bollobás [6]. We know something of their connectivity properties, Molloy and Reed

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[16, 17].(See also Cooper and Frieze [8] for the connectivity properties of random *digraphs* with a fixed degree sequence). They have been used in the context of *massive graph models* of telephone networks and the WWW, Aiello, Chung and Lu [3].

In this paper we will concern ourselves with the chromatic number of $G_{n,\mathbf{d}}$. What should we expect? For r-regular graphs it is known that when r is large $\chi(G_{n,r}) \sim \frac{r}{2\ln r}$ **whp**[1], see Frieze and Łuczak [10] and Achlioptas and Moore [1] (see also Cooper et al. [9] for extension of some of these results to the case where $r = r(n) \leq n^{1-\eta}$, for an arbitrary small constant $\eta > 0$). For the random graph $G_{n,p}$ where the average degree $d = (n-1)p$ is large, it is known that $\chi(G_{n,p}) \sim \frac{d}{2\ln d}$, see Bollobás [7], Łuczak [12] and Achlioptas and Naor [2]. So if we let

$$d = \frac{d_1 + d_2 + \cdots + d_n}{n}$$

be the average degree in $G_{n,\mathbf{d}}$ then we might hope to prove that $\chi(G_{n,\mathbf{d}}) \sim \frac{d}{2\ln d}$ **whp**. This is too much to expect given the variety of possible degree sequences. Indeed if $\nu = \frac{n}{\ln n}$ and

$$d_i = \begin{cases} 1 & i \leq n - \nu \\ d\ln n & i > n - \nu \end{cases}$$

then the average degree $\sim d$, but **whp** the sub-graph H induced by the ν largest degree vertices has average degree close to $d\ln n$ when d is large. Then **whp** $\chi(H) \sim \frac{d\ln n}{2\ln\ln n}$ for $\ln d = o(\ln\ln n)$ and so we will have to be less ambitious in our goals[2]. Let

$$D_k = d_n + d_{n-1} + \cdots + d_{n-k+1}$$

be the sum of the k largest degrees.

Let

$$M_1 = D_n = dn \text{ and } M_2 = \sum_{i=1}^{n} d_i(d_i - 1) \leq \Delta M_1 \text{ where } \Delta = d_n.$$

Theorem 1

1. *Suppose that there exist constants $1/2 < \alpha < 1$, ϵ, $K_0 > 0$ and $\omega = \omega(n) \to \infty$ such that*

   (a)

   $$D_k \leq K_0 dn(k/n)^\alpha \tag{1}$$

   for $k \leq \epsilon n$.

   (b) $\Delta^5 \leq M_2/\omega$.

[1] A sequence of events $\mathcal{E}_n$ occurs *with high probability* (**whp**) if $\lim_{n\to\infty} \mathbf{Pr}(\mathcal{E}_n) = 1$.
[2] The methods of [9] can be used to verify this claim.

Then there exists b_1 dependent only on α, ϵ, K_0 such that **whp**

$$\chi(G_{n,\mathbf{d}}) \leq b_1 \frac{d}{\ln d}.$$

2. *Suppose only that $\Delta^4 \leq M_1/\omega$ (a weaker condition than 1(b)), then there exists b_2 such that* **whp**

$$\chi(G_{n,\mathbf{d}}) \geq b_2 \frac{d}{\ln d}.$$

Notice that if Condition (a) of the theorem holds for α, ϵ, K_0 then it also holds when K_0 is replaced by a larger constant. As we increase K_0 our bound b_1 will decrease. We are therefore justified in assuming throughout that K_0 is sufficiently large that some inequalities are valid.

Condition (b) is chosen so that we can use the results of [15]. It may be possible to prove our results under the less stringent conditions of [14], but there are difficulties, as will be pointed to later.

A referee has suggested that the quantity $d' = \max_k D_k^2/dnk$ (which can be viewed as the probabilistic version of the "maximum average degree") might be a better predictor of chromatic number. This could well be true and it makes for an interesting research question. We also suspect that if $d_i \sim df(i/n)$ where f is some "nice" real valued function on $[0,1]$ such that $\int_0^1 f(x)dx = 1$ then we might be able to determine the chromatic number of $G_{n,\mathbf{d}}$ asymptotically.

The proof of the upper bound is given in Section 3. The proof of the lower bound is given in Section 4. We have made no attempt to optimize constants.

2 Degree sequences that satisfy (1)

It is natural to ask whether there many types of degree sequence that satisfy the conditions of the first part of the theorem.

We first consider a degree sequence with a *power law* tail. For integer $\ell \geq 1$ we let ν_ℓ denote the number of vertices of degree ℓ. Our assumption is that there are some constants $A > 0$ and $\gamma > 3$ such that for $\ell \geq (A/\epsilon)^{1/(\gamma-1)}$

$$\nu_\ell \leq \begin{cases} 0 & \ell \leq 1 \\ \lfloor Ad\ell^{-\gamma} n \rfloor & 2 \leq \ell \leq n^{1/5}/\ln n \\ 0 & \ell > n^{1/5}/\ln n \end{cases}.$$

Here we have $\alpha = \frac{\gamma-2}{\gamma-1} > 1/2$.

