Compatible Hamilton cycles in Dirac graphs

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Abstract

A graph is Hamiltonian if it contains a cycle passing through every vertex exactly once. A celebrated theorem of Dirac from 1952 asserts that every graph on $n \geq 3$ vertices with minimum degree at least $n/2$ is Hamiltonian. We refer to such graphs as Dirac graphs. In this paper we obtain the following strengthening of this result. Given a graph $G = (V, E)$, an *incompatibility system* $\mathcal{F}$ over G is a family $\mathcal{F} = \{F_v\}_{v \in V}$ such that for every $v \in V$, the set F_v is a family of unordered pairs $F_v \subseteq \{\{e, e'\} : e \neq e' \in E, e \cap e' = \{v\}\}$. An incompatibility system is Δ-*bounded* if for every vertex v and an edge e incident to v, there are at most Δ pairs in F_v containing e. We say that a cycle C in G is *compatible* with $\mathcal{F}$ if every pair of incident edges e, e' of C satisfies $\{e, e'\} \notin F_v$, where $v = e \cap e'$. This notion is partly motivated by a concept of transition systems defined by Kotzig in 1968, and can be viewed as a quantitative measure of robustness of graph properties. We prove that there is a constant $\mu > 0$ such that for every μn-bounded incompatibility system $\mathcal{F}$ over a Dirac graph G, there exists a Hamilton cycle compatible with $\mathcal{F}$. This settles in a very strong form a conjecture of Häggkvist from 1988.

1 Introduction

A *Hamilton cycle* in a graph G is a cycle passing through each vertex of G, and a graph is *Hamiltonian* if it contains a Hamilton cycle. Hamiltonicity, named after Sir Rowan Hamilton who studied it in the 1850s, is a very important and extensively studied concept in graph theory. The study of Hamiltonicity has mainly been concerned with looking for simple sufficient conditions implying Hamiltonicity. One of the most important results in this direction is Dirac's theorem asserting that every n-vertex graph, $n \geq 3$, of minimum degree at least $\frac{n}{2}$ contains a Hamilton cycle. In this context, we define a *Dirac graph* as an n-vertex graph of minimum degree at least $\frac{n}{2}$. Note that the bound $\frac{n}{2}$ is tight, as can be seen by the following two examples: first is a graph obtained by taking two vertex-disjoint complete graphs of order k and identifying one vertex from each of them, and second is the complete bipartite graph with parts of sizes k and $k - 1$. Both graphs have $2k - 1$ vertices and minimum degree $k - 1$, but are not Hamiltonian.

Recently there has been an increasing interest in studying robustness of graph properties, aiming to strengthen classical results in extremal and probabilistic combinatorics. For example, consider the Hamiltonicity property of Dirac graphs. Then one can ask, "How strongly do Dirac graphs possess the property of being Hamiltonian?". There are several ways to answer this question using different measures of robustness. For example, one can try to show that a Dirac graph has many

Hamilton cycles or that Maker can win a Hamiltonicity game played on edges of a Dirac graph. These extensions and other similar questions have been answered in [8] for the number of Hamilton cycles, in [6] and [10] for the number of edge-disjoint Hamilton cycles, in [11] for the cycle space generated by Hamilton cycles, and in [14] for Hamiltonicity of random subgraphs and for the Maker-Breaker games on Dirac graphs. Also, very recently, a number of related important problems on regular Dirac graphs, such as the existence of decomposition of its edge set into Hamilton cycles, have been settled in [7], using a result proved in [16]. In fact, the study of robustness of graph properties can be identified as one of the central themes in extremal graph theory, and its scope extends far beyond Hamiltonicity and Dirac graphs (see, e.g., [21]).

In this paper, we are interested in yet another type of robustness, and study the robustness of Hamiltonicity of Dirac graphs with respect to a new measure.

Definition 1.1. *Let $G = (V, E)$ be a graph.*

(i) *An* incompatibility system $\mathcal{F}$ *over G is a family $\mathcal{F} = \{F_v\}_{v \in V}$ such that for every $v \in V$, F_v is a family of unordered pairs $F_v \subseteq \{\{e, e'\} : e \neq e' \in E, e \cap e' = \{v\}\}$.*

(ii) *If $\{e, e'\} \in F_v$ for some edges e, e' and vertex v, then we say that e and e' are* incompatible *in $\mathcal{F}$. Otherwise, they are* compatible *in $\mathcal{F}$. A subgraph $H \subseteq G$ is* compatible *in $\mathcal{F}$, if all its pairs of edges e and e' are compatible.*

(iii) *For a positive integer Δ, an incompatibility system $\mathcal{F}$ is Δ-bounded if for each vertex $v \in V$ and an edge e incident to v, there are at most Δ other edges e' incident to v that are incompatible with e.*

The definition is motivated by two concepts in graph theory. First, it generalizes *transition systems* introduced by Kotzig [13] in 1968, where a transition system is a 1-bounded incompatibility system. Kotzig's work was motivated by a problem of Nash-Williams on cycle covering of Eulerian graphs (see, e.g. Section 8.7 of [3]).

Incompatibility systems and compatible Hamiton cycles also generalize the concept of properly colored Hamilton cycles in edge-colored graphs. The problem of finding properly colored Hamilton cycles in edge-colored graph was first introduced by Daykin [9]. He asked if there exists a constant μ such that for large enough n, there exists a properly colored Hamilton cycle in every edge-coloring of a complete graph K_n where each vertex has at most μn edges incident to it of the same color (we refer to such coloring as a *μn-bounded edge coloring*). Daykin's question has been answered independently by Bollobás and Erdős [2] with $\mu = \frac{1}{69}$, and by Chen and Daykin [5] with $\mu = \frac{1}{17}$. Bollobás and Erdős further conjectured that all $(\lfloor \frac{n}{2} \rfloor - 1)$-bounded edge coloring of K_n admits a properly colored Hamilton cycle. After subsequent improvements by Shearer [20] and by Alon and Gutin [1], Lo [17] recently settled the conjecture asymptotically, proving that for any positive ε, every $(\frac{1}{2} - \varepsilon)n$-bounded edge coloring of K_n admits a properly colored Hamilton cycle.

Note that a μn-bounded edge coloring naturally defines a μn-bounded incompatibility system, and thus the question mentioned above can be considered as a special case of the problem of finding compatible Hamilton cycles. However, in general, the restrictions introduced by incompatibility systems need not come from an edge-coloring of graphs, and thus results on properly colored Hamilton cycles do not necessarily generalize easily to incompatibility systems.

In this paper we study compatible Hamilton cycles in Dirac graphs. Our work is motivated by the following conjecture of Häggkvist from 1988 (see [3, Conjecture 8.40]).

Conjecture 1.2. *Let G be a Dirac graph. For every 1-bounded incompatibility system $\mathcal{F}$ over G, there exists a Hamilton cycle compatible with $\mathcal{F}$.*

Here, we settle this conjecture, in fact in a very strong form.

Theorem 1.3. *There exists a constant μ such that the following holds for large enough n. For every n-vertex Dirac graph G and a μn-bounded incompatibility system $\mathcal{F}$ defined over G, there exists a Hamilton cycle in G compatible with $\mathcal{F}$.*

Our theorem shows that Dirac graphs are very robust against incompatibility systems, i.e., one can find a Hamilton cycle even after forbidding a quadratic number of pairs of edges incident to each vertex from being used together in the cycle. The order of magnitude is clearly best possible since we can simply forbid all pairs incident to some vertex from being used together to disallow a compatible Hamilton cycle. However, it is not clear what the best possible value of μ should be. Our proof shows the existence of a positive constant μ (approximately 10^{-16} although no serious attempt has been made to optimize the constant), and a variant of a construction of Bollobás and Erdős [2] shows that $\mu \leq \frac{1}{4}$. See Section 5 for further discussion. An alternative proof of Conjecture 1.2 can be obtained by using our result in [14] on the Maker-Breaker Hamiltonicity game played on Dirac graphs. However this approach gives a weaker statement than Theorem 1.3, see Section 5 for further details.

The rest of the paper is organized as follows. In Section 2, we discuss and develop our main tool, based on Pósa's rotation-extension technique. The proof of Theorem 1.3 consists of several cases, and will be given in Sections 3 and 4. In Section 5, we conclude this paper with some remarks.

Notation. A graph $G = (V, E)$ is given by a pair of its vertex set $V = V(G)$ and edge set $E = E(G)$. For a set X, let $e(X)$ be the number of edges whose both endpoints are in X. For a pair X, Y of sets of vertices, we define $e(X, Y)$ as the number of edges with one endpoint in X and the other in Y counted with multiplicity. Hence, $e(X, X) = 2e(X)$.

For a path P (or a cycle C), we let $|P|$ (or $|C|$, respectively) be the number of vertices in the path (cycle, respectively). Also, we define the length of a path, or a cycle, as its number of edges. Throughout the paper, we assume that the number of vertices n is large enough, and omit floor and ceiling signs whenever these are not crucial.

2 Rotation-extension technique and its modification

2.1 Pósa's rotation-extension technique

Our main tool in proving Theorem 1.3 is Pósa's rotation-extension technique, which first appeared in [19] (see also [18, Ch. 10, Problem 20]).
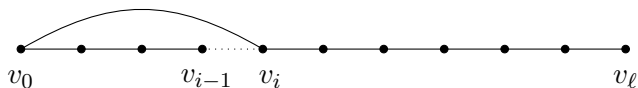

Figure 1: Rotating a path.

Let G be a connected graph and let $P = (v_0, \ldots, v_\ell)$ be a path on some subset of vertices of G. Consider the neighbors of v_0. If there exists an edge $\{v_0, w\}$ for $w \notin V(P)$, then we can *extend P*

to find a longer path $(w, v_0, \ldots, v_\ell)$. Otherwise, if $\{v_0, v_\ell\}$ is an edge of the graph, then we can use it to close P into a cycle, in which case since G is connected, we either get a Hamilton cycle, or can find a path longer than P.

Now assume that we cannot directly extend P as above, and that G contains an edge $\{v_0, v_i\}$ for some i. Then $P' = (v_{i-1}, \ldots, v_0, v_i, v_{i+1}, \ldots, v_\ell)$ forms another path of length ℓ in G (see Figure 1). We say that P' is obtained from P by a *rotation* with *fixed endpoint* v_ℓ, *pivot point* v_i, and *broken edge* (v_{i-1}, v_i). Note that after performing this rotation, we can now close a cycle of length ℓ also using the edge $\{v_{i-1}, v_\ell\}$ if it exists. As we perform more and more rotations, we will get more such candidate edges (call them *closing edges*). The rotation-extension technique is employed by repeatedly rotating the path until one can extend the path, or find a closing edge in the graph to find a cycle (in which case we either find a Hamilton cycle, or can proceed further by finding a longer path).

Let P'' be a path obtained from P by several rounds of rotations. An important observation that we will use later is that for every interval $I = (v_j, \ldots, v_k)$ of vertices of P $(1 \leq j < k \leq \ell)$, if no edges of I were broken during these rotations, then I appears in P'' either exactly as it does in P, or in the reversed order. We define the *orientation*, or *direction*, of a path P'' with respect to an interval I to be *positive* in the former situation, and *negative* in the latter situation.

2.2   The class of smoothly compatible paths

There are several difficulties in naively applying the rotation-extension technique to find compatible Hamilton cycles. Suppose that we are given a graph G and an incompatibility system $\mathcal{F}$ over G. Let $P = (v_0, v_1, \ldots, v_\ell)$ be a path compatible with $\mathcal{F}$. First of all, even if v_0 has a neighbor outside P, we might not be able to extend the path P to a longer path, since the edge connecting v_0 to its neighbor outside P can be incompatible with the edge $\{v_0, v_1\}$. Second, even if all the neighbors of v_0 are in P, we might not be able to perform a single round of rotation since the pair of edges $\{v_0, v_i\}$ and $\{v_i, v_{i+1}\}$ can be incompatible for each i.

Note, however, that the first problem is less of an issue, since we can find a longer path compatible with $\mathcal{F}$ as long as v_0 has greater than μn neighbors outside P. The second problem is more serious, and to overcome the difficulty, we consider only a special type of paths. First, we identify the problematic vertices with respect to a given path.

Definition 2.1. *Let γ be a positive real. Let $P = (v_0, v_1, \ldots, v_\ell)$ be a path and $w \in V$ be a vertex (w need not be in $V(P)$).*

   (i) *A vertex $v_i \in N(w)$ is a* bad neighbor *of w in P, if $\{w, v_i\}$ is incompatible with $\{v_i, v_{i-1}\}$ or $\{v_i, v_{i+1}\}$. A vertex $v_i \in N(w)$ is a* good neighbor *of w in P otherwise.*

   (ii) *w is γ-bad* for P *if there are at least $\gamma|P|$ bad neighbors of w in P, and w is γ-good* for P *otherwise.*

  (iii) *Similarly define* bad neighbors, *γ-bad and γ-good* vertices with respect to cycles.

Note that the relation of 'being a bad neighbor' is not symmetric, i.e., x being a bad neighbor of y in P does not imply that y is a bad neighbor of x in P (similar for good neighbors). The definition above was made with the intention of forcing both endpoints of the path to be γ-good (for an appropriately chosen γ), throughout the process of rotation and extension, hoping to resolve the latter problem mentioned above. Indeed, suppose that v_0 is γ-good for P. Then by the rotation-extension technique, all but at most $\gamma|P|$ of the neighbors of v_0 in P can be used as a pivot point to

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give another path P' compatible with $\mathcal{F}$. This (weakly) resolves the issue mentioned above regarding the rotation-extension technique. Indeed, we can perform one round of rotation, but afterwards, we have no guarantee that the two new endpoints are good with respect to the new path P' (the new endpoint might be a bad vertex, or the number of bad neighbors of the fixed endpoint might have increased). In order to resolve this issue, we make the following definition.

