Component Games on Random Graphs

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Abstract

In the $(1 : b)$ component game played on a graph G, two players, MAKER and BREAKER, alternately claim 1 and b previously unclaimed edges of G, respectively. MAKER's aim is to maximise the size of a largest connected component in her graph, while BREAKER is trying to minimise it. We show that the outcome of the game on the binomial random graph is strongly correlated with the appearance of a nonempty $(b + 2)$-core in the graph.

For any integer k, the k-core of a graph is its largest subgraph of minimum degree at least k. Pittel, Spencer and Wormald showed in 1996 that for any $k \geq 3$ there exists a constant c_k, which they determine, such that $p = c_k/n$ is the threshold function for the appearance of the k-core in $G \sim \mathcal{G}(n, p)$. More precisely, $G \sim \mathcal{G}(n, c/n)$ has WHP a linear-size k-core for constant $c > c_k$, and an empty k-core when $c < c_k$.

We show that for any positive constant integer b, when playing the $(1 : b)$ component game on $G \sim \mathcal{G}(n, c/n)$, MAKER can WHP build a linear-size component if $c > c_{b+2}$, while BREAKER can WHP prevent MAKER from building larger than polylogarithmic-size components if $c < c_{b+2}$.

For the strategy of MAKER when $c > c_{b+2}$, we utilise known results on the k-core. For BREAKER when $c < c_{b+2}$, we make use of a result of Achlioptas and Molloy (sketching its proof) that states that after deleting all vertices of degree less than k, and repeating this step a constant number of times, G is WHP shattered into pieces of polylogarithmic size.

1   Introduction

Let X be a finite set, let $\mathcal{F} \subseteq 2^X$ be a family of subsets of X, and let b be a positive integer. In the $(1 : b)$ MAKER–BREAKER game $(X, \mathcal{F})$, two players, called MAKER and BREAKER, take turns in

claiming previously unclaimed elements of X. On MAKER's move, she claims one element of X, and on BREAKER's move, he claims b elements (if less than b elements remain before BREAKER's last move, he claims all of them). The game ends when all of the elements have been claimed by either of the players. MAKER wins the game $(X, \mathcal{F})$ if by the end of the game she has claimed all the elements of some $F \in \mathcal{F}$; otherwise BREAKER wins. The description of the game is completed by stating which of the players is the first to move, though usually it makes no real difference. For convenience, we typically assume that $\mathcal{F}$ is closed upwards, and specify only the inclusion-minimal elements of $\mathcal{F}$. Since these are finite, perfect information games with no possibility of draw, for each setup of $\mathcal{F}, b$ and the choice of the identity of the first player, one of the players has a strategy to win regardless of the other player's strategy. Therefore, for a given game we may say that the game is MAKER's win, or alternatively that it is BREAKER's win. The set X is referred to as the *board* of the game, and the elements of $\mathcal{F}$ are referred to as the *winning sets*.

When $b = 1$, we say that the game is *unbiased*; otherwise it is *biased*, and b is called the *bias* of BREAKER. It is easy to see that MAKER–BREAKER games are *bias monotone*. That is, if MAKER wins some game with bias $(1 : b)$, she also wins this game with bias $(1 : b')$ for every $b' \leq b$. Similarly, if BREAKER wins a game with bias $(1 : b)$, he also wins this game with bias $(1 : b')$ for every $b' \geq b$. This bias monotonicity allows us to define the *threshold bias*: for a given game $\mathcal{F}$, the threshold bias b^* is the value for which BREAKER wins the game $\mathcal{F}$ with bias $(1 : b)$ if and only if $b > b^*$. It is quite easy to observe that it is never a disadvantage in a MAKER–BREAKER game to be the first player, and that if a player has a winning strategy as the second player, essentially the same strategy can be used to also win the game as the first player. Hence, when we describe a strategy for MAKER we assume that she is the second player, implying that under the conditions described she can win as either a first or a second player. The same goes for BREAKER's strategy.

In this paper, our attention is dedicated to the $(1 : b)$ MAKER–BREAKER s-component game on the binomial random graph $\mathcal{G}(n, p)$, in which each of the $\binom{n}{2}$ possible edges appears independently with probability $p = p(n)$; that is, the board is the edge set of $G \sim \mathcal{G}(n, p)$ and the (inclusion-minimal) winning sets are the trees of G with s vertices. Since the board is random, our results hold with high probability (WHP), i.e., with probability tending to 1 as n tends to infinity.

For more on MAKER–BREAKER games as well as other positional games, please see the books by Beck [3] and by Hefetz et al. [13].

1.1 Previous results

A natural case to consider is $s = n$; that is, the winning sets are the spanning trees of the graph the game is played on. This $(1 : b)$ n-component game is known as the *connectivity* game.

The unbiased game was completely solved by Lehman [18], who showed that MAKER, as a second player, wins the $(1 : 1)$ connectivity game on a graph G if and only if G contains two edge-disjoint spanning trees. It follows easily from [22, 27] that if G is $2k$-edge-connected then it contains k pairwise independent spanning trees; thus, MAKER wins the $(1 : 1)$ connectivity game on 4-regular 4-edge-connected graphs, whereas BREAKER trivially wins the $(1 : 1)$ connectivity game on graphs with less than $2n - 2$ edges, i.e., average degree under 4. For denser graphs, since MAKER wins the unbiased game by such a large margin, it only seems fair to even out the odds by strengthening BREAKER, giving him a bias $b \geq 2$. The first and most natural board to consider is the edge set of the complete graph K_n. Chvátal and Erdős [5] showed that $\left(\frac{1}{4} - o(1)\right) n / \log n \leq b^*(K_n) \leq (1 + o(1)) n / \log n$; the upper bound was proved to be tight by Gebauer and Szabó [11]; that is, $b^*(K_n) = (1 + o(1)) n / \log n$.