We have forced Condition (b) and so we only need to check Condition (a). Suppose now that $k = \kappa n \le \epsilon n$. Then

$$D_k \le Adn \sum_{\ell \ge r} \ell^{-(\gamma-1)} = (1+\delta_r)\frac{Adn}{\gamma-2} r^{2-\gamma}$$

where $\delta_r \to 0$ as $r \to \infty$ and

$$r = \min\left\{\rho :\ Ad\sum_{\ell\ge\rho} \ell^{-\gamma} \ge \kappa\right\} = (1+\delta'_\kappa)\left(\frac{Ad}{(\gamma-1)\kappa}\right)^{1/(\gamma-1)}$$

where $\delta'_\kappa \to 0$ as $\kappa \to 0$. Thus,

$$D_k \le (1+\delta_r)(1+\delta'_\kappa)^{2-\gamma}(\gamma-2)^{-1}A\left(\frac{\gamma-1}{Ad}\right)^\alpha dn\kappa^\alpha$$

and we have the condition of Theorem 1.

A similar argument holds if we assume that our degree sequence has an *exponential* tail viz. for some constants $A > 0$ and $0 < \epsilon \ll \gamma < 1$ we have for $\ell \ge \lfloor \ln_{1/\gamma}(Ad/\epsilon)\rfloor$

$$\nu_\ell \le Ad\gamma^\ell n.$$

Note that **whp** the degree sequence of $G_{n,p}$, $p = c/n$, c constant, satisfies this condition.

Suppose now that $k = \kappa n \le \epsilon n$. Then

$$D_k \le Adn\sum_{\ell\ge r} \ell\gamma^\ell \le Adn\int_{x=r-1}^{\infty} x\gamma^x dx = \frac{Adn\gamma^{r-1}}{\ln 1/\gamma}\left(r-1+\frac{1}{\ln 1/\gamma}\right)$$

where

$$r = \min\left\{\rho :\ Ad\sum_{\ell\ge\rho}\ell\gamma^\ell \ge \kappa\right\} \sim \ln_{1/\gamma}\left(\frac{Ad}{\kappa \ln 1/\gamma}\right)$$

as $\epsilon \to 0$.

Thus,

$$D_k \le 2\kappa n$$

and we have the condition of Theorem 1 with room to spare.

3 Upper bound

3.1 Configurations

We will work initially in the configuration model and then show how our results can be justified in the uniform model. Let $W = [nd]$ be our set of *points* and let $W_i = [d_1 +$

$\cdots + d_{i-1} + 1, d_1 + \cdots + d_i]$, $i \in [n]$, partition W. The function $\phi : W \to [n]$ is defined by $w \in W_{\phi(w)}$. Given a pairing F (i.e. a partition of W into $m = dn/2$ pairs) we obtain a (multi-)graph G_F with vertex set $[n]$ and an edge $(\phi(u), \phi(v))$ for each $\{u, v\} \in F$. Choosing a pairing F uniformly at random from among all possible pairings of the points of W produces a random (multi-)graph G_F.

This model is valuable because of the following easily proven fact: Suppose $G \in \mathcal{G}_{n,\mathbf{d}}$. Then

$$\mathbf{Pr}(G_F = G \mid G_F \text{ is simple}) = \frac{1}{|\mathcal{G}_{n,\mathbf{d}}|}.$$

It follows that if G is chosen randomly from $\mathcal{G}_{n,\mathbf{d}}$, then for any graph property $\mathcal{P}$

$$\mathbf{Pr}(G \in \mathcal{P}) \leq \frac{\mathbf{Pr}(G_F \in \mathcal{P})}{\mathbf{Pr}(G_F \text{ is simple})}. \tag{2}$$

Before starting the proof of the upper bound in Theorem 1, we introduce some notation that we will use throughout the course of the proof. Let

$$\theta = \max\left\{d^{2/(2\alpha+1)}, \frac{16}{2\alpha - 1}\right\}$$

and

$$\epsilon_1 = \min\left\{\epsilon, \left(\frac{(2\alpha-1)\theta}{K_1 d}\right)^{2/(2\alpha-1)}\right\}. \tag{3}$$

for some sufficiently large K_1.

For a set $U \subseteq V(G_F)$ we denote by $D(U)$ the sum of the degrees of U in G_F.

We first prove

Lemma 1 **Whp** *G_F does not contain a set of vertices S with $k = |S| \leq \epsilon_1 n$, such that S induces a sub-graph $G[S]$ with minimum degree at least θ.*

Proof Suppose that there exists a vertex set S of size $k \in [\theta, \epsilon_1 n]$ that induces a graph $G[S]$ with minimum degree at least θ. Let S_1 be the $k/2$ lowest degree vertices of S and let $S_2 = S \setminus S_1$. For vertex sets S, T let $e(S)$ denote the number of edges contained in S and let $e(S : T)$ denote the number of edges joining S and T.

At least one of the following two events must occur:

$\mathcal{E}_1$**:** $e(S_1 : S_2) \geq \theta k/4$;

$\mathcal{E}_2$**:** $e(S_1) \geq \theta k/8$.