Definition 2.2. *Let γ be a positive real. Two vertices v_1 and v_2 are γ-correlated if there are at least γn vertices w such that the pair of edges $\{v_1, w\}$ and $\{v_2, w\}$ are incompatible. We say that v_1 and v_2 are γ-uncorrelated if they are not γ-correlated.*

Note that two vertices being γ-correlated is a global condition; it does not depend on individual paths. We now define a special type of paths and cycles by utilizing the two concepts defined above.

Definition 2.3. *Let G be an n-vertex graph with a μn-bounded incompatibility system $\mathcal{F}$. A path $P = (v_0, v_1, \cdots, v_\ell)$ is smoothly compatible with $\mathcal{F}$ (or smooth in short if $\mathcal{F}$ is clear from the context) if*

  (i) P is compatible with $\mathcal{F}$,
  (ii) both endpoints v_0 and v_ℓ are $8\sqrt{\mu}$-good for P,
  (iii) the pair of vertices v_0 and v_ℓ is $\sqrt{\mu}$-uncorrelated.

Note that we made the two endpoints to be $8\sqrt{\mu}$-good, instead of making them $\sqrt{\mu}$-good. This choice was made for technical reasons, and will be crucial later. We conclude this subsection with a proposition asserting that good vertices and uncorrelated pairs are abundant, thus showing that the definition of smooth paths is not too restrictive.

Proposition 2.4. *Let G be an n-vertex graph with a μn-bounded incompatibility system $\mathcal{F}$.*

  (i) For every path P, there are at most $\sqrt{\mu} n$ vertices that are $2\sqrt{\mu}$-bad for P. The same statement holds for a cycle C.
  (ii) For every vertex v, there are at most $\sqrt{\mu} n$ vertices that are $\sqrt{\mu}$-correlated with v.

Proof. (i) We prove the claim only for paths as the claim for cycles follows from the same proof. Let $P = (v_0, v_1, \cdots, v_\ell)$. We count the number of edges of the form $\{w, v_i\}$ such that v_i is a bad neighbor of w in P in two different ways.

First, for a fixed vertex v_i, since $\mathcal{F}$ is μn-bounded, there are at most $2\mu n$ vertices w for which v_i is a bad neighbor of w; at most μn vertices for which $\{w, v_i\}$ is incompatible with $\{v_i, v_{i+1}\}$, and at most μn vertices for which $\{w, v_i\}$ is incompatible with $\{v_i, v_{i-1}\}$. Hence there are at most $|P| \cdot 2\mu n$ such edges. Second, if we define N as the number of vertices with at least $2\sqrt{\mu}|P|$ bad neighbors in P, then by definition, the number of such edges is at least $N \cdot 2\sqrt{\mu}|P|$. By combining the two bounds, we see that $N \leq \sqrt{\mu} n$.

(ii) Let M be the number of vertices that are $\sqrt{\mu}$-correlated with v. We count the number of pairs of vertices v', w such that $\{v, w\}$ and $\{v', w\}$ are incompatible in two different ways. On one hand, each vertex that is $\sqrt{\mu}$-correlated with v gives at least $\sqrt{\mu} n$ such pairs, and thus the number of pairs is at least $M \cdot \sqrt{\mu} n$. On the other hand, for each fixed vertex w, there are at most μn vertices v' such that $\{v, w\}$ and $\{v', w\}$ are incompatible. Hence the number of pairs is at most $n \cdot \mu n$. Therefore $M \leq \sqrt{\mu} n$. $\qquad\square$

2.3 Rotating smooth paths

Our first lemma shows how the rotation part of Pósa's rotation-extension technique extends to the class of smooth paths. This lemma is the most important building block of our proof.

Lemma 2.5. *Let μ be a positive real satisfying $\mu \leq \frac{1}{225}$. Let G be an n-vertex graph of minimum degree at least $15\sqrt{\mu}n$ with a μn-bounded incompatibility system $\mathcal{F}$. Suppose that $P = (v_0, v_1, \ldots, v_\ell)$ is a smooth path in G, where there is no vertex $w \notin V(P)$ for which $(w, v_0, \ldots, v_\ell)$ is a smooth path. Then there exists a subset X of the set of good neighbors of v_0 in P, of size at least*

$$|X| \geq d(v_0) - 14\sqrt{\mu}n$$

such that for every vertex $v_i \in X$, the path $(v_{i-1}, \ldots, v_1, v_0, v_i, v_{i+1}, \ldots, v_\ell)$ is smooth.

Proof. Let $V = V(G)$. Define

$$B_1 = \Big\{ w : \{v_0, w\} \text{ is incompatible with } \{v_0, v_1\}, \{v_\ell, w\}, \text{ or } \{v_0, v_\ell\} \text{ (if in } E(G)) \Big\}, \text{ and}$$

$$B_2 = \Big\{ w : w \text{ is } 2\sqrt{\mu}\text{-bad for } P \Big\} \cup \Big\{ w : w \text{ is } \sqrt{\mu}\text{-correlated with } v_\ell \Big\}.$$

We have $|B_1| \leq 2\mu n + \sqrt{\mu}n$ since $\mathcal{F}$ is μn-bounded, and the pair of vertices v_0, v_ℓ are $\sqrt{\mu}$-uncorrelated. We have $|B_2| \leq 2\sqrt{\mu}n$ by Proposition 2.4.

Suppose that v_0 has a neighbor w in $V \setminus (V(P) \cup B_1 \cup B_2)$. We claim that the path $P' = (w, v_0, v_1, \cdots, v_\ell)$ is smooth. First, the path P' is compatible since $w \notin B_1$ implies that the pair of edges $\{w, v_0\}$ and $\{v_0, v_1\}$ is compatible. Second, since $w \notin B_2$, we see that w and v_ℓ are $\sqrt{\mu}$-uncorrelated. It remains to show that w and v_ℓ are both $8\sqrt{\mu}$-good for P'. For w, a vertex v_i is a bad neighbor of w in P' if and only if it is in P. Hence by the fact that $w \notin B_2$, we see that w is $2\sqrt{\mu}$-good for P'. For the other endpoint v_ℓ, its set of bad neighbors in P can be different from that in P' in at most two vertices v_0 and w. However, since $w \in B_1$, w cannot be a bad neighbor of v_ℓ in P', and v_0 can be a bad neighbor of v_ℓ in P' if and only if it were in P. Hence P' is a smooth path, contradicting our assumption. This shows that all the neighbors of v_0 are in $V(P) \cup B_1 \cup B_2$.

Further define

$$B_3 = \Big\{ w \in V(P) : w \text{ is a bad neighbor of } v_0 \text{ in } P \Big\}.$$

We have $|B_3| \leq 8\sqrt{\mu}n$, since v_0 is $8\sqrt{\mu}$-good for P. Define $B_2^+ = \{v_{i+1} \,|\, v_i \in B_2 \cap V(P)\}$, and $X = \Big(N(v_0) \cap V(P) \Big) \setminus (B_1 \cup B_2^+ \cup B_3)$. Since all neighbors of v_0 are in $V(P) \cup B_1 \cup B_2$, we have

$$|X| \geq |N(v_0)| - |B_1| - |B_2| - |B_2^+| - |B_3| \geq |N(v_0)| - |B_1| - 2|B_2| - |B_3|$$
$$\geq |N(v_0)| - 13\sqrt{\mu}n - 2\mu n \geq d(v_0) - 14\sqrt{\mu}n \,.$$

We prove that X is the set claimed in the statement of the lemma. It suffices to prove that for every $v_i \in X$, the path $P'' = (v_{i-1}, \cdots, v_1, v_0, v_i, v_{i+1}, \cdots, v_\ell)$ is smooth. First, to check compatibility, we need to check the two pairs of edges $\Big(\{v_0, v_1\}, \{v_0, v_i\} \Big)$ and $\Big(\{v_0, v_i\}, \{v_i, v_{i+1}\} \Big)$. The first pair is compatible since $v_i \notin B_1$, and the second pair is compatible since $v_i \notin B_3$. Second, since $v_i \notin B_2^+$, we see that $v_{i-1} \notin B_2$, and thus the pair of vertices v_{i-1} and v_ℓ are $\sqrt{\mu}$-uncorrelated. It remains to show that the two endpoints of P'' are both $8\sqrt{\mu}$-good for P''.

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Note that for each vertex, the set of its bad neighbors in P'' can be different from that in P in at most two vertices, v_0 and v_i. Since $v_i \notin B_2^+$, we see that $v_{i-1} \notin B_2$, and thus v_{i-1} has less than $2\sqrt{\mu}|P|$ bad neighbors in P. This proves that v_{i-1} has less than $2\sqrt{\mu}|P| + 2$ bad neighbors in P''; thus v_{i-1} is $8\sqrt{\mu}$-good for P'' (we used the fact $|P| \geq |X| \geq \sqrt{\mu}n$ following from the given minimum degree condition). For the other endpoint v_ℓ, consider its relation with the two vertices v_0 and v_i. Since $v_i \notin B_1$, the pairs of edges $\left(\{v_\ell, v_i\}, \{v_0, v_i\}\right)$ and $\left(\{v_\ell, v_0\}, \{v_0, v_i\}\right)$ are compatible (note that $\{v_\ell, v_i\}$ and $\{v_\ell, v_0\}$ may not be edges of G, in which case there is no need to consider the compatibility of pairs involving them). Therefore, the set of bad neighbors of v_ℓ in P'' is a subset of that in P. This proves our claim that P'' is smooth. $\qquad\square$

3 Proof of Theorem 1.3

We prove Theorem 1.3 in two steps. In the first step, we show that the given graph contains a compatible Hamilton cycle unless it has a special structure.

Theorem 3.1. *Let μ be a positive real satisfying $\sqrt{\mu} < \frac{1}{400}$. Let G be an n-vertex graph of minimum degree at least $\frac{n}{2}$ with a μn-bounded incompatibility system $\mathcal{F}$. Then at least one of the following holds:*

  (i) *There exists a Hamilton cycle compatible with $\mathcal{F}$, or*
  (ii) *there exist two sets A and B of sizes $|A|, |B| \geq (\frac{1}{2} - 200\sqrt{\mu})n$ such that $e(A, B) \leq 16\sqrt{\mu}n^2$.*

In the second step, we show that there exists a compatible Hamilton cycle even in the case when the graph has a special structure.

Theorem 3.2. *Let μ, ν and η be reals satisfying $110\nu + 250\eta + 10\sqrt{\mu} < \frac{1}{2000}$. Let G be an n-vertex graph of minimum degree at least $\frac{n}{2}$ with a μn-bounded incompatibility system $\mathcal{F}$, and suppose that there are two sets A, B of sizes $|A|, |B| \geq (\frac{1}{2} - \nu)n$ such that $e(A, B) \leq \eta n^2$. Then G contains a Hamilton cycle compatible with $\mathcal{F}$.*

The two theorems imply Theorem 1.3 with $\mu = 10^{-16}$.

3.1 Step I : Theorem 3.1

In the following lemma, we prove, by utilizing Lemma 2.5, that every smooth path can be closed into a compatible cycle, after several rounds of rotations and extensions. We consider graphs that have minimum degree close to $\frac{n}{2}$, but not necessarily at least $\frac{n}{2}$. This extra flexibility will be useful for our later application.

Lemma 3.3. *Let α and μ be non-negative reals satisfying $\alpha \leq \sqrt{\mu} \leq \frac{1}{400}$. Let G be an n-vertex graph of minimum degree at least $\left(\frac{1}{2} - \alpha\right)n$ with a μn-bounded incompatibility system $\mathcal{F}$. Then at least one of the following holds:*

  (i) *For every smooth path P, there exists a compatible cycle C of length $|C| \geq (\frac{1}{2} + 20\sqrt{\mu})n$ for which $V(P) \subseteq V(C)$, and $|E(C) \setminus E(P)| \leq 3|V(C) \setminus V(P)| + 4$.*
  (ii) *There exist two sets A and B of sizes $|A|, |B| \geq (\frac{1}{2} - 200\sqrt{\mu})n$ such that $e(A, B) \leq 15\sqrt{\mu}n^2$.*

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Proof. Let G be a given graph for which (ii) does not hold. Given an arbitrary smooth path $P = (v_0, v_1, \cdots, v_\ell)$ in G, we will either extend P in at most two rotations, or show that $|P| \geq (\frac{1}{2} + 20\sqrt{\mu})n$ and close P into a cycle in at most three rotations. If the former event happens, then repeat the above until the latter event happens. Since we use at most three new edges in the former case, and four new edges in the latter case, the final cycle C that we obtain will satisfy

$$|E(C) \setminus E(P)| \leq 3|V(C) \setminus V(P)| + 4.$$

This implies (i). To prove our claim, it suffices to assume that P cannot be extended in at most two rotations, and show that under this assumption, it can be closed into a cycle in at most three rotations. For a set $X \subseteq V(P)$, define $X^+ = \{v_{i+1} \mid v_i \in X, i \leq \ell - 1\}$ and $X^- = \{v_{i-1} \mid v_i \in X, i \geq 1\}$.

Let S be the subset of $V(P)$ having the following property: for every $v_i \in S$, there exists a smooth path P' of length ℓ between v_i and v_ℓ, which is obtained from P in at most three rounds of rotations having v_ℓ as a fixed endpoint and starts with edge $\{v_i, v_{i-1}\}$ or $\{v_i, v_{i+1}\}$. Note that if $v_{\ell-1} \in S$, then v_ℓ must have been used as a pivot point. However, since v_ℓ is a fixed endpoint of the path, this implies the existence of a cycle of length $|V(P)|$ obtained from P by adding at most four edges. Thus we may assume that $v_{\ell-1} \notin S$, which in particular implies that for all paths P' as above, the edge incident to v_ℓ is still $\{v_{\ell-1}, v_\ell\}$.

We prove the lemma by proving that $|S| \geq (\frac{1}{2} + 20\sqrt{\mu})n$. Assume for the moment that this bound holds. By Lemma 2.5, there exists a subset of the set of good neighbors of v_ℓ in P, of size at least $\left(\frac{1}{2} - \alpha\right)n - 14\sqrt{\mu}n$. Since

$$\left(\frac{1}{2} + 20\sqrt{\mu}\right)n + \left(\frac{1}{2} - \alpha - 14\sqrt{\mu}\right)n > n,$$

we can find a vertex $v_i \in S$ that is a good neighbor of v_ℓ, and is connected to v_ℓ by an edge compatible with $\{v_\ell, v_{\ell-1}\}$. Since $v_i \in S$, by definition, there exists a smooth path P_i from v_i to v_ℓ obtained from P whose edge incident to v_i is $\{v_i, v_{i+1}\}$ or $\{v_i, v_{i-1}\}$. This shows that we may use the edge $\{v_\ell, v_i\}$ to close P_i into a cycle C compatible with $\mathcal{F}$. Note that $|C| \geq |S| \geq (\frac{1}{2} + 20\sqrt{\mu})n$.

Hence it suffices to prove $|S| \geq (\frac{1}{2} + 20\sqrt{\mu})n$. Assume to the contrary that $|S| < (\frac{1}{2} + 20\sqrt{\mu})n$. We show that S must have some specific structure under this assumption.

Claim. Suppose that P cannot be extended in at most two rotations. If $|S| < (\frac{1}{2} + 20\sqrt{\mu})n$, then $|S^- \cup S^+| \leq \left(\frac{1}{2} + 200\sqrt{\mu}\right)n$.

The proof of this claim will be given later. By Lemma 2.5 and our assumption on P not being extendable, there exists a set $X \subseteq N(v_0) \cap V(P)$ of size at least $d(v_0) - 14\sqrt{\mu}n$ such that for every vertex $v_i \in X$, the path $(v_{i-1}, \ldots, v_0, v_i, v_{i+1}, \ldots, v_\ell)$ is a smooth path. For each vertex $v_i \in X$, we similarly obtain a set $X_i \subseteq N(v_{i-1})$ of size $|X_i| \geq d(v_{i-1}) - 14\sqrt{\mu}n$ such that each vertex $v_j \in X_i \setminus \{v_i, v_{i+1}\}$ can be used as a pivot point to give either v_{j-1} or v_{j+1} as another endpoint in S. Hence the definition of S implies that $X_i \setminus \{v_i, v_{i+1}\} \subseteq S^- \cup S^+$. Since $X_i \subseteq N(v_{i-1})$, this implies that the number of edges between X^- and $S^- \cup S^+$ satisfies

$$e(X^-, S^- \cup S^+) \geq \sum_{x \in X^-} (d(x) - 14\sqrt{\mu}n - 2) \geq e(X^-, V) - 14\sqrt{\mu}n \cdot |X| - 2n,$$

and thus

$$e\left(X^-, V \setminus (S^- \cup S^+)\right) \leq 15\sqrt{\mu}n^2.$$

However, this gives sets $A = X^-$ and $B = V \setminus (S^- \cup S^+)$ satisfying (ii), thus contradicting our assumption. $\qquad \square$

It remains to prove the claim. The intuition behind this somewhat peculiar looking claim comes from the following examples of graphs that have minimum degree close to $\frac{n}{2}$ but are not Hamiltonian. First is the graph G on $2k+1$ vertices consisting of two complete graphs K_{k+1} sharing a single vertex. There exists a Hamilton path P in this graph, but it cannot be closed into a Hamilton cycle; note that the set S as in the proof of the lemma above consists of the first half of the path, and we have $|S^+ \cup S^-| \approx \frac{n}{2}$. Second is the complete bipartite graph $K_{k,k+1}$. Again, there exists a Hamilton path P that cannot be closed into a Hamilton cycle.; the set S consists of every other vertex along the path, and we have $|S^+ \cup S^-| \approx \frac{n}{2}$. Thus informally, our claim asserts that the given graph resembles such graphs when $|S| \approx \frac{n}{2}$.

Proof of Claim. Recall that $P = (v_0, \ldots, v_\ell)$ is a smooth path that cannot be extended in at most two rotations. We defined S as the subset of vertices of $V(P)$ having the following property: for every $v_i \in S$, there exists a smooth path P' of length ℓ between v_i and v_ℓ, which is obtained from P in at most three rounds of rotations having v_ℓ as a fixed endpoint and starts with edge $\{v_i, v_{i-1}\}$ or $\{v_i, v_{i+1}\}$. Moreover, we assumed that $|S| \leq (\frac{1}{2} + 20\sqrt{\mu})n$. Recall that for a set $X \subseteq V(P)$, we defined $X^+ = \{v_{i+1} \mid v_i \in X, \, i \leq \ell - 1\}$ and $X^- = \{v_{i-1} \mid v_i \in X, \, i \geq 1\}$. An *interval* is a set of vertices $I \subseteq V(P)$ of the form $\{v_j \mid j \in [a,b]\}$ for some $0 \leq a < b \leq \ell$. Throughout the proof, we sometimes add constants to inequalities, such as in $|I \cap X| \leq |I \cap X^-| + 1$ for an interval I and a set $X \subseteq V(P)$, in order to account for potential boundary effects.

We first prove the existence of a vertex $w \in V(P)$, a 'large' interval $I \subseteq V(P)$, and two smooth paths P_1 and P_2 of length ℓ between w and v_ℓ, where P_1 traverses the interval I positively and P_2 traverses it negatively (see Figure 2). By Lemma 2.5 and our assumption on P not being extendable, there exists a set $X \subseteq N(v_0) \cap V(P)$ of size at least $d(v_0) - 14\sqrt{\mu}n$ such that for every vertex $v_i \in X$, the path $(v_{i-1}, \ldots, v_0, v_i, v_{i+1}, \ldots, v_\ell)$ is smooth. By definition of S, we thus have $X^- \subseteq S$. Let $\beta = 70\sqrt{\mu} + 3\alpha$, and take the vertex $v_j \in X$ whose index j is βn-th largest among those in X. Such index exists since $|X| \geq \delta(G) - 14\sqrt{\mu}n \geq \beta n$, by the given condition on parameters.

Let $I_1 = \{v_0, v_1, \ldots, v_{j-1}\}$ and $I_2 = \{v_j, \ldots, v_\ell\}$. Note that

$$|I_1 \cap S| = |S| - |I_2 \cap S| \leq |S| - |I_2 \cap X^-| \leq \left(\frac{1}{2} + 20\sqrt{\mu} - \beta\right)n + 1. \tag{1}$$

Let $P' = (v_{j-1}, v_{j-2}, \cdots, v_1, v_0, v_j, v_{j+1}, \cdots, v_\ell)$. Note that P' cannot be extended since P cannot be extended in at most two rotations. Thus by applying Lemma 2.5 to P', we see, as above, that there exists a set $Y \subseteq N(v_{j-1}) \setminus \{v_j, v_{j+1}\}$ of size $|Y| \geq d(v_{j-1}) - 14\sqrt{\mu}n - 2$ such that $I_1 \cap Y^+ \subseteq S$ and $I_2 \cap Y^- \subseteq S$. If there exists a vertex $w \in I_2 \cap X^- \cap Y^-$, then there exist two paths P_+, P_- and
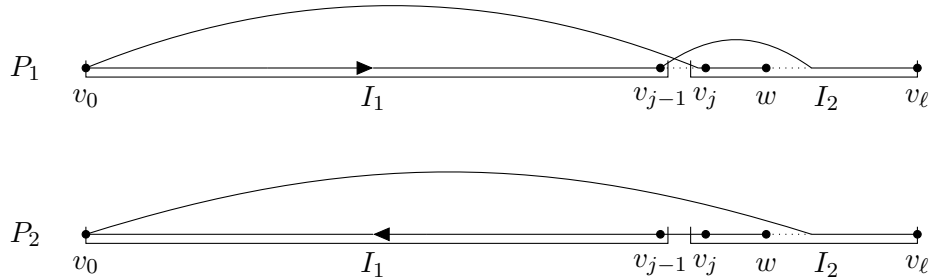

Figure 2: Obtaining the same endpoint in two different ways.

an interval $I = I_1$ as claimed (see Figure 2). Assume for the contrary that $I_2 \cap X^- \cap Y^- = \emptyset$. Then since $I_2 \cap X^-$ and $I_2 \cap Y^-$ are both subsets of $I_2 \cap S$, we see that

$$
\begin{aligned}
|X| + |Y| &= |I_1 \cap X| + |I_2 \cap X| + |I_1 \cap Y| + |I_2 \cap Y| \\
&\leq |I_1 \cap X^-| + |I_2 \cap X^-| + |I_1 \cap Y^+| + |I_2 \cap Y^-| + 4 \\
&\leq 2 \cdot |I_1 \cap S| + |I_2 \cap S| + 4,
\end{aligned}
$$

where we used the fact that $I_1 \cap X^-$, $I_1 \cap Y^+$, $I_2 \cap X^-$, and $I_2 \cap Y^-$ are all subsets of S, and that $I_2 \cap X^-$ and $I_2 \cap Y^-$ are disjoint. Since $|I_1 \cap S| + |I_2 \cap S| = |S|$, we see by (1) that

$$
|X| + |Y| \leq |I_1 \cap S| + |S| + 4 \leq (1 + 40\sqrt{\mu} - \beta)n + 5.
$$

which is a contradiction, since

$$
\min\{|X|, |Y|\} \geq \delta(G) - 14\sqrt{\mu}n - 2 \geq \left(\frac{1}{2} - \alpha - 14\sqrt{\mu}\right)n - 2,
$$

and $\beta = 70\sqrt{\mu} + 3\alpha$.

Hence we proved the existence of a vertex $w \in V(P)$, an interval $I_1 \subseteq V(P)$, and two smooth paths P_1 and P_2 of length ℓ between w and v_ℓ, where P_1 traverses the interval I_1 positively and P_2 traverses it negatively. Note that if a vertex $v_i \in N(w) \cap I_1$ for $0 < i < j - 1$ is used as a pivot point in P_1, then we obtain v_{i-1} as a new endpoint, and if used in P_2, then we obtain v_{i+1} as a new endpoint. Since $0 < i < j - 1$, then the edge incident to the new endpoint belongs to the original path P for both cases.

Since P cannot be extended in at most two rotations, by Lemma 2.5, there exists a set $Z \subseteq N(w)$ of size at least $|Z| \geq |N(w)| - 14\sqrt{\mu}n$ that can be used as pivot points of P_1. Then $Z_1 = Z \cap I_1$ satisfies

$$
|Z_1| = |Z \cap I_1| \geq |N(w) \cap I_1| - |N(w) \setminus Z| \geq |N(w) \cap I_1| - 14\sqrt{\mu}n.
$$

Note that vertices in Z_1 can be used as pivot points of P_1. Similarly there exists a set $Z_2 \subseteq N(w) \cap I_1$ of size at least $|N(w) \cap I_1| - 14\sqrt{\mu}n$ that can be used as pivot points of P_2. Therefore, the set $Z = Z_1 \cap Z_2$ can be used as pivot points for both paths, and has size

$$
|Z| \geq |N(w) \cap I_1| - 28\sqrt{\mu}n.
$$

Moreover, the observation from the previous paragraph shows that $Z \setminus \{v_0, v_{j-1}\} \subseteq S^- \cap S^+$. Therefore

$$
|S^- \cap S^+| \geq |Z| - 2 \geq |N(w) \cap I_1| - 28\sqrt{\mu}n - 2.
$$

Let $Z_w \subseteq N(w) \cap V(P)$ be the set obtained by applying Lemma 2.5 to the path P_2. Since $Z_w \subseteq N(w)$ and $|Z_w| \geq (\frac{1}{2} - \alpha)n - 14\sqrt{\mu}n$,

$$
\begin{aligned}
|S^- \cap S^+| &\geq |Z_w \cap I_1| - 28\sqrt{\mu}n - 2 = |Z_w| - |Z_w \cap I_2| - 28\sqrt{\mu}n - 2 \\
&\geq \left(\frac{1}{2} - \alpha\right)n - 14\sqrt{\mu}n - |Z_w \cap I_2| - 28\sqrt{\mu}n - 2.
\end{aligned}
$$

Since all vertices in $(Z_w \cap I_2) \setminus \{v_j, w^+, w^{++}\}$ can be used as pivot points for path P_2 to give new endpoints in $I_2 \cap S$, we see that $|Z_w \cap I_2| \leq |I_2 \cap S| + 3$. Therefore,

$$
|S^- \cap S^+| \geq \left(\frac{1}{2} - \alpha\right)n - |I_2 \cap S| - 42\sqrt{\mu}n - 5.
$$

Since

$$|I_2 \cap S| \leq |I_2 \cap X^-| + |S \setminus X^-| = (\beta n - 1) + (|S| - |X^-|)$$
$$\leq \left(\frac{1}{2} + 20\sqrt{\mu} + \beta\right)n - \left(\frac{1}{2} - \alpha - 14\sqrt{\mu}\right)n \leq (\alpha + \beta + 34\sqrt{\mu})n,$$

we see that

$$|S^- \cap S^+| \geq \left(\frac{1}{2} - 2\alpha - \beta - 77\sqrt{\mu}\right)n.$$

From this, since $\beta = 70\sqrt{\mu} + 3\alpha$, we obtain

$$|S^- \cup S^+| = |S^-| + |S^+| - |S^- \cap S^+| \leq 2\left(\frac{1}{2} + 20\sqrt{\mu}\right)n - \left(\frac{1}{2} - 2\alpha - \beta - 77\sqrt{\mu}\right)n$$
$$= \left(\frac{1}{2} + 5\alpha + 187\sqrt{\mu}\right)n \leq \left(\frac{1}{2} + 200\sqrt{\mu}\right)n.$$

$\square$

By using Lemma 3.3, we can now prove Theorem 3.1.