This follows because

$$2e(S_1)+e(S_1:S_2)=\sum_{v\in S_1} d_{G[S]}(v)\geq \theta k/2.$$

Now $k\leq \epsilon_1 n$ and so,

$$dn-\theta k/2\geq dn-\theta\epsilon_1 n/2\geq \frac{99}{100}dn$$

provided

$$K_1\geq 16\left(\frac{1600}{2\alpha-1}\right)^{\alpha-1/2}.$$

(We have chosen K_1 large enough so that whichever of $d^{2/(2\alpha+1)}$, $\frac{16}{2\alpha-1}$ is the larger, $\theta\epsilon_1<1/100$.)

And so

$$\begin{aligned}
\mathbf{Pr}(\mathcal{E}_1) &\leq \sum_{k=\theta}^{\epsilon_1 n}\sum_{|S|=k}\binom{D(S_1)}{\theta k/4}\left(\frac{D(S_2)}{dn-\theta k/2}\right)^{\theta k/4} \qquad (4)\\
&\leq \sum_{k=\theta}^{\epsilon_1 n}\binom{n}{k}\binom{D_{k/2}}{\theta k/4}\left(\frac{D_{k/2}}{dn-\theta k/2}\right)^{\theta k/4}\\
&\leq \sum_{k=\theta}^{\epsilon_1 n}\left(\frac{en}{k}\right)^k\left(\frac{5eD_{k/2}^2}{\theta k dn}\right)^{\theta k/4}\\
&\leq \sum_{k=\theta}^{\epsilon_1 n}\left(\frac{en}{k}\right)^k\left(\frac{5eK_0^2 d}{2^{2\alpha}\theta}\cdot\left(\frac{k}{n}\right)^{2\alpha-1}\right)^{\theta k/4}\\
&\leq \sum_{k=\theta}^{\epsilon_1 n}\left(\left(\frac{k}{n}\right)^{(2\alpha-1)\theta/4-1}\left(\frac{5e^2K_0^2 d}{2^{2\alpha}\theta}\right)^{\theta/4}\right)^k\\
&\leq \sum_{k=\theta}^{\epsilon_1 n}\left(\left(\frac{k}{n}\right)^{3(2\alpha-1)\theta/16}\left(\frac{5e^2K_0^2 d}{2^{2\alpha}\theta}\right)^{\theta/4}\right)^k\\
&\leq \sum_{k=\theta}^{\epsilon_1 n}\left(\left(\frac{k}{n}\right)^{(2\alpha-1)/4}\frac{5e^2K_0^2}{2^{2\alpha}}\right)^{\theta k/4}\\
&= o(1),
\end{aligned}$$

if we choose $K_1\geq 100e^4K_0^4$ (we used the assumption $k/n\leq\epsilon_1\leq((2\alpha-1)\theta/(K_1 d))^{2/(2\alpha-1)}\leq(16/K_1)^{2/(2\alpha-1)}$ in the last line).

Explanation of (4)**:** At least $\theta k/4$ of the points corresponding to vertices of S_1 are to be paired with points corresponding to vertices of S_2. When pairing points, the probability of pairing with a point corresponding to S_2 is never more than $\frac{D(S_2)}{dn-\ell}$, given up to $\ell/2-1$ previous pairings.

In the following estimate, $\ell = \ell(S_1)$ is the total degree of the subgraph $G[S_1]$. Here $\theta k/4 \leq \ell \leq K_0\epsilon_1^\alpha dn$. Now the maximum degree in S_1 is at most $\frac{D_{k/2}}{k/2} \leq K_0 d(2n/k)^{1-\alpha}$ and so

$$\begin{aligned}
&\mathbf{Pr}(\mathcal{E}_2)\\
&\leq \sum_{k=\theta}^{\epsilon_1 n} \sum_{S_1=\{i_1<\cdots<i_{k/2}\}} \sum_{\ell=\theta k/4}^{K_0\epsilon_1^\alpha dn} \sum_{a_1+\cdots+a_{k/2}=\ell} \frac{\binom{\binom{k/2}{2}}{\ell/2}}{(dn-\ell)^{\ell/2}} \prod_{t=1}^{k/2} d_{i_t}^{a_t} \qquad (5)
\end{aligned}$$

$a_1, a_2, \ldots a_{k/2}$ is the degree sequence of $G[S_1]$ (more explanation below)

$$\begin{aligned}
&\leq \sum_{k=\theta}^{\epsilon_1 n} \sum_{S_1=\{i_1<\cdots<i_{k/2}\}} \sum_{\ell=\theta k/4}^{K_0\epsilon_1^\alpha dn} \sum_{a_1+\cdots+a_{k/2}=\ell} \left(\frac{k^2}{\ell dn}\right)^{\ell/2} \prod_{t=1}^{k/2} (K_0 d(2n/k)^{(1-\alpha)})^{a_t}\\
&\leq \sum_{k=\theta}^{\epsilon_1 n} \sum_{S_1=\{i_1<\cdots<i_{k/2}\}} \sum_{\ell=\theta k/4}^{K_0\epsilon_1^\alpha dn} \sum_{a_1+\cdots+a_{k/2}=\ell} \left(\frac{16K_0^2 dk^{2\alpha-1}}{\theta n^{2\alpha-1}}\right)^{\ell/2}\\
&\leq \sum_{k=\theta}^{\epsilon_1 n} \sum_{S_1=\{i_1<\cdots<i_{k/2}\}} \sum_{\ell=\theta k/4}^{K_0\epsilon_1^\alpha dn} \left(\frac{64K_0^2 dk^{2\alpha-1}}{\theta n^{2\alpha-1}}\right)^{\ell/2}\\
&\leq dn \sum_{k=\theta}^{\epsilon_1 n} \binom{n}{k/2} \left(\frac{64K_0^2 dk^{2\alpha-1}}{\theta n^{2\alpha-1}}\right)^{\theta k/8}\\
&\leq dn \sum_{k=\theta}^{\epsilon_1 n} \left(\frac{2en}{k}\left(\frac{64K_0^2 dk^{2\alpha-1}}{\theta n^{2\alpha-1}}\right)^{\theta/4}\right)^{k/2}\\
&\leq dn \sum_{k=\theta}^{\epsilon_1 n} \left(\frac{2^{12}K_0^4 d^2 k^{2\alpha-1}}{\theta^2 n^{2\alpha-1}}\right)^{\theta k/16}\\
&= o(1).
\end{aligned}$$