Proof of Theorem 3.1. Remove from G all the edges whose two endpoints are $\sqrt{\mu}$-correlated. By Proposition 2.4, we know that the resulting graph G' has minimum degree at least $\left(\frac{1}{2} - \sqrt{\mu}\right)n$. If there exist two sets A and B of sizes $|A|, |B| \geq (\frac{1}{2} - 200\sqrt{\mu})n$ such that $e_{G'}(A, B) \leq 15\sqrt{\mu}n^2$, then we have $e_G(A, B) \leq 16\sqrt{\mu}n^2$, and thus alternative (ii) holds. Hence we may assume that there are no such two sets A and B. From now on, we will only consider the graph G'. Thus by abusing notation, we let G be the graph G'. Note that Lemma 3.3 (i) applies to our graph for all μn-bounded incompatibility systems.

Let $C = (v_0, \ldots, v_\ell, v_0)$ be a maximum length compatible cycle in G. Throughout the proof, for a set $X \subset V$, define $X^+ = \{v_{i+1} : v_i \in X \cap V(C)\}$, and $X^- = \{v_{i-1} : v_i \in X \cap V(C)\}$ (where index addition and subtraction are modulo $\ell + 1$). By Lemma 3.3, we have $|C| \geq (\frac{1}{2} + 20\sqrt{\mu})n$. If C is a Hamilton cycle, then we are done. Otherwise, there exists a vertex z not in the cycle.

Define B_0 as the set of vertices that are $2\sqrt{\mu}$-bad for C. By Proposition 2.4, we know that $|B_0| \leq \sqrt{\mu}n$. Hence if $|C| < (1 - \sqrt{\mu})n$, then we may take z to be a vertex not in B_0. In this case, define B_1 as the set of bad neighbors of z in C. By definition, we have $|B_1| \leq 2\sqrt{\mu}|C|$. Otherwise if $|C| \geq (1 - \sqrt{\mu})n$, then let z be an arbitrary vertex not in C, and define $B_1 = \emptyset$. Since

$$|N(z) \cap V(C)| \geq d(z) + |V(C)| - n \geq \left(\frac{1}{2} - \sqrt{\mu}\right)n + \left(\frac{1}{2} + 20\sqrt{\mu}\right)n - n \geq 7\sqrt{\mu}n$$

and

$$3|B_0| + |B_1| \leq 5\sqrt{\mu}n,$$

the set $T = \{v_i \in V(C) \,|\, v_i \in N(z), v_{i-1}, v_{i+1} \notin B_0, v_i \notin B_0 \cup B_1\}$ has cardinality

$$|T| \geq 2\sqrt{\mu}n.$$

Take a vertex $v_i \in T$. Since $\mathcal{F}$ is μn-bounded, there are at most μn vertices $x \in T$ for which the edges $\{z, v_i\}$ and $\{z, x\}$ are incompatible. Also, by Proposition 2.4, there are at most $\sqrt{\mu}n$ vertices $x \in T$ for which x^+ is $\sqrt{\mu}$-uncorrelated with v_{i+1}. Therefore, since $|T| \geq 2\sqrt{\mu}n$, we can find a vertex

$v_j \in T$ for which the edges $\{z, v_i\}$ and $\{z, v_j\}$ are compatible, the pair of vertices v_{i+1} and v_{j+1} is $\sqrt{\mu}$-uncorrelated, and $v_j \neq v_i, v_{i-1}, v_{i+1}$. Consider the path $P = (v_{i+1}, \ldots, v_j, z, v_i, v_{i-1}, \ldots, v_{j+1})$. First, the two endpoints of P are $\sqrt{\mu}$-uncorrelated by the choice of v_i and v_j, and second, both v_{i+1} and v_{j+1} have at most $2\sqrt{\mu}|C| + 3$ bad neighbors in P, since $v_{i+1}, v_{j+1} \notin B_0$ and the set of bad neighbors in P and in C can differ only in at most three vertices v_j, z, and v_i.

To check whether P is compatible, it suffices to check the compatibility of the three pairs $\Big(\{v_{j-1}, v_j\}, \{v_j, z\}\Big)$, $\Big(\{v_j, z\}, \{z, v_i\}\Big)$, and $\Big(\{z, v_i\}, \{v_i, v_{i-1}\}\Big)$. The pair of edges $\{v_j, z\}$ and $\{z, v_i\}$ is compatible by our choice of v_i and v_j. If $|C| < (1 - \sqrt{\mu})n$, then by the choice of z and the set B_1, since $v_i, v_j \notin B_1$ (this follows from $v_i, v_j \in T$), we further see that the two other pairs are both compatible. Hence P is compatible, and therefore smooth. By Lemma 3.3, this gives a compatible cycle longer than C, contradicting the maximality of C.

Therefore, we must have $|C| \geq (1 - \sqrt{\mu})n$. In this case, P is 'almost' smoothly compatible with $\mathcal{F}$, in the sense that it satisfies all the conditions except for possibly the compatibility of two pairs of edges. Define $\mathcal{F}_1$ as the incompatibility system obtained from $\mathcal{F}$ by making the pairs of edges $\Big(\{v_{j-1}, v_j\}, \{v_j, z\}\Big)$ and $\Big(\{z, v_i\}, \{v_i, v_{i-1}\}\Big)$ to be compatible. Note that P is smoothly compatible with $\mathcal{F}_1$. Hence by Lemma 3.3, we can find a cycle C_1 compatible with $\mathcal{F}_1$, with $V(C_1) \supseteq V(P) \supseteq V(C)$ and

$$|E(C_1) \setminus E(C)| \leq |E(C_1) \setminus E(P)| + 2 \leq 3|V(C_1) \setminus V(C)| + 6.$$

Let P_1 be the path obtained from C_1 by removing the edge $\{v_{j-1}, v_j\}$ if it is in C_1 (if it is not in P_1, then skip the rest of this paragraph). We claim that P_1 is smoothly compatible with $\mathcal{F}_1$. First, it is compatible with $\mathcal{F}_1$, since C_1 is. Second, the two endpoints are $\sqrt{\mu}$-uncorrelated, since we started by removing all edges whose two endpoints are $\sqrt{\mu}$-correlated. Third, since $v_j \notin B_0$, we know that v_j is $2\sqrt{\mu}$-good for C. Since $V(C_1) \supseteq V(C)$, it follows that v_j has at most

$$2\sqrt{\mu}|C| + 2|E(C_1) \setminus E(C)| \leq 2\sqrt{\mu}|C| + 6(|C_1| - |C|) + 12 \leq 8\sqrt{\mu}|C_1| = 8\sqrt{\mu}|P_1|$$

bad neighbors in P_1, where the final inequality follows from $|C| \geq (1 - \sqrt{\mu})n$. A similar estimate holds for the other endpoint v_{j-1}. Let $\mathcal{F}_2$ be the incompatibility system obtained from $\mathcal{F}$ by making only the pair $\Big(\{z, v_i\}, \{v_i, v_{i-1}\}\Big)$ to be compatible. Note that P_1 is smoothly compatible with $\mathcal{F}_2$ as well, since it does not contain the edge $\{v_{j-1}, v_j\}$. Hence we can find, by Lemma 3.3, a cycle C_2 compatible with $\mathcal{F}_2$.

Repeat the argument above for C_2 and $\mathcal{F}_2$, to find a path P_2 that is smoothly compatible with $\mathcal{F}_2$, not containing the edge $\{v_i, v_{i-1}\}$. This path is smoothly compatible with $\mathcal{F}$, and thus by Lemma 3.3, we can find a cycle compatible with $\mathcal{F}$ whose vertex set contains $V(C) \cup \{z\}$, contradicting the maximality of C. Therefore, the given graph contains a Hamilton cycle compatible with $\mathcal{F}$. $\quad\square$

3.2 Step II : Theorem 3.2

In this subsection, we consider the case when G contains two large subsets A and B with few edges between them. We first show that in this case A and B are either almost disjoint, or almost identical. Afterwards, for each case, we further process the graph to convert the problem into a problem of establishing 'compatible Hamilton connectivity' of almost complete graphs, and a problem of establishing 'compatible Hamiltonicity' of almost bipartite graphs. The proof of these final pieces are very similar in structure to the proof of Theorem 3.1, but are different in detail, and will be given in the following section.

Suppose that G is an n-vertex graph of minimum degree at least $\frac{n}{2}$, and let $V = V(G)$. Suppose that there exist two sets A and B of sizes $|A|, |B| \geq (\frac{1}{2} - \nu)n$ such that $e(A, B) \leq \eta n^2$. We have

$$
\begin{aligned}
e(A, B) &\geq e(A \cap B, A \cup B) \\
&\geq |A \cap B| \cdot (\delta(G) - (n - |A \cup B|)) \\
&= |A \cap B| \cdot (\delta(G) + |A| + |B| - |A \cap B| - n).
\end{aligned}
$$

The cardinalities of A, B, the bound $e(A, B) \leq \eta n^2$, and the bound $\delta(G) \geq \frac{n}{2}$ imply that

$$
|A \cap B| \cdot \left(\frac{n}{2} - 2\nu n - |A \cap B| \right) \leq \eta n^2.
$$

If $3\eta n \leq |A \cap B| \leq (\frac{1}{2} - 2\nu - 3\eta)n$, then the left-hand-side above is greater than ηn^2, since $3\eta \cdot (\frac{1}{2} - 2\nu - 3\eta) > \eta$. Hence, we must have

$$
|A \cap B| < 3\eta n \quad \text{or} \quad |A \cap B| > \left(\frac{1}{2} - 2\nu - 3\eta \right)n.
$$

We consider the two cases separately.

Case 1. $|A \cap B| < 3\eta n$.

If $|A \cap B| < 3\eta n$, then there exist disjoint sets $A' \subseteq A$ and $B' \subseteq B$ satisfying

$$
|A'|, |B'| \geq \left(\frac{1}{2} - \nu \right)n - \frac{|A \cap B|}{2} \geq \left(\frac{1}{2} - \nu - \frac{3}{2}\eta \right)n,
$$

and $e(A', B') \leq e(A, B) \leq \eta n^2$. Let $A'' \cup B''$ be a random partition obtained by adding the vertices not in $A' \cup B'$ either to A' or to B' uniformly and independently at random. Then by linearity of expectation,

$$
\begin{aligned}
\mathbb{E}[e(A'', B'')] &\leq e(A', B') + \frac{1}{2}e(V \setminus (A' \cup B'), V) \\
&\leq \eta n^2 + \frac{1}{2} \cdot (2\nu + 3\eta)n \cdot n = \left(\nu + \frac{5}{2}\eta \right)n^2.
\end{aligned}
$$

Hence there exists a particular choice of partition $A'' \cup B''$ of V for which $e(A'', B'') \leq \left(\nu + \frac{5}{2}\eta \right)n^2$ holds. Starting from the partition $A'' \cup B''$, repeatedly move vertices that have at most $\frac{n}{6}$ neighbors in their own part to the other part. Since G has minimum degree at least $\frac{n}{2}$, such vertex has at least $\frac{n}{3}$ neighbors in the other part prior to moving, and thus each time we move a vertex, the number of edges across the partition decreases by at least $\frac{n}{6}$. Hence the process ends in at most $(6\nu + 15\eta)n$ steps, producing a partition $W \cup W^c$. In this partition, both parts have size between $(\frac{1}{2} - 7\nu - \frac{33}{2}\eta)n \geq \frac{n}{3}$ and $(\frac{1}{2} + 7\nu + \frac{33}{2}\eta)n \leq \frac{2n}{3}$, and we have $e(W, W^c) \leq e(A'', B'') \leq (\nu + \frac{5}{2}\eta)n^2$. Moreover, the minimum degrees of $G[W]$ and $G[W^c]$ are both at least $\frac{n}{6} \geq \frac{1}{5}\max\{|W|, |W^c|\}$, since $\max\{|W|, |W^c|\} \leq \frac{2}{3}n$.

Without loss of generality, assume that $|W| \leq \frac{n}{2}$. Note that $e(W)$ is at least

$$
\frac{1}{2}\left(|W| \cdot \frac{n}{2} - e(W, W^c) \right) \geq \binom{|W|}{2} - \left(\frac{1}{2}\nu + \frac{5}{4}\eta \right)n^2 \geq \binom{|W|}{2} - \left(\frac{9}{2}\nu + \frac{45}{4}\eta \right)|W|^2,
$$

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where the last inequality follows from the bound $|W| \geq \frac{n}{3}$. Also, $e(W^c)$ is at least

$$\frac{1}{2}\left(|W^c| \cdot \frac{n}{2} - e(W, W^c)\right) \geq \binom{|W^c|}{2} - \frac{1}{2}|W^c|\left(|W^c| - \frac{n}{2}\right) - \left(\frac{1}{2}\nu + \frac{5}{4}\eta\right)n^2$$

$$\geq \binom{|W^c|}{2} - \left(\frac{7}{2}\nu + \frac{33}{4}\eta\right)n|W^c| - \left(\frac{1}{2}\nu + \frac{5}{4}\eta\right)n^2$$

$$\geq \binom{|W^c|}{2} - \left(9\nu + 22\eta\right)|W^c|^2,$$

where the last inequality follows from the bound $|W^c| \geq \frac{n}{2}$. Therefore, in the end, we obtain a partition $W \cup W^c$ with the following properties:

- $\frac{n}{3} \leq |W| \leq \frac{n}{2}$,
- For $X = W$ and W^c, the graph $G[X]$ has minimum degree at least $\frac{|X|}{5}$, and
- For $X = W$ and W^c, we have $e(X) \geq \binom{|X|}{2} - \left(9\nu + 22\eta\right)|X|^2$.

We find a Hamilton cycle compatible with $\mathcal{F}$ by first finding two vertex disjoint edges $e_1 = \{x_1, y_1\}$ and $e_2 = \{x_2, y_2\}$ such that $x_1, x_2 \in W$ and $y_1, y_2 \in W^c$, and then finding a Hamilton path in $G[W]$ whose two endpoints are x_1 and x_2, and a Hamilton path in $G[W^c]$ whose two endpoints are y_1 and y_2; of course we need to ensure the compatibility of the so obtained cycle.