Here we choose $K_1 \geq 2^{10}K_0^2$ so that $2^{12}K_0^4 d^2 \epsilon_1^{2\alpha-1} < \theta^2/2$.

Explanation of (5)**:** Fix $S_1 = \{i_1 < \cdots < i_{k/2}\}$. Now fix the degrees $a_1, \ldots, a_{k/2}$ of $i_1, \ldots, i_{k/2}$. $\binom{\binom{k/2}{2}}{\ell/2}$ is a crude upper bound on the number of graphs with this set of degrees. Then we use the fact that the probability an edge exists between two vertices of degrees d_r, d_s is never more than $\frac{d_r d_s}{dn-\ell}$, given up to $\ell/2 - 1$ previous pairings. $\square$

Now let

$$t_1 = 1 + \lceil \log_3(2/\epsilon_1) \rceil$$

and

$$I_t = \left\{n - \frac{n}{3^t} + 1, \ldots, n - \frac{n}{3^{t+1}}\right\} \text{ for } t = 0, 1, \ldots, t_1.$$

Let G_t be the sub-graph of G_F induced by I_t and for $k \in I_t$ let $\tilde{d}_k$ be the degree of k in G_t.

Let
$$\Delta_t = \frac{K_2 d}{2^{(2\alpha-1)t}}$$
where $K_2 = 3\epsilon^{-1}2^{(2\alpha-1)t_0} + K_2' + K_2''$ where
$$t_0 = \lfloor \log_3 1/\epsilon \rfloor$$
and K_2', K_2'' are defined below.

For $0 \le t \le t_1$ let $B_t = \{k \in I_t : \ \tilde{d}_k \ge \Delta_t\}$, $Z_t = |B_t|$ and $Z = Z_0 + \cdots + Z_{t_1}$.

Lemma 2 $Z \le \epsilon_1 n/2$ **whp**.

Proof If $k \in I_t$, $t \le t_0$, then there are at least $n - |\bigcup_{i=1}^t I_i| \ge 3\epsilon n$ vertices of degree at least d_k, and thus $d_k \le 3\epsilon^{-1}d$ and so $B_t = \emptyset$, $0 \le t \le t_0$. For $t > t_0$ and $k \in I_t$ we see that $\tilde{d}_k$ is stochastically dominated by the binomial $B\left(d_k, \frac{D_{n/3^t}}{dn}\right)$. Furthermore, since $k \ge n - \epsilon n$ we have
$$d_k \le \frac{D_{n/3^{t+1}}}{n/3^{t+1}} \le K_0 3^{(t+1)(1-\alpha)} d \tag{6}$$
and
$$\frac{D_{n/3^t}}{dn} \le K_0 3^{-\alpha t}.$$

So, if $K_2' = K_0^2 e^2 3^{1-\alpha}$ then
$$\mathbf{Pr}\left(\tilde{d}_k \ge \Delta_t\right) \le \left(\frac{K_0^2 3^{1-\alpha} e}{K_2'(3/2)^{(2\alpha-1)t}}\right)^{\Delta_t} \le e^{-\Delta_t}.$$

(We used: $\mathbf{Pr}[B(n,p) \ge k] \le \binom{n}{k} p^k \le \left(\frac{enp}{k}\right)^k$.)

Thus,
$$\mathbf{E}(Z) \le n \sum_{t=1}^{t_1} 3^{-t} e^{-\Delta_t} < n e^{-\Delta_{t_1}}.$$

Now,
$$2^{(2\alpha-1)t_1} \le 2^{(2\alpha-1)(2+\log_3(2/\epsilon)+\frac{2}{2\alpha-1}\log_3\left(\frac{2K_1 d}{(2\alpha-1)\theta}\right))} \le 2^{2+\log_3(2/\epsilon)+\frac{2}{2\alpha-1}\log_3(2K_1/(2\alpha-1))} d^{\gamma}$$

where
$$\gamma = \frac{1}{2\alpha+1}\frac{\ln 4}{\ln 3} \in \left(\frac{2}{5}, \frac{2}{3}\right). \tag{7}$$

So, if
$$K_2'' = d^{-1/3} 2^{2+\log_3(2/\epsilon)+\frac{2}{2\alpha-1}\log_3(2K_1/(2\alpha-1))}\left(\ln 4 + \frac{2}{2\alpha-1}\ln(K_1 d/(2\alpha-1))\right),$$

then

$$\mathbf{E}(Z) < e^{-d^{2/3-\gamma}(2/(2\alpha-1))\ln(K_1 d/(2\alpha-1))} n/4 \leq \epsilon_1 n/4.$$