To find two vertex disjoint edges, if $|W| < \frac{n}{2}$, the minimum degree condition of G implies that each vertex in W has at least $\lceil \frac{n}{2} \rceil - (|W| - 1) \geq 2$ neighbors in W^c. Thus in this case, we can easily find two vertex disjoint edges. Otherwise, if $|W| = |W^c| = \frac{n}{2}$, then the bipartite graph induced by the partition $W \cup W^c$ has minimum degree at least 1, which for $|W| \geq 2$ implies that the minimum vertex cover is of size at least 2. Therefore we can find two vertex disjoint edges in this case as well. Let $e_1 = \{x_1, y_1\}$ and $e_2 = \{x_2, y_2\}$ be the vertex disjoint edges that we have found, where $x_1, x_2 \in W$ and $y_1, y_2 \in W^c$.

Let $G_1 = G[W]$, and add the edge $\{x_1, x_2\}$ if it is not already in G_1. Define an incompatibility system $\mathcal{F}_1$ over G_1 as follows. For two edges $e_1, e_2 \in E(G_1)$ both different from $\{x_1, x_2\}$, let e_1 and e_2 be incompatible in $\mathcal{F}_1$ if and only if they are incompatible in $\mathcal{F}$. For an edge $e \neq \{x_1, x_2\}$ containing x_1, the edges $\{x_1, x_2\}$ and e are incompatible in $\mathcal{F}_1$ if and only if $\{x_1, y_1\}$ and e are incompatible in $\mathcal{F}$. For an edge $e \neq \{x_1, x_2\}$ containing x_2, the edges $\{x_1, x_2\}$ and e are incompatible in $\mathcal{F}_1$ if and only if $\{x_2, y_2\}$ and e are incompatible in $\mathcal{F}$. One can easily check that $\mathcal{F}_1$ is a μn-bounded incompatibility system. Similarly let $G_2 = G[W^c]$ (with the edge $\{y_1, y_2\}$ added) and define a μn-bounded incompatibility system $\mathcal{F}_2$ over G_2.

Suppose that we find a Hamilton cycle C_1 in G_1 containing $\{x_1, x_2\}$ and compatible with $\mathcal{F}_1$, and C_2 in G_2 containing $\{y_1, y_2\}$ and compatible with $\mathcal{F}_2$. The two cycles C_1, C_2 together with the two edges e_1 and e_2 give a Hamilton cycle C in G that is compatible with $\mathcal{F}$, due to the way we defined the incompatibility systems $\mathcal{F}_1$ and $\mathcal{F}_2$. Therefore, it suffices to prove the following theorem (we will apply it with $\beta_{3.4} = 9\nu + 22\eta$ and $\mu_{3.4} = 3\mu$, where the subscripts indicate that the constants will be applied to Theorem 3.4. The factor of 3 in $\mu_{3.4}$ has been introduced since W can be as small as $\frac{n}{3}$). Its proof is similar to that of Theorem 3.1 (and is in fact much simpler), and will be given separately in the following section.

Theorem 3.4. *Let β and μ be positive reals satisfying $\beta + 2\sqrt{\mu} \leq \frac{1}{1200}$. Let G be an n-vertex graph with minimum degree at least $\frac{n}{5}$ and at least $\binom{n}{2} - \beta n^2$ edges, and $\mathcal{F}$ be a μn-bounded incompatibility*

system over G. Then for every edge e of G, there exists a Hamilton cycle containing e that is compatible with $\mathcal{F}$.

Case 2. $|A \cap B| \geq \left(\frac{1}{2} - 2\nu - 3\eta\right)n.$

Let $A' = A \cap B$ and $B' = V \setminus A'$. Note that $e(A', A') \leq e(A, B) \leq \eta n^2$. Therefore,

$$e(A', B') \geq \delta(G) \cdot |A'| - e(A', A') \geq \frac{n}{2} \cdot \left(\frac{1}{2} - 2\nu - 3\eta\right)n - \eta n^2 = \left(\frac{1}{4} - \nu - \frac{5}{2}\eta\right)n^2.$$

Repeatedly move vertices having at most $\frac{n}{6}$ neighbors across the partition to the other part. Since G has minimum degree at least $\frac{n}{2}$, such vertex has at least $\frac{n}{3}$ neighbors in its own part prior to moving. Hence each time we move a vertex, the number of edges across the partition increases by at least $\frac{n}{6}$. Since the maximum possible number of edges across a partition is $\frac{1}{4}n^2$, the process ends in at most $(6\nu + 15\eta)n$ steps, producing a partition $W \cup W^c$. Both parts have size between $|A \cap B| - (6\nu + 15\eta)n \geq (\frac{1}{2} - 8\nu - 18\eta)n$ and $(\frac{1}{2} + 8\nu + 18\eta)n$, and satisfy $e(W, W^c) \geq e(A', B') \geq \left(\frac{1}{4} - \nu - \frac{5}{2}\eta\right)n^2$. Moreover, each vertex has at least $\frac{n}{6}$ neighbors across the partition. Without loss of generality, assume that $|W| \geq \frac{n}{2}$. Then, while the bound $|W| > \lceil\frac{n}{2}\rceil$ holds, repeatedly move vertices $w \in W$ having at least $\frac{n}{16}$ neighbors in W, to the other part. Note that we move at most $(8\nu + 18\eta)n$ vertices during this process. In the end, we obtain a partition with the following properties.

- the bipartite graph induced by G on $W \cup W^c$ has minimum degree at least $\left(\frac{1}{16} - 8\nu - 18\eta\right)n$,
- $e(W, W^c) \geq \left(\frac{1}{4} - \nu - \frac{5}{2}\eta\right)n^2 - \left(8\nu + 18\eta\right)n^2 \geq \left(\frac{1}{4} - 9\nu - 21\eta\right)n^2$,
- $\lceil\frac{n}{2}\rceil \leq |W| \leq \left(\frac{1}{2} + 8\nu + 18\eta\right)n$, and
- if $|W| > \lceil\frac{n}{2}\rceil$, then $G[W]$ has maximum degree less than $\frac{n}{16}$.

Let $|W| = \frac{n+t}{2}$ and $|W^c| = \frac{n-t}{2}$, for a non-negative integer $t \leq (16\nu + 36\eta)n$. The Hamilton cycle that we find will make use of exactly t edges within the set W, and all other edges will be between W and W^c. We thus must first find a set E_0 of t edges within W. If $t = 0$, then let $E_0 = \emptyset$. Otherwise, since G has minimum degree at least $\frac{n}{2}$, we see that each vertex $w \in W$ has at least $\frac{t}{2}$ neighbors in W. Therefore, $e(W) \geq \frac{1}{2} \cdot \frac{t}{2}|W| \geq \frac{nt}{8}$. If $t = 1$, then let e_1 be an arbitrary edge whose both endpoints are in W. Otherwise, if $t > 1$, then since $G[W]$ has maximum degree less than $\frac{n}{16}$, it has covering number greater than $2t$, and therefore contains t disjoint edges $e_1, \ldots, e_t$. Let $E_0 = \{e_1, \ldots, e_t\}$.

Let $W_1 \subseteq W$ be a set of size $|W_1| = \frac{n-t}{2}$ that intersects each edge $e_i \in E_0$ in exactly one vertex, and let $W_2 = W^c$. Note that $|W_1| = |W_2|$. Our next step towards establishing Hamiltonicity is to find a perfect matching between W_1 and W_2. This perfect matching will later play an important role in finding a Hamilton cycle. Consider a bipartite subgraph H of G obtained by the following process. First, take only the edges between the two sets W_1 and W_2. Then, for each $e_i \in E_0$ and its endpoint $v_i \in W_1$, remove all edges incident to v_i that are incompatible with e_i. We claim that H satisfies Hall's condition. Note that H has minimum degree at least

$$\delta(H) \geq \left(\frac{1}{16} - 8\nu - 18\eta\right)n - t - \mu n \geq \left(\frac{1}{16} - 24\nu - 54\eta - \mu\right)n \geq \frac{n}{20}.$$

Furthermore, since the complement of G has at most $(9\nu + 21\eta)n^2$ edges between W and W^c and H is obtained from an induced subgraph of G by removing at most $\mu n|W_1| \leq \mu n^2$ edges, the complement of H has at most $(9\nu + 21\eta)n^2 + \mu n^2$ edges between W_1 and W_2. Therefore,

$$e(H) \geq |W_1||W_2| - (9\nu + 21\eta)n^2 - \mu n^2 > |W_1||W_2| - \frac{1}{400}n^2. \tag{2}$$

By the minimum degree condition of H, it suffices to consider the expansion of sets $W_1' \subseteq W_1$ of size

$$\frac{n}{20} \leq |W_1'| \leq \frac{n-t}{2} - \frac{n}{20}.$$

If a set W_1' does not expand, then there exists a set W_2' of size $|W_2'| \geq \frac{n-t}{2} - |W_1'|$, where

$$e_H(W_1', W_2') = 0.$$

However, this implies that $e(H) \leq |W_1||W_2| - \frac{1}{400}n^2$, contradicting the lower bound (2) on $e(H)$. Hence, H satisfies Hall's condition, and thus contains a perfect matching.

The perfect matching of H together with the t edges of E_0 gives t paths (x_i, v_i, w_i) of length 2, and $\frac{n-3t}{2}$ edges (v_j, w_j), all vertex-disjoint, thus covering all vertices of the graph G exactly once. Moreover, each path of length 2 consists of a pair of edges that are compatible. Consider the graph H' and incompatibility system $\mathcal{F}'$ obtained from G and $\mathcal{F}$ by the following process. First, consider only the edges between W and W^c. Then, for each index i with $1 \leq i \leq t$, remove the vertex v_i, add the edge $\{x_i, w_i\}$ (if it was not an edge in G), and make $\{x_i, w_i\}$ incompatible with the edges in G that are incident to x_i and incompatible with $\{x_i, v_i\}$, and are incident to w_i and incompatible with $\{v_i, w_i\}$. One can easily check that H' is a balanced bipartite graph with bipartition $\{x_1, \ldots, x_t, v_{t+1}, \ldots, v_m\} \cup \{w_1, \ldots, w_m\}$, where $m = \frac{n-t}{2}$, and that $\mathcal{F}'$ is a μn-bounded incompatibility system. Moreover, H' has minimum degree at least

$$\delta(H') \geq \left(\frac{1}{16} - 8\nu - 18\eta\right)n - t \geq \frac{n}{20},$$

and its number of edges is at least

$$e(H') \geq \left(\frac{n-t}{2}\right)^2 - \left(9\nu + 21\eta\right)n^2 \geq \left(1 - (45\nu + 105\eta)\right)\left(\frac{n-t}{2}\right)^2.$$

Suppose that we find a Hamilton cycle C' in H' containing the edges $\{x_1, w_1\}, \ldots, \{x_t, w_t\}$, that is compatible with $\mathcal{F}'$. Consider the cycle C obtained from C' by replacing each edge $\{x_i, w_i\}$ with the path (x_i, v_i, w_i). Note that C is a Hamilton cycle in G, and is compatible with $\mathcal{F}$ by our definition of $\mathcal{F}'$. Therefore, this case can be settled through the following theorem, whose proof will be given in the following section (we will apply it with $\gamma_{3.5} = 64\nu + 144\eta$, $\beta_{3.5} = 45\nu + 105\eta$, and $\mu_{3.5} = 3\mu$, where the subscripts indicate that the constants are being applied to Theorem 3.5. The value of $\mu_{3.5}$ is chosen as above to ensure that $\mu_{3.5}m = \mu_{3.5}\frac{n-t}{2} \geq \mu n$).

Theorem 3.5. *Let μ, β, γ be positive reals satisfying $\gamma + \beta + \sqrt{\mu} < \frac{1}{2000}$. Let G be a bipartite graph with bipartition $A \cup B$ and minimum degree at least $\frac{m}{10}$, where $|A| = |B| = m$ and $e(A, B) \geq (1 - \beta)m^2$. Let $\mathcal{F}$ be a μm-incompatibility system over G. Further suppose that there is a perfect matching consisting of edges $e_1, \ldots, e_m$. Then there exists a Hamilton cycle containing the edges $e_1, \ldots, e_{\gamma m}$, that is compatible with $\mathcal{F}$.*

4   Extremal cases

In this section, we prove Theorems 3.4 and 3.5, thereby completing the proof of Theorem 3.2, and thus Theorem 1.3. Both proofs are very similar to the proof of Theorem 3.1 in structure.

4.1 Almost complete graph

In this subsection, we prove Theorem 3.4. The first step is to prove the following lemma, which can be seen as an alternative version of Lemma 3.3. We intentionally impose a slightly weaker minimum degree condition of $\frac{n}{6}$ compared to that of Theorem 3.4 with later usage in mind.

Lemma 4.1. *Suppose that β and μ are reals satisfying $\beta + \sqrt{\mu} < \frac{1}{1200}$. Let G be an n-vertex graph with minimum degree at least $\frac{n}{6}$ and at least $\binom{n}{2} - \beta n^2$ edges, and $\mathcal{F}$ be a μn-bounded incompatibility system over G. Then for every edge e of G and smooth path P that contains e, there exists a cycle C compatible with $\mathcal{F}$ with the following properties:*

- *C contains e,*
- *C has length $|C| \geq (\frac{6}{7} - 14\sqrt{\mu})n$,*
- *$V(C) \supseteq V(P)$, and*
- *$|E(C) \setminus E(P)| \leq 2|V(C) \setminus V(P)| + 3$.*

Proof. Let G be a given graph, and e be an edge of G. Let P be a smooth path in G that contains e. It suffices to prove that either there exists a smooth path P' containing e with $|P'| \geq |P| + 1$ and $|E(P') \setminus E(P)| \leq 2$, or a compatible cycle C containing e with $|C| \geq (\frac{6}{7} - 14\sqrt{\mu})n$ and $|E(C) \setminus E(P)| \leq 3$. Since then, we can repeatedly find a longer path to eventually find a cycle with the claimed properties. Assume that the former event does not occur.

Let L be the set of vertices that have degree at least $\frac{6}{7}n$ in G, and note that

$$\frac{1}{2} \cdot \frac{1}{7}n \cdot |V \setminus L| \leq e(G^c) \leq \beta n^2.$$

Hence $|L| \geq (1 - 14\beta)n$. By Lemma 2.5, there exists a set $X \subset N(v_0) \cap V(P)$ of size

$$|X| \geq d(v_0) - 14\sqrt{\mu}n \geq \frac{n}{6} - 14\sqrt{\mu}n > 14\beta n + 1$$

such that for every vertex $v_i \in X$, the path $(v_{i-1}, \ldots, v_0, v_i, v_{i+1}, \ldots, v_\ell)$ is smooth. Thus in particular, we may choose v_i so that $v_{i-1} \in L$ and $\{v_{i-1}, v_i\} \neq e$. Let $P' = (w_0, \ldots, w_\ell)$ be the path obtained in this way. By our assumption on P, the path P' contains e and cannot be extended by adding one edge.

By Lemma 2.5, there exists a set $Y \subset N(w_0) \cap V(P')$ of size $|Y| \geq \frac{6}{7}n - 14\sqrt{\mu}n$ such that for every vertex $w_i \in Y$, the path $(w_{i-1}, \ldots, w_0, w_i, w_{i+1}, \ldots, w_\ell)$ is smooth. Similarly, there exists a set $Z \subset N(w_\ell) \cap V(P')$ of size $|Z| \geq \delta(G) - 14\sqrt{\mu}n$ such that for every vertex $w_j \in Z$, the path $(w_0, w_1, \ldots, w_j, w_\ell, w_{\ell-1}, \ldots, w_{j+1})$ is smooth. In particular, for each vertex $w_j \in Z$, we see that $\{w_\ell, w_j\}$ is compatible with both $\{w_j, w_{j-1}\}$ and $\{w_\ell, w_{\ell-1}\}$. Since

$$|Y| + |Z| \geq \left(\frac{6}{7}n - 14\sqrt{\mu}n\right) + (\delta(G) - 14\sqrt{\mu}n) \geq \frac{43}{42}n - 28\sqrt{\mu}n > n + 2,$$

there exists an index i such that $w_{i-1} \in Z$, $w_i \in Y$, and both vertices w_{i-1}, w_i are not incident to e. For this index, the cycle $C = (w_{i-1}, \ldots, w_0, w_i, w_{i+1}, \ldots, w_\ell, w_{i-1})$ is compatible with $\mathcal{F}$ and contains e. Also, $|C| \geq |Y| \geq \frac{6}{7}n - 14\sqrt{\mu}n$. Moreover, C is obtained from P by adding at most three extra edges. □

We now present the proof of Theorem 3.4, which we restate here for reader's convenience.

Theorem. *Let β and μ be positive reals satisfying $\beta + 2\sqrt{\mu} \leq \frac{1}{1200}$. Let G be an n-vertex graph with minimum degree at least $\frac{n}{5}$ and at least $\binom{n}{2} - \beta n^2$ edges, and $\mathcal{F}$ be a μn-bounded incompatibility system over G. Then for every edge e of G, there exists a Hamilton cycle containing e that is compatible with $\mathcal{F}$.*

Proof. Let G be a given graph, and e be an edge of G. Consider the graph obtained from G by removing all edges whose two endpoints are $\sqrt{\mu}$-correlated (except e). By Proposition 2.4, the resulting graph has minimum degree at least $\left(\frac{1}{5} - \sqrt{\mu}\right)n \geq \frac{n}{6}$, and has at least $\binom{n}{2} - (\beta + \sqrt{\mu})n^2$ edges. By abusing notation, we use G to denote this graph. Note that Lemma 4.1 can be applied to this graph since $(\beta + \sqrt{\mu}) + \sqrt{\mu} \leq \frac{1}{1200}$.

Let $C = (v_0, v_1, \ldots, v_\ell)$ be a cycle in G containing e and compatible with $\mathcal{F}$, of maximum length. By Lemma 4.1, we have $|C| \geq \left(\frac{6}{7} - 14\sqrt{\mu}\right)n$. Throughout the proof, for a set $X \subset V$, define $X^+ = \{v_{i+1} : v_i \in X \cap V(C)\}$, and $X^- = \{v_{i-1} : v_i \in X \cap V(C)\}$ (where index addition and subtraction are modulo $\ell + 1$). If C is a Hamilton cycle, then we are done. Otherwise, there exists a vertex z not in the cycle.

Define B_0 as the set of vertices that are $2\sqrt{\mu}$-bad for C. By Proposition 2.4, we know that $|B_0| \leq \sqrt{\mu}n$. Hence if $|C| < (1 - \sqrt{\mu})n$, then we may take z to be a vertex not in B_0. In this case, define B_1 as the set of bad neighbors of z in C. By definition, we have $|B_1| \leq 2\sqrt{\mu}|C|$. Otherwise if $|C| \geq (1 - \sqrt{\mu})n$, then let z be an arbitrary vertex not in C, and define $B_1 = \emptyset$.

Since

$$|N(z) \cap V(C)| \geq d(z) + |V(C)| - n \geq \left(\frac{n}{5} - \sqrt{\mu}n\right) + \left(\frac{6}{7}n - 14\sqrt{\mu}n\right) - n \geq 7\sqrt{\mu}n$$

and

$$|B_0| + |B_0^+| + |B_0^-| + |B_1| \leq 5\sqrt{\mu}n,$$

the set $T = \{v_i \in V(C) \,|\, v_i \in N(z), v_{i-1}, v_{i+1} \notin B_0, v_i \notin B_0 \cup B_1\}$ has cardinality

$$|T| \geq |N(z) \cap V(C)| - (|B_0| + |B_0^+| + |B_0^-| + |B_1|) \geq 2\sqrt{\mu}n.$$

Take a vertex $v_i \in T$ not incident to e. Since $\mathcal{F}$ is μn-bounded, there are at most μn vertices $x \in T$ for which the pair of edges $\{z, v_i\}$ and $\{z, x\}$ is incompatible. Also, by Proposition 2.4, there are at most $\sqrt{\mu}n$ vertices that are $\sqrt{\mu}$-correlated with v_{i+1}. Therefore, since $|T| \geq 2\sqrt{\mu}n > \mu n + \sqrt{\mu}n + 5$, we can find a vertex $v_j \in T$ not incident to e and not v_i, v_{i+1}, or v_{i-1} for which the pair of edges $\{z, v_i\}$ and $\{z, v_j\}$ is compatible, and the pair of vertices v_{i+1} and v_{j+1} is $\sqrt{\mu}$-uncorrelated. Consider the path $P = (v_{i+1}, \ldots, v_j, z, v_i, v_{i-1}, \ldots, v_{j+1})$. First, the two endpoints of P are $\sqrt{\mu}$-uncorrelated by the choice of v_i and v_j. Second, both v_{i+1} and v_{j+1} have at most $2\sqrt{\mu}|C| + 3$ bad neighbors in P, since $v_{i+1}, v_{j+1} \notin B_0$, and the set of bad neighbors in P and in C can differ only in at most three vertices v_j, z, and v_i. Third, P contains e since C does, and v_i, v_j are not incident to e.

To check whether P is compatible, it suffices to check the compatibility of three pairs of edges $\left(\{v_{j-1}, v_j\}, \{v_j, z\}\right)$, $\left(\{v_j, z\}, \{z, v_i\}\right)$, and $\left(\{z, v_i\}, \{v_i, v_{i-1}\}\right)$. The pair of edges $\{v_j, z\}$ and $\{z, v_i\}$ is compatible by our choice of v_i and v_j. If $|C| < (1 - \sqrt{\mu})n$, then by the choice of z and the set B_1, since $v_i, v_j \notin B_1$ (this follows from $v_i, v_j \in T$), we further see that the other two pairs of edges are both compatible, thus implying that P is compatible, and therefore smooth. This by Lemma 4.1 gives a compatible cycle containing e that is longer than C, and contradicts the maximality of C.

Therefore, we must have $|C| \geq (1 - \sqrt{\mu})n$. In this case, P is 'almost' smoothly compatible with $\mathcal{F}$, in the sense that it satisfies all the conditions except for possibly the compatibility of two pairs

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of edges. Define $\mathcal{F}_1$ as the incompatibility system obtained from $\mathcal{F}$ by making the pairs of edges $\Big(\{v_{j-1}, v_j\}, \{v_j, z\}\Big)$ and $\Big(\{z, v_i\}, \{v_i, v_{i-1}\}\Big)$ to be compatible. Note that P is smoothly compatible with $\mathcal{F}_1$. Hence by Lemma 4.1, we can find a cycle C_1 containing e, compatible with $\mathcal{F}_1$, with $V(C_1) \supseteq V(P) \supseteq V(C)$ and

$$|E(C_1) \setminus E(C)| \le |E(C_1) \setminus E(P)| + 2 \le 2|V(C_1) \setminus V(C)| + 5.$$

Let P_1 be the path obtained from C_1 by removing the edge $\{v_{j-1}, v_j\}$ if it is in C_1 (if not, then skip the rest of the paragraph). We claim that P_1 is smoothly compatible with $\mathcal{F}_1$. First, it is compatible with $\mathcal{F}_1$, since C_1 is. Second, the two endpoints are $\sqrt{\mu}$-uncorrelated, since we started by removing all edges whose two endpoints are $\sqrt{\mu}$-correlated. Third, since $v_j \notin B_0$, we know that v_j is $2\sqrt{\mu}$-good for C. Since $V(C_1) \supseteq V(C)$, it follows that v_j has at most

$$2\sqrt{\mu}|C| + 2|E(C_1) \setminus E(C)| \le 2\sqrt{\mu}|C| + 4(|C_1| - |C|) + 10 \le 6\sqrt{\mu}|C_1| = 6\sqrt{\mu}|P_1|$$

bad neighbors in P_1, where the final inequality follows from $|C| \ge (1 - \sqrt{\mu})n$. A similar estimate holds for the other endpoint v_{j-1}. Let $\mathcal{F}_2$ be the incompatibility system obtained from $\mathcal{F}$ by making the pair $\Big(\{z, v_i\}, \{v_i, v_{i-1}\}\Big)$ to be compatible. Note that P_1 is smoothly compatible with $\mathcal{F}_2$ as well, since P_1 is smoothly compatible with $\mathcal{F}_1$ and does not contain the edge $\{v_{j-1}, v_j\}$. Thus by Lemma 4.1, we can find a cycle C_2 containing e that is compatible with $\mathcal{F}_2$, whose vertex set contains $V(P_1)$.

Let P_2 be the path obtained from C_2 by removing the edge $\{v_i, v_{i-1}\}$ if it is in C_2 (if not, then it contradicts the maximality of C). Similarly as before, the path P_2 is smoothly compatible with $\mathcal{F}$, and thus by Lemma 4.1, we can find a cycle whose vertex set contains $V(C) \cup \{z\}$, contradicting the maximality of C. Therefore, the cycle C is a Hamilton cycle. $\qquad\square$

4.2 Almost complete bipartite graph

Let μ, β, γ be positive reals satisfying

$$\gamma + \beta + \sqrt{\mu} < \frac{1}{2000}.$$

Let G be a $2m$-vertex bipartite graph with bipartition $A \cup B$ such that $|A| = |B| = m$, with minimum degree at least $\frac{m}{10}$ and at least $(1 - \beta)m^2$ edges. Let $\mathcal{F}$ be a μm-bounded incompatibility system defined over G. Further suppose that a perfect matching $e_1 = \{a_1, b_1\}, \ldots, e_m = \{a_m, b_m\}$ satisfying $a_i \in A$ and $b_i \in B$ is given. Let f be a bijection between A and B defined by the relation $f(a_i) = b_i$ and $f(b_i) = a_i$ for each $i = 1, \ldots, m$. We will fix these notations throughout the section. All lemmas and results in this subsection are based on these notations.

Definition 4.2. *A path or a cycle H of G is* proper *if it contains all edges $e_1, \ldots, e_{\gamma m}$ and satisfies $f(V(H) \cap A) = V(H) \cap B$.*

We restrict our attention to proper paths and cycles. The condition $f(V(H) \cap A) = V(H) \cap B$ ensures that the two endpoints of the path are in A and in B, respectively. We consider proper paths because it is a convenient way of forcing such property while using the rotation-extension technique. For example, if for a proper path $P = (v_0, v_1, \ldots, v_\ell)$ the vertex v_0 is adjacent to some vertex $x \notin V(P)$, then the path $(f(x), x, v_0, v_1, \ldots, v_\ell)$ forms a proper path that is longer than P. To ensure that the new path is compatible with $\mathcal{F}$, we need the compatibility of the two pairs

$(\{f(x), x)\}, \{x, v_0\})$ and $(\{x, v_0\}, \{v_0, v_1\})$, but it might be the case that there are no neighbors of v_0 giving the compatibility of these pairs. Thus we slightly modify the definition of smooth paths. Define X_A as the set of vertices $x \in A$ for which there are at least $\sqrt{\mu}m$ indices i such that the pair of edges $e_i = \{a_i, b_i\}$ and $\{x, b_i\}$ is incompatible. By counting the number of pairs of edges $\{x, b_i\}$ and $\{a_i, b_i\}$ that are incompatible in two ways, we obtain the inequality

$$|X_A| \cdot \sqrt{\mu}m \le \mu m \cdot m \implies |X_A| \le \sqrt{\mu}m. \tag{3}$$

Similarly define $X_B \subseteq B$, and we get $|X_B| \le \sqrt{\mu}m$. Throughout this section, we will use the following definition of smooth paths.