It is straightforward to show that Z is concentrated about its mean. A random pairing can be obtained by choosing a random permutation $\omega = (w_1, w_2, \ldots, w_{dn})$ of W and pairing w_{2i-1} with w_{2i}, for $1 \leq i \leq dn/2$. Interchanging two elements in ω can change Z by at most two. Lemma 11 of [11] states: *Let S be a set with $|S| = N$. Let Ω be the set of $N!$ permutations of S. Let ω be chosen uniformly from Ω. Let $Z = Z(\omega)$ be such that $|Z(\omega) - Z(\omega')| \leq 1$ when ω' is obtained from ω by interchanging two elements of the permutation.*

$$\mathbf{Pr}(|Z - \mathbf{E}(Z)| \geq t) \leq 2e^{-2t^2/N}. \tag{8}$$

Using (8) with $N = dn$ and $t = (dn)^{1/2} \ln n$ yields the lemma. □

We now appeal to a result of Alon, Krivelevich and Sudakov [4] to show that we can colour the graphs $\Gamma_t = G_t - B_t$ with few colours. The main result of that paper is

Theorem 2 *There exists an absolute positive constant C_0 such that the following holds. Let G be a graph with maximum degree Δ in which the neighbourhood of any vertex v spans at most Δ^2/f edges. Then the chromatic number of G is at most $C_0 \frac{\Delta}{\ln f}$.*

We use this to prove

Lemma 3 *There exists $\beta > 0$ such that*

$$\mathbf{Pr}\left(\exists t \in [0, t_1] : \ \chi(\Gamma_t) \geq \frac{C_0 \Delta_t}{\beta \ln d}\right) = o(1).$$

Proof Fix t and $v \in I_t$ and condition on the neighbours of v in G_F so that $v \notin B_t$. In so doing we fix at most Δ pairs of the configuration. With this conditioning the number of edges ξ_v in the Γ_t-neighbourhood of v is stochastically dominated by the Binomial $B(\Delta_t^2/2, \Delta/(dn - 2\Delta)$. So, if $\beta = 4/3 - 2\gamma > 0$ (γ as in (7)) then

$$\mathbf{Pr}\left(\xi_v \geq \frac{\Delta_t^2}{d^\beta}\right) \leq \binom{\Delta_t^2/2}{\Delta_t^2/d^\beta} \left(\frac{2\Delta}{dn}\right)^{\Delta_t^2/d^\beta} \leq \left(\frac{e\Delta}{d^{1-\beta}n}\right)^{\Delta_t^2/d^\beta}.$$

But $\Delta \leq n^{1/3}$ and $\Delta_t^2/d^\beta \geq 10$ for sufficiently large K_1 and so

$$\mathbf{Pr}\left(\xi_v \geq \frac{\Delta_t^2}{d^\beta}\right) = o(1/n).$$

Applying Theorem 2 with $\Delta = \Delta_t$ and $f = d^\beta$ yields the lemma. □

Lemma 4 *If F is chosen uniformly from the set Ω of pairings then* **whp**

$$\chi(G_F) \leq b_1 \frac{d}{\ln d}.$$

Proof Let $B_t, G_t, \Gamma_t\,, t \in [0, t_1]$, be as defined above. It follows from Lemma 3 that **whp** the number of colours needed to colour $\bigcup_{t=0}^{t_1} \Gamma_t$ is at most

$$\sum_{t=0}^{t_1} \frac{C_0 K_2 d}{2^{(2\alpha-1)t} \beta \ln d} = O\left(\frac{d}{\ln d}\right).$$

Finally note that the number of vertices in $B = \bigcup_{t=0}^{t_1} B_t \cup \{i \geq n - \frac{n}{3^{t_1+1}}\}$ is **whp** less than ϵn and then, by Lemma 1, the graph induced by B is $\theta = d^{2/(2\alpha+1)}$-degenerate and so can be coloured with at most $\theta + 1 = O(d/\ln d)$ colours. □

3.2 From configurations to graphs

It is at this point that we appeal to some results from Mckay and Wormald [15]. Where possible, we will use the terminology and notation of that paper. A *loop* of a pairing F is a pair $\{u, v\}$ such that $\phi(u) = \phi(v)$. A *double pair* of F is a pair $\{u_1, v_1\}, \{u_2, v_2\} \in F$ such that $\phi(u_1) = \phi(u_2)$ and $\phi(v_1) = \phi(v_2)$. A *double loop* of F is a pair of pairs $\{u_1, v_1\}, \{u_2, v_2\}$ such that $\phi(u_1) = \phi(v_1) = \phi(u_2) = \phi(v_2)$. A *triple pair* is a triple of pairs $\{u_i, v_i\}\,, i = 1, 2, 3$ such that $\phi(u_1) = \phi(u_2) = \phi(u_3)$ and $\phi(v_1) = \phi(v_2) = \phi(v_3)$.

In the lemmas that follow, we will assume always that Condition (b) of Theorem 1 holds.