Definition 4.3. *A proper path $P = (v_0, v_1, \cdots, v_\ell)$ with $v_0 \in A$ and $v_\ell \in B$ is* smoothly compatible with $\mathcal{F}$ *(or* smooth *in short if $\mathcal{F}$ is clear from the context) if*
  (i) P is compatible with $\mathcal{F}$,
  (ii) both endpoints v_0 and v_ℓ are $8\sqrt{\mu}$-good for P,
  (iii) $v_0 \notin X_A$ and $v_\ell \notin X_B$.

Note that we no longer impose the two endpoints to be $\sqrt{\mu}$-uncorrelated. This is because the pair of vertices $v_0 \in A$ and $v_\ell \in B$ always have no common neighbors. Hence all proper paths automatically satisfy the condition that the two endpoints are $\sqrt{\mu}$-uncorrelated.

The following modification of Lemma 2.5 will be used.

Lemma 4.4. *Let G be given as above. Suppose that $P = (v_0, v_1, \ldots, v_\ell)$ is a proper smooth path in G, where there is no vertex $x \notin V(P)$ for which $(f(x), x, v_0, \ldots, v_\ell)$ is a proper smooth path. Then there exists a set $Z \subseteq N(v_0) \cap V(P)$ of good neighbors of v_0 in P of size at least*

$$|Z| \ge d(v_0) - (25\sqrt{\mu} + 2\gamma)m$$

such that for every vertex $v_i \in Z$, the path $(v_{i-1}, \ldots, v_1, v_0, v_i, v_{i+1}, \ldots, v_\ell)$ is a proper smooth path.

Proof. Define

$$
\begin{aligned}
B_0 &= \Big\{b \in B : \{v_0, b\} \text{ and } \{b, f(b)\} \text{ are incompatible}\Big\} \\
&\quad \cup \Big\{b \in B : \{v_\ell, f(b)\} \text{ and } \{b, f(b)\} \text{ are incompatible}\Big\}, \\
B_1 &= \Big\{b \in B : \{v_0, b\} \text{ is incompatible with } \{v_0, v_1\} \text{ or } \{v_0, v_\ell\} \text{ (if exists)}\Big\}, \text{ and} \\
B_2 &= \Big\{a \in A : a \text{ is } 2\sqrt{\mu}\text{-bad for } P\Big\} \cup X_A.
\end{aligned}
$$

We have $|B_0| \le 2\sqrt{\mu}m$ since $v_0 \notin X_A$ and $v_\ell \notin X_B$, $|B_1| \le 2\mu m$ since $\mathcal{F}$ is μm-bounded, and $|B_2| \le 3\sqrt{\mu}m$ by Proposition 2.4 (with $n = 2m$) and (3). Suppose that there exists a neighbor b of v_0 such that $b \notin V(P) \cup B_0 \cup B_1 \cup f(B_2)$, and consider the path $P' = (f(b), b, v_0, v_1, \ldots, v_\ell)$ (note that $f(b) \notin V(P)$ since P is proper). Note that P' is a proper path. We claim that it in fact is a proper smooth path. It is compatible since $b \notin B_0 \cup B_1$. The set of bad neighbors of $f(b)$ in P' is identical to the set of bad neighbors of $f(b)$ in P since $f(b)$ and v_0 are not adjacent. Similarly, the set of bad neighbors of v_ℓ in P' is identical to that in P since $b \notin B_0 \cup B_1$, and v_ℓ is not adjacent to b. Therefore, we see that the two endpoints of P' are $8\sqrt{\mu}$-good. Finally $v_\ell \notin X_B$ since P is a smooth path, and $f(b) \notin X_A$ since $b \notin f(B_2)$. Therefore P' in fact is a proper smooth path, contradicting our assumption.

Hence all neighbors of v_0 are in $V(P) \cup B_0 \cup B_1 \cup f(B_2)$. Further define

$$B_3 \;=\; \Big\{ w \in V(P) \,:\, w \text{ is a bad neighbor of } v_0 \text{ in } P, \text{ or intersects some edge } e_1, e_2, \ldots, e_{\gamma m} \Big\}.$$

We have $|B_3| \le 8\sqrt{\mu}|P| + 2\gamma m$, since v_0 is $8\sqrt{\mu}$-good for P. Define $B_2^+ = \{v_{i+1} \,|\, v_i \in B_2 \cap V(P)\}$, and $Z = \Big(N(v_0) \cap V(P) \Big) \setminus (B_1 \cup B_2^+ \cup B_3)$. Since all neighbors of v_0 are in $V(P) \cup B_0 \cup B_1 \cup f(B_2)$,

$$\begin{aligned}
|Z| &\ge |N(v_0)| - |B_0| - |B_1| - 2|B_2| - |B_3| \\
&\ge |N(v_0)| - 2\sqrt{\mu}m - 2\mu m - 6\sqrt{\mu}m - (8\sqrt{\mu}\cdot 2m + 2\gamma m) \ge d(v_0) - 25\sqrt{\mu}m - 2\gamma m \,.
\end{aligned}$$

One can check as in the proof of Lemma 2.5 that Z satisfies our claim. $\qquad\square$

We start by showing that the class of proper smooth paths is non-empty. This will be achieved in two steps: first proving the existence of a proper path, and then of a proper smooth path.

Proposition 4.5. *Let G be a graph given as above.*

(i) There exists a proper compatible path P with $|V(P)| \le 10\gamma m - 9$.
(ii) There exists a proper smooth path P' with $|V(P')| \le 10\gamma m$.

Proof. Let $Y_A \subseteq A$ be the set of vertices in A of degree less than $\frac{3}{4}m$. Since

$$\frac{m}{4} \cdot |Y_A| \le e_{G^c}(A, B) \le \beta m^2,$$

we see that $|Y_A| \le 4\beta m$. Similarly define $Y_B \subseteq B$, and we get $|Y_B| \le 4\beta m$.

(i) We prove the following statement for $t = 1, \ldots, \gamma m$ using induction on t: there exists a compatible path P_t of length at most $10t - 9$ containing the edges $e_1, \ldots, e_t$ but not intersecting $e_{t+1}, \cdots, e_{\gamma m}$ and satisfying $f(V(P_t) \cap A) = V(P_t) \cap B$. The statement is trivially true for $t = 1$. Suppose that we are given a compatible path P_t as above.

Let $b \in B$ be an endpoint of P_t, and let $\{a, b\}$ be an edge of P_t incident to b. Let $A_1 \subseteq A$ be the neighbors a_i of b with the following properties: (i) $\{b, a_i\}$ is compatible with $\{a, b\}$, (ii) $a_i \notin X_A \cup Y_A$ and $b_i = f(a_i) \notin X_B \cup Y_B$, and (iii) $a_i, b_i \notin V(P_t) \cup \bigcup_{\ell=t+1}^{\gamma m} \{a_\ell, b_\ell\}$. Note that

$$\begin{aligned}
|A_1| &\ge |N(b)| - \mu m - (|X_A| + |X_B|) - (|Y_A| + |Y_B|) - (|V(P_t)| + 2\gamma m) \\
&\ge \frac{1}{10}m - \mu m - 2\sqrt{\mu}m - 8\beta m - 12\gamma m > \frac{1}{11}m.
\end{aligned}$$

Let $B_1 \subseteq B$ be the neighbors b_j of a_{t+1} with the following properties: (i) $\{a_{t+1}, b_j\}$ is compatible with e_{t+1}, (ii) $a_j \notin X_A \cup Y_A$ and $b_j \notin X_B \cup Y_B$, an (iii) $a_j, b_j \notin V(P_t) \cup \bigcup_{\ell=t+1}^{\gamma m} \{a_\ell, b_\ell\}$. A computation similar to above shows that $|B_1| > \frac{1}{11}m$.

Since $e(A, B) \ge m^2 - \beta m^2 > m^2 - (|A_1||B_1| - m)$, there exists an edge $\{b_i, a_j\}$ such that $b_i \in f(A_1), a_j \in f(B_1)$ and $i \ne j$. Our goal is to find two indices k and ℓ for which the path

$$P_{t+1} = (P, b, a_i, b_k, a_k, b_i, a_j, b_\ell, a_\ell, b_j, a_{t+1}, b_{t+1})$$

is compatible. By the definitions of b_i and a_j, it suffices to show the existence of distinct indices k and ℓ for which the path $(b, a_i, b_k, a_k, b_i, a_j)$ and the path $(b_i, a_j, b_\ell, a_\ell, b_j, a_{t+1})$ are both compatible.

The compatibility of the path $(b, a_i, b_k, a_k, b_i, a_j)$ depends only on the index k. We must first have a_i adjacent to b_k and b_i adjacent to a_k, and avoid having a_k, b_k in the set $V(P_t)$ or in $\{a_i, b_i, a_j, b_j\} \cup \bigcup_{\ell=t+1}^{\gamma m} \{a_\ell, b_\ell\}$. Since $a_i \notin Y_A$ and $b_i \notin Y_B$, the number of possible indices satisfying the restriction is at least

$$\frac{3}{4}m + \frac{3}{4}m - m - (|V(P_t)| + 2 + 2\gamma m) \geq \frac{1}{2}m - 12\gamma m - 2.$$

Moreover, since $a_i \notin X_A$ and $b_i \notin X_B$, the compatibility of the pairs of edges further forbid $2\sqrt{\mu}m + 2\mu m$ indices k. Thus we can find an index k for which the path $(b, a_i, b_k, a_k, b_i, a_j)$ is compatible. Similarly, we can find an index $\ell \neq k$ for which the path $(b_i, a_j, b_\ell, a_\ell, b_j, a_{t+1})$ is compatible. Note that for this choice of k and ℓ, the path P_{t+1} satisfies $|P_{t+1}| = |P_t| + 10$, P_{t+1} does not intersect $e_{t+1}, \cdots, e_{\gamma m}$, and $f(V(P_{t+1}) \cap A) = V(P_{t+1}) \cap B$. This completes the proof of the inductive step.

(ii) By part (i), there exists a proper compatible path P with $|V(P)| \leq 10\gamma m - 9$. Let a and b be the two endpoints of P, where $a \in A$ and $b \in B$. Let B' be the set of vertices $x \in N(a) \setminus (V(P) \cup X_B \cup Y_B)$ that are (i) connected to a by an edge compatible with the edge incident to a in P, for which (ii) $f(x) \notin X_A \cup Y_A$, and (iii) $f(x)$ is $2\sqrt{\mu}$-good for P. By Proposition 2.4 (with $n = 2m$),

$$|B'| \geq |N(a)| - (|V(P)| + |X_B| + |Y_B|) - \mu m - (|X_A| + |Y_A|) - 2\sqrt{\mu}m$$
$$\geq \frac{m}{10} - (10\gamma m + \sqrt{\mu}m + 4\beta m) - \mu m - (4\beta m + \sqrt{\mu}m) - 2\sqrt{\mu}m > 0.$$

Let b' be a vertex in B', and let $a' = f(b')$. Let I be the set of indices i such that $(b, P, a, b', a_i, b_i, a')$ is a compatible path. Since $b' \notin X_B \cup Y_B$ and $a' \notin X_A \cup Y_A$, we have

$$|I| \geq \frac{3}{4}m + \frac{3}{4}m - m - 2\sqrt{\mu}m - \mu m - |V(P)| > 0.$$

Fix an arbitrary index $i \in I$. Similarly as above, let A' be the set of vertices $y \in N(b) \setminus (V(P) \cup X_A \cup Y_A \cup \{a', a_i\})$ that are (i) connected to b by an edge compatible with the edge incident to b in P and for which (ii) $f(y) \notin X_B \cup Y_B$, and (iii) $f(y)$ is $2\sqrt{\mu}$-good for P. By Proposition 2.4 (with $n = 2m$), we have

$$|A'| \geq |N(b)| - (|V(P)| + |X_A| + |Y_A| + 2) - \mu m - (|X_B| + |Y_B|) - 2\sqrt{\mu}m > 0.$$

Let a'' be a vertex in A', and let $b'' = f(a'')$. Similarly as above, we can find an index $j \neq i$ such that the path $P' = (b'', a_j, b_j, a'', b, P, a, b', a_i, b_i, a')$ is compatible and proper. To show that P' is smooth, it suffices to show that the two endpoints are $8\sqrt{\mu}$-good for P'. This easily follows from the fact that a' and b'' are $2\sqrt{\mu}$-good for P and that $|V(P') \setminus V(P)| = 8$ (note that $|V(P)| \geq \gamma m$). $\quad\square$

The following lemma is a variant of Lemmas 3.3 and 4.1.

Lemma 4.6. *Let G be given as above. For every proper smooth path P, there exists a proper compatible cycle C satisfying the following properties:*

- *C has length $|C| \geq (\frac{19}{10} - 50\sqrt{\mu} - 4\gamma)m$,*
- *$V(C) \supseteq V(P)$, and*
- *$|E(C) \setminus E(P)| \leq \frac{3}{2}|V(C) \setminus V(P)| + 3$.*

Proof. Let G be a given graph. Let $P = (v_0, \ldots, v_\ell)$ be a proper smooth path in G, where $v_0 \in A$ and $v_\ell \in B$. It suffices to prove that either there exists a proper smooth path P' with $|P'| \geq |P| + 2$ and $|E(P') \setminus E(P)| \leq 3$, or a proper compatible cycle C with $|C| \geq (\frac{19}{10} - 50\sqrt{\mu} - 4\gamma)m$ and $|E(C) \setminus E(P)| \leq 3$. Since then, we can repeatedly find a longer path to eventually find a cycle with the claimed properties. Assume that the former event does not occur.