Lemma 5 *(Lemma 2 of [15])*
The probability that F contains at least one triple pair is $O(\Delta^2 M_2^2/M_1^3) = o(1)$ and the probability of at least one double loop is $O(\Delta^2 M_2/M_1^2) = o(1)$.

Let now l denote the number of loops and r denote the number of double pairs in F.

Lemma 6 *(Lemma 3 of [15])*
If $\lambda(n) \to \infty$ then **whp**

$$l \leq 2\Delta + \lambda \text{ and } r \leq \Delta^2 + \lambda. \tag{9}$$

We define the following two operations on a pairing: If $\phi(u) = i$ then we say that u is in *cell* i.

I l-switching.

Take pairs $\{p_1, p_6\}, \{p_2, p_3\}, \{p_4, p_5\}$ where $\{p_2, p_3\}$ is a loop, and $p_1, \ldots, p_6$ are in five different cells. Replace these pairs by $\{p_1, p_2\}, \{p_3, p_4\}, \{p_5, p_6\}$. In this operation, none of the pairs created or destroyed is permitted to be part of a double pair. (See Figure 1).

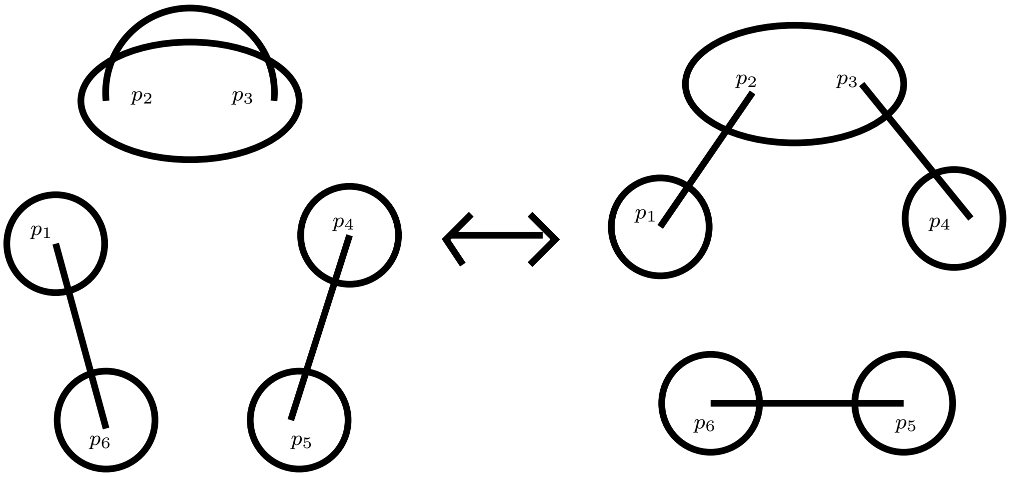

Figure 1:

II r-switching.

Take pairs $\{p_1, p_5\}, \{p_2, p_6\}, \{p_3, p_7\}, \{p_4, p_8\}$ where $\phi(p_2) = \phi(p_3)$ and $\phi(p_6) = \phi(p_7)$, but the cells containing $p_1, p_2, p_4, p_5, p_6, p_8$ are all distinct. Replace these pairs by $\{p_1, p_2\}, \{p_3, p_4\}, \{p_5, p_6\}, \{p_7, p_8\}$. In this operation, none of the pairs created or destroyed (other than the pairs $\{p_2, p_6\}, \{p_3, p_7\}$) is permitted to be part of a multiple pair. (See Figure 2).

A *forward* l-switching is an l-switching as described, and a *backward* l-switching is the reverse operation. We use the same convention for r-switchings. Note that a forward l-switching always reduces the number of loops by one and does not create or destroy double pairs. Similarly, a forward r-switching reduces the number of double pairs by one and neither creates nor destroys loops.

Now let $\mathcal{C}_{l,r}$ denote the set of pairings F with l loops, r double pairs and no triple pairs or double loops.

Lemma 7 *(Lemma 4 of [15])*
Denote an operation taking an element of $\mathcal{C}_{i,j}$ to an element $\mathcal{C}_{k,l}$ by $\mathcal{C}_{i,j} \to \mathcal{C}_{k,l}$. For each of the following operations, we bound the number, m, of ways of applying the operation to a fixed F.

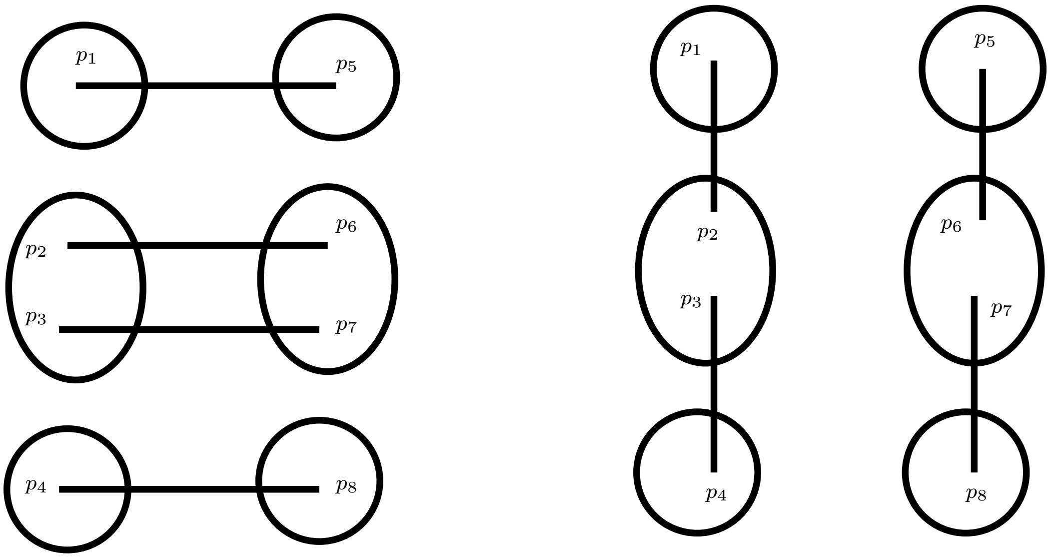

Figure 2:

(1) *Forward l-switching $\mathcal{C}_{l,r} \rightarrow \mathcal{C}_{l-1,r}$:*

$$2lM_1^2 \geq m \geq 2lM_1^2 \left(1 - O\left(\frac{\Delta^2 + l + r}{M_1}\right)\right).$$

(2) *Backward l-switching $\mathcal{C}_{l-1,r} \rightarrow \mathcal{C}_{l,r}$:*

$$M_1M_2 \geq m \geq M_1M_2 \left(1 - \frac{\Delta(6(l+2r) + \Delta l)}{M_2} - \frac{2\Delta(\Delta+2)}{M_1}\right).$$

(3) *Forward r-switching $\mathcal{C}_{0,r} \rightarrow \mathcal{C}_{0,r-1}$:*

$$4rM_1^2 \geq m \geq 4rM_1^2 \left(1 - O\left(\frac{\Delta^2 + r}{M_1}\right)\right).$$

(4) *Backward r-switching $\mathcal{C}_{0,r-1} \rightarrow \mathcal{C}_{0,r}$:*

$$M_2^2 \geq m \geq M_2^2 \left(1 - \frac{\Delta(16r + 9\Delta + 3 + \Delta^2)}{M_2}\right).$$

Now consider the following algorithm for generating a member of $\mathcal{G}_{n,\mathbf{d}}$:

1. Generate a random pairing F.
2. If there is a double loop or a triple pair, output $\perp$ – construed as failure.

3. If the number of loops $l \geq 2\Delta + \log n$ or the number of double pairs $r \geq \Delta^2 + \log n$, output $\perp$ – construed as failure.

4. $F_0 \leftarrow F$.

5. For $i = 1$ to l choose a random forward l-switching on F_{i-1}, creating $F_i \in \mathcal{C}_{l-i,r}$.

6. For $i = l+1$ to $l+r$ choose a random forward r-switching on F_{i-1}, creating $F_i \in \mathcal{C}_{0,r-(i-l)}$.

7. Output $G^* = G_{F_{l+r}} \in \mathcal{G}_{n,\mathbf{d}}$.

For each l, r satisfying (9), with $\lambda = \log n$, and $G \in \mathcal{G}_{n,\mathbf{d}}$, there are by Lemma 7(2),(4)

$$(M_1M_2)^l M_2^r \left(1 + O\left(\frac{\Delta l(\Delta l + r)}{M_2} + \frac{\Delta^2 l}{M_1} + \frac{\Delta r(\Delta^2 + r)}{M_2}\right)\right)$$

sequences of switchings which yield G. Each of these has probability

$$((2M_1^2)^l l!(4M_1^2)^r r!)^{-1}\left(1 + O\left(\frac{l(\Delta^2 + l + r)}{M_1} + \frac{r(\Delta^2 + r)}{M_1}\right)\right)$$

of being followed by the algorithm, given l, r.

Thus if Condition (b) holds, then **whp** the algorithm outputs a graph in $\mathcal{G}_{n,\mathbf{d}}$ and

$$\mathbf{Pr}(G^* = G) = (1 + o(1)) \sum_{l=0}^{2\Delta + \log n} \sum_{r=0}^{\Delta^2 + \log n} \frac{M_1^l M_2^{r+l}}{2^l M_1^{2(l+r)} l! r!} \mathbf{Pr}(l \textit{ loops}, r \textit{ double pairs})$$

and so for $G_1, G_2 \in \mathcal{G}_{n,\mathbf{d}}$

$$\mathbf{Pr}(G^* = G_1) = (1 + o(1))\mathbf{Pr}(G^* = G_2).$$

Given this, we only have to show that **whp** $\chi(G^*) = O(\chi(G_F))$. Thus let H be the graph induced by the edges that are *added* by the switchings. We will show that **whp**

$$\Delta(H) \leq 5. \tag{10}$$

Since $\chi(G^*) \leq \chi(H)\chi(G_F) \leq (\Delta(H) + 1)\chi(G_F)$, this will complete the proof of our upper bound.

Now every edge at distance ≥ 2 from the loop or double edge can be used as one of the two edges destroyed by the two types of switching. Thus, vertex i has probability

$$O\left(\frac{d_i}{M_1 - \Delta^2}\right) = O\left(\frac{\Delta}{M_1}\right)$$

of having an H-edge created in any switching, regardless of the history of the switchings to this point.