Let $L_A \subseteq A$ be the set of vertices in A that have degree at least $\frac{19}{20}m$ in G, and note that

$$\frac{1}{20}m \cdot |V \setminus L_A| \leq e_{G^c}(A, B) \leq \beta m^2.$$

Hence $|L_A| \geq m - 20\beta m$. By Lemma 4.4, there exists a set $X \subset N(v_0) \cap V(P)$ of size

$$|X| \geq d(v_0) - 25\sqrt{\mu}m - 2\gamma m > 20\beta m$$

such that for all vertices $v_i \in X$, the path $(v_{i-1}, \ldots, v_0, v_i, v_{i+1}, \ldots, v_\ell)$ is proper and smooth. Since $|X| > m - |L_A|$, there exists a vertex $v_i \in X$ for which $v_{i-1} \in L_A$. Let $P' = (w_0, w_1, \ldots, w_\ell)$ be the proper smooth path obtained by taking v_i as a pivot point (where $w_0 \in L_A$ and $w_\ell \in B$).

By our assumption on P, we know that P' cannot be extended into a longer proper smooth path by adding at most two edges. Hence by Lemma 4.4, since $w_0 \in L_A$, there exists a set $Y \subset N(w_0) \cap V(P')$ of size $|Y| \geq \frac{19}{20}m - (25\sqrt{\mu} + 2\gamma)m$ consisting of good neighbors of w_0 such that for all vertices $w_i \in Y$, the path $(w_{i-1}, \ldots, w_0, w_i, w_{i+1}, \ldots, w_\ell)$ is proper and smooth. If $w_\ell \in Y$, then we immediately find a cycle with the claimed properties, and hence we may assume that $w_\ell \notin Y$. Then for each vertex $w_i \in Y$, we see that the edge $\{w_0, w_i\}$ is compatible with both $\{w_0, w_1\}$ and $\{w_i, w_{i+1}\}$, and that $\{w_{i-1}, w_i\}$ is not one of the edges $e_1, \ldots, e_{\gamma m}$. Similarly, there exists a set $Z \subset N(w_\ell) \cap V(P)$ of size $|Z| \geq \delta(G) - (25\sqrt{\mu} + 2\gamma)m$ such that for all vertices $w_j \in Z$, the path $(w_0, w_1, \ldots, w_j, w_\ell, w_{\ell-1}, \ldots, w_{j+1})$ is proper and smooth. In particular, for each vertex $w_j \in Z$, we see that the edge $\{w_j, w_\ell\}$ is compatible with both $\{w_\ell, w_{\ell-1}\}$ and $\{w_j, w_{j-1}\}$.

Since G is a bipartite graph, w_0 is adjacent only to vertices w_i with odd index i, and w_ℓ is adjacent only to vertices w_i with even index i. Therefore, since

$$|Y| + |Z| \geq \left(\frac{19}{20}m - (25\sqrt{\mu} + 2\gamma)m\right) + \left(\frac{m}{10} - (25\sqrt{\mu} + 2\gamma)m\right) > m,$$

there exists an index i such that $w_{i-1} \in Z$ and $w_i \in Y$. For this index i, the cycle $C = (w_{i-1}, \ldots, w_0, w_i, w_{i+1}, \ldots, w_\ell, w_{i-1})$ is a proper cycle compatible with $\mathcal{F}$. Also,

$$|C| \geq 2|Y| \geq 2 \cdot \left(\frac{19}{20}m - (25\sqrt{\mu} + 2\gamma)m\right).$$

Moreover, C is obtained from P by adding at most three extra edges. $\qquad\square$

We now present the proof of Theorem 3.5, which asserts the existence of a proper Hamilton cycle compatible with $\mathcal{F}$.

Proof of Theorem 3.5. Let G be a graph satisfying the conditions given in Theorem 3.5. Let $C = (v_0, v_1, \ldots, v_\ell)$ be a proper cycle in G compatible with $\mathcal{F}$, of maximum length. The existence of such a cycle follows from Proposition 4.5 and Lemma 4.6. Moreover, Lemma 4.6 shows that $|C| \geq (\frac{19}{10} - (50\sqrt{\mu} + 4\gamma))m$. Throughout the proof, for a set $X \subset V$, define $X^+ = \{v_{i+1} : v_i \in X \cap V(C)\}$, and $X^- = \{v_{i-1} : v_i \in X \cap V(C)\}$ (where index addition and subtraction are modulo $\ell + 1$). If

C is a Hamilton cycle, then we are done. Otherwise, there exists an edge e in the matching, not intersecting the cycle.

Define B_0 as the set of vertices that are $2\sqrt{\mu}$-bad for C. By Proposition 2.4 (with $n = 2m$), we know that $|B_0| \leq 2\sqrt{\mu}m$. Hence if $|C| < (2 - 4\sqrt{\mu})m$, then we may take an edge $e = \{a, b\}$ in the matching so that $a, b \notin B_0$. In this case, define B_a as the set of bad neighbors of a, and B_b as the set of bad neighbors of b. By definition, we have $|B_a|, |B_b| \leq 2\sqrt{\mu}|C|$. Otherwise if $|C| \geq (2 - 4\sqrt{\mu})m$, then let $e = \{a, b\}$ be an arbitrary edge of the matching not intersecting C, and define $B_a = B_b = \emptyset$ (in both cases, we assume that $a \in A$ and $b \in B$).

Since G is a balanced bipartite graph, we have

$$|N(a) \cap V(C)| \geq d(a) + \frac{1}{2}|V(C)| - m \geq \frac{1}{10}m + \frac{1}{2}\Big(\frac{19}{10} - (50\sqrt{\mu} + 4\gamma)\Big)m - m \geq (3\gamma + 14\sqrt{\mu})m.$$

Define $T_a = \{v_i \in N(a) \cap V(C) : v_{i-1}, v_{i+1} \notin X_A \cup B_0, v_i \notin X_B \cup B_0 \cup B_a\}$, and note that since

$$2(|X_A| + |B_0|) + |X_B| + |B_0| + |B_a| \leq 13\sqrt{\mu}m,$$

we have $|T_a| \geq (3\gamma + \sqrt{\mu})m$. We can similarly define a set $T_b \subseteq N(b) \cap V(C)$ of size $|T_b| \geq (3\gamma + \sqrt{\mu})m$.

Take a vertex $v_i \in T_a$ not incident to $e_1, \ldots, e_{\gamma m}$, for which the pair of edges $\{a, v_i\}$ and $\{a, b\}$ is compatible (such vertex exists since $|T_a| \geq (3\gamma + \sqrt{\mu})m$ and $\mathcal{F}$ is μm-bounded). Then similarly take a vertex $v_j \in T_b$ not incident to $e_1, \ldots, e_{\gamma m}$, for which the pair of edges $\{a, b\}$ and $\{b, v_j\}$ is compatible and $v_j \neq v_{i-1}, v_{i+1}$. Consider the path $P = (v_{i+1}, v_{i+2}, \ldots, v_j, b, a, v_i, v_{i-1}, \ldots, v_{j+1})$. First, P is a proper path since v_i and v_j are not incident to $e_1, \ldots, e_{\gamma m}$. Second, $v_{i+1} \notin X_A$ and $v_{j+1} \notin X_B$ by the definitions of T_a and T_b, and third, both v_{i+1} and v_{j+1} have at most $2\sqrt{\mu}|C| + 4$ bad neighbors in P, since $v_{i+1}, v_{j+1} \notin B_0$, and the set of bad neighbors in P and in C can differ only in at most four vertices v_j, a, b, and v_i.

To check if P is compatible, it suffices to check the compatibility of four pairs $\Big(\{v_{j-1}, v_j\}, \{v_j, b\}\Big)$, $\Big(\{v_j, b\}, \{b, a\}\Big)$, $\Big(\{b, a\}, \{a, v_i\}\Big)$, and $\Big(\{a, v_i\}, \{v_i, v_{i-1}\}\Big)$. The two pairs of edges $\Big(\{v_j, b\}, \{b, a\}\Big)$ and $\Big(\{b, a\}, \{a, v_i\}\Big)$ are both compatible by our choice of v_i and v_j. If $|C| < (2 - 4\sqrt{\mu})m$, then by the choice of e and of the sets B_a, B_b, since $v_i \notin B_a$ and $v_j \notin B_b$, we further see that the other pairs of edges are both compatible, thus implying that P is compatible, and is therefore smooth. This by Lemma 4.6 gives a proper compatible cycle longer than C and contradicts the maximality of C. Therefore, we must have $|C| \geq (2 - 4\sqrt{\mu})m$.

In this case, P is 'almost' smoothly compatible with $\mathcal{F}$, in the sense that it satisfies all the conditions except for possibly the compatibility of two pairs of edges. Define $\mathcal{F}_1$ as the incompatibility system obtained from $\mathcal{F}$ by making the pairs of edges $\Big(\{v_{j-1}, v_j\}, \{v_j, b\}\Big)$ and $\Big(\{a, v_i\}, \{v_i, v_{i-1}\}\Big)$ to be compatible. Note that P is smoothly compatible with $\mathcal{F}_1$. Hence by Lemma 4.6, we can find a proper cycle C_1 compatible with $\mathcal{F}_1$, with $V(C_1) \supseteq V(P) \supseteq V(C)$ and

$$|E(C_1) \setminus E(C)| \leq |E(C_1) \setminus E(P)| + 3 \leq \frac{3}{2}|V(C_1) \setminus V(C)| + 6.$$

Let P_1 be the path obtained from C_1 by removing the edge $\{v_{i-1}, v_i\}$ if it is in C_1 (if not, then skip this paragraph). Note that P_1 is a proper path. We claim that P_1 is smoothly compatible with $\mathcal{F}_1$. First, it is compatible with $\mathcal{F}_1$, since C_1 is. Second, $v_{i-1} \notin X_A$ and $v_i \notin X_B$ since $v_i \in T_a$. Third, since $v_i \notin B_0$, we know that v_i is $2\sqrt{\mu}$-good for C. Since $V(C_1) \supseteq V(C)$ and G is bipartite,

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it follows that v_i has at most

$$2\sqrt{\mu}|C| + |E(C_1) \setminus E(C)| \leq 2\sqrt{\mu}|C| + \frac{3}{2}(|C_1| - |C|) + 6 \leq 8\sqrt{\mu}|C_1| = 8\sqrt{\mu}|P_1|$$

bad neighbors in P_1, where the second inequality follows from $|C| \geq (2 - 4\sqrt{\mu})m$. A similar estimate holds for the other endpoint v_{i-1}. Let $\mathcal{F}_2$ be the incompatibility system obtained from $\mathcal{F}$ by making the pair $\left(\{a, v_j\}, \{v_j, v_{j-1}\}\right)$ to be compatible. Note that P_1 is compatible with $\mathcal{F}_2$, since P_1 is compatible with $\mathcal{F}_1$ and does not contain the edge $\{v_{i-1}, v_i\}$. By Lemma 4.6, there exists a proper cycle C_2 compatible with $\mathcal{F}_2$ whose vertex set contains $V(C) \cup \{a, b\}$.

Let P_2 be the path obtained from C_2 by removing the edge $\{v_j, v_{j-1}\}$ if it is in C_2. An argument similar to above shows that P_2 is a proper path smoothly compatible with $\mathcal{F}$. By Lemma 4.6, we can find a proper cycle compatible with $\mathcal{F}$ whose vertex set contains $V(C) \cup \{a, b\}$, contradicting the maximality of C. Therefore, the cycle C is a Hamilton cycle. $\qquad\square$

5 Concluding remarks

• We have proven the existence of a constant $\mu > 0$ such that the following holds for large enough n: for every n-vertex Dirac graph G with a given μn-bounded incompatibility system $\mathcal{F}$, there exists a Hamilton cycle in G compatible with $\mathcal{F}$. The value of μ that we obtain is quite small ($\mu = 10^{-16}$), and determining the best possible value of μ is an interesting open problem remaining to be solved. It is not clear what this value should be. The following variant of a construction of Bollobás and Erdős [2] shows that μ is at most $\frac{1}{4}$. Let n be an integer of the form $4k - 1$, and let G be an edge-disjoint union of two $\frac{n+1}{4}$-regular graphs G_1 and G_2 on the same n-vertex set. Color the edges of G_1 in red, and of G_2 in blue. Note that G does not contain a properly colored Hamilton cycle since a Hamilton cycle of G is of odd length. Let $\mathcal{F}$ be an incompatibility system defined over G, where incident edges of the same color are incompatible. Then there exists a Hamilton cycle compatible with $\mathcal{F}$ if and only if there exists a properly colored Hamilton cycle. Since there is no properly colored Hamilton cycle, we see that there is no Hamilton cycle compatible with $\mathcal{F}$.

• As mentioned in the introduction, the motivation for our work came from a conjecture of Häggkvist (Conjecture 1.2). We note that the conjecture can be answered using a result in [14] that studied Hamiltonicity Maker-Breaker game played on Dirac graphs. The theorem proven there asserts the existence of a positive constant β such that Maker has a winning strategy in a $(1 : \beta n / \log n)$ Hamiltonicity Maker-Breaker game played on Dirac graphs. To see how this implies the conjecture, given a graph G and a 1-bounded incompatibility system $\mathcal{F}$, consider a Breaker's strategy claiming at each turn the edges that are incompatible with the edge that Maker claimed in the previous turn; this strategy forces Maker's graph to be compatible with $\mathcal{F}$ at all stages. Since Maker has a winning strategy for a $(1 : 2)$ game, we see that there exists a Hamilton cycle compatible with $\mathcal{F}$. This analysis gives a weaker version of our main theorem asserting the existence of a compatible Hamilton cycle for every $\frac{1}{2}\beta n / \log n$-bounded incompatibility system.

• The concept of incompatibility systems appears to provide a new and interesting take on robustness of graph properties. Further study of how various extremal results can be strengthened using this notion appears to be a promising direction of research. For example in the forthcoming paper [15], we show that there exists a constant $\mu > 0$ such that with high probability over the choice of a random graph $G = G(n, p)$ with $p \gg \frac{\log n}{n}$, for any μnp-bounded system $\mathcal{F}$ over G, there is a compatible Hamilton cycle. This extends classical Hamiltonicity results for random graphs.

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