Thus, for some constant $c > 0$, assuming due to Lemma 6 that G^* satisfies (9):

$$\mathbf{Pr}(\Delta(H) \geq 5) \leq n \binom{\Delta^2 + 2\Delta + 2\log n}{5} \left(\frac{c\Delta}{M_1}\right)^5 \leq n M_1^{5/2-15/4} = o(1)$$

and this completes the proof of the upper bound in Theorem 1.

Note that we cannot immediately use the stronger result of [13]. This is because in that paper Lemma 7(4) is replaced by an *average* bound.

4 Lower bound

We will work with the configuration model.

We first observe that

$$\mathbf{Pr}(G_F \text{ is simple}) \geq e^{-\Delta^2}. \tag{11}$$

This follows directly from the formula for the asymptotic number of labelled graphs with a degree sequence in which $\Delta = o(M_1^{1/4})$, see [13].

Let

$$t = b_2 \frac{d}{\ln d},$$

where $b_2 > 0$ is a sufficiently small constant. Fix a partition $(V_1, \ldots, V_t)$ of the vertex set of G_F into t parts. We will show:

$$\mathbf{Pr}((V_1, \ldots, V_t) \text{ is a proper colouring of } G_F) \leq t^{-2n} .$$

This will be enough to beat both the union bound for the number of t-partitions (which is $\leq t^n$) and the inverse probability that G_F is simple (which is $\leq e^{\Delta^2}$ – see (11)).

Observe first that

$$\mathbf{Pr}(V_i \text{ is independent in } G_F) \leq \left(1 - \frac{D(V_i)/2 - \Delta}{dn}\right)^{D(V_i)/2}.$$

Explanation: Match the first $D(V_i)/2$ points of V_i in the configuration model. When matching a point, it should be matched either to a point belonging to a vertex outside of V_i, or to a point of the same vertex.

Now, if in the partition $(V_1, \ldots, V_t)$ one of the sets V_i satisfies $D(V_i) \geq nd/4$, by the above argument we get (taking b_2 small enough if necessary):

$$\mathbf{Pr}((V_1, \ldots, V_t) \text{ is a } t\text{-colouring}) \leq \mathbf{Pr}(V_i \text{ is independent}) \leq e^{-(1-o(1))nd/64} \ll t^{-2n}.$$

Assume therefore that $|V_i| \le dn/4$ for all $1 \le i \le t$. Assume w.l.o.g. that $D(V_1) \le D(V_2) \le \cdots \le D(V_t)$. Choose $1 \le k \le t$ so that

$$\frac{dn}{4} \le \sum_{i=1}^{k} D(V_i) \le \frac{dn}{2}\,;$$

such a k exists as $D(V_i) \le nd/4$ for all i.

We will estimate from above the probability that all of the sets V_i, $i = 1, \ldots, k$, are independent. In order to do this, we will expose the pairings of the points of V_i, each time choosing a set V_i that has at least half of its points unpaired, for as long as possible. Let σ describe the order in which the sets $V_1, \ldots, V_k$ are exposed. Observe that

$$\begin{aligned}&\mathbf{Pr}(V_{\sigma(i)} \text{ is independent}|V_{\sigma(1)}, \ldots, V_{\sigma(i-1)} \text{ are independent})\\ &\qquad \le \left(1 - \frac{D(V_{\sigma(i)})/4 - \Delta}{dn}\right)^{D(V_{\sigma(i)})/4} \le e^{-\frac{D(V_{\sigma(i)})^2 - 4\Delta D(V_{\sigma(i)})}{16dn}}\end{aligned}$$

(match the first half of yet unmatched points of $V_{\sigma(i)}$).

Suppose that we have in this way exposed the pairings of sets $V_{\sigma(1)}, \ldots, V_{\sigma(l)}$. If

$$\sum_{i=1}^{l} D(V_{\sigma(i)}) \le \frac{dn}{16}\,,$$

then at this moment the number of paired points does not exceed $dn/8$, and therefore at least one of the unexposed sets V_i, $i \notin \{\sigma(1), \ldots, \sigma(l)\}$, has at least a half of its points unpaired, and we can proceed. Thus, if by the end of the process we have exposed pairings of sets $V_{\sigma(1)}, \ldots, V_{\sigma(l^*)}$, it follows that $\sum_{i=1}^{l^*} D(V_{\sigma(i)}) \ge dn/16$ and therefore

$$\sum_{i=1}^{l^*} (D(V_{\sigma(i)})^2 - 4\Delta D(V_{\sigma(i)})) \ge \frac{(1-o(1))d^2n^2}{256 l^*} \ge \frac{(1-o(1))d^2n^2}{256t}\,.$$

Hence

$$\begin{aligned}&\mathbf{Pr}((V_{\sigma(1)}, \ldots, V_{\sigma(l^*)} \text{ are independent})\\ &\qquad \le \exp\left\{-\frac{\sum_{i=1}^{l^*}(D(V_{\sigma(i)})^2 - 4\Delta D(V_{\sigma(i)}))}{16dn}\right\} \le \exp\left\{-\frac{dn}{4100t}\right\}.\end{aligned}$$

Recalling the expression for t and choosing the constant b_2 there to be small enough, we get the desired estimate.

This completes the proof of Theorem 1. $\square$

Acknowledgement: We thank Paul Horn for pointing out an error in an earlier draft of the paper.

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