Returning to $\mathcal{G}(n, p)$, Stojaković and Szabó [26] showed that WHP $b^*(\mathcal{G}(n, p)) = \Theta(np/\log n)$, where BREAKER's win holds for any $0 \leq p \leq 1$, while MAKER's win requires large enough p (MAKER cannot win for small p since $G \sim \mathcal{G}(n, p)$ is WHP disconnected). This was improved by Ferber et al. in [8], who showed for $p = \omega(\log n/n)$ that WHP $b^*(\mathcal{G}(n, p)) = (1 + o(1)) np/\log n$.

A different random graph model, the random d-regular graph $\mathcal{G}(n, d)$ on n vertices, was considered by Hefetz et al. in [12]. They showed that WHP $b^*(\mathcal{G}(n, d)) \geq (1 - \epsilon) d/\log_2 n$ for $d = o(\sqrt{n})$. Note that when $d = \Omega(\sqrt{n})$, the model $\mathcal{G}(n, d)$ is quite close to $\mathcal{G}(n, p)$ for $p = d/n$, since for this value of p all degrees in $G \sim \mathcal{G}(n, p)$ are WHP $(1 + o(1))d$. Moreover, they showed that $b^*(G) \leq \max\{2, \bar{d}/\log n\}$ for any graph G of average degree $\bar{d}$, so the result is asymptotically tight.

BREAKER's strategy in practically all results mentioned above is to deny connectivity by isolating a single vertex. Much less is known, however, for the case $s < n$. It seems that even if BREAKER is able to isolate a vertex in a constant number of moves, it does little to prevent MAKER from winning the s-component game for $s = \Omega(n)$.

Instead of considering the threshold bias b^*, we shift the focus to the maximal component size s achievable by MAKER in the $(1 : b)$ component game, for a given bias b (assuming optimal play by both players). Let us denote this quantity by $s_b^*(G)$. Bednarska and Łuczak considered in [2] the $(1 : b)$ component game on the complete graph. They showed that $s_b^*(K_n)$ undergoes a certain type of phase transition around $b = n$; specifically, that $s_{n+t}^*(K_n) = (1 - o(1)) n/t$ for $\sqrt{n} \ll t \ll n$ but $s_{n-t}^*(K_n) = t + O(\sqrt{n})$ for $0 \leq t \leq n/100$.

The component game on d-regular graphs for fixed $d \geq 3$ was considered by Hod and Naor in [15]. They showed a similar phase transition of s_b^* around $b = d - 2$: for any d-regular graph G on n vertices,

$$s_b^*(G) = \begin{cases} O(1), & b \geq d - 1; \\ O(\log n) & b = d - 2 \end{cases}$$

whereas $s_{d-3}^*(\mathcal{G}(n, d)) = \Omega(n)$ WHP.

1.2 Our results

Given previous results, it is not surprising that the component game on the binomial random graph $\mathcal{G}(n, p)$ also undergoes a phase transition. Writing $p = c/n$ — so the expected average degree in $\mathcal{G}(n, p)$ is asymptotically equal to c — we could perhaps guess the phase transition occurs around $c = b + 2$ in accord with the results for d-regular graphs. Another plausible approach would be to consider the so-called *random graph intuition*, as first observed by Chvátal and Erdős in [5]: it turns out that in many cases, the winner of a game in which both sides play to their best is the same as if both sides were to play randomly. We may thus guess that the transition occurs around $c = b + 1$, since in the random players scenario MAKER would end up with the edge set of $\mathcal{G}(n, p/(b + 1))$, which contains a linear-size component if and only if $c > b + 1$.

It turns out, however, that neither of these heuristics gives the correct answer. The key graph parameter is degeneracy, which is related to the minimum degree of subgraphs, rather than average degree; consequently, the critical c is (somewhat) larger than $b + 2$.

Definition. For an integer $k \geq 1$, the *k-core* of a graph $G = (V, E)$ is its largest subgraph K of minimum degree $\delta(K) \geq k$. If no such subgraph exists, we say G has an empty k-core, or that G has no k-core, or that G is $(k - 1)$-*degenerate*.

Pittel, Spencer and Wormald [23] (see also [7, 16, 25]) proved that for every $k \geq 3$ there exists a threshold constant c_k for the appearance of the k-core in the binomial random graph. More specifically, $G \sim \mathcal{G}(n, c/n)$ has WHP an empty k-core for $c < c_k$ and a linear-size k-core for $c > c_k$. It was previously shown by Łuczak [19, 20] that $G \sim \mathcal{G}(n, p)$ WHP has either an empty or a linear-size k-core, for every fixed $k \geq 3$. The constant c_k is implicitly defined and satisfies $c_k = k + \sqrt{k \log k} + O\left(\sqrt{k/\log k}\right)$ (see [24, Lemma 1]). For small values of k we have $c_3 \approx 3.351$, $c_4 \approx 5.149$, $c_5 \approx 6.799$, $c_6 \approx 8.365$.

The following theorems show that for any constant b, the phase transition for the $(1 : b)$ component game on $G \sim \mathcal{G}(n, c/n)$ occurs at $c = c_{b+2}$.

Theorem 1.1. *For any two constants b and $c < c_{b+2}$ we have WHP $s_b^*(\mathcal{G}(n, c/n)) = o(\log^3 n)$.*

Theorem 1.2. *For any two constants b and $c > c_{b+2}$ we have WHP $s_b^*(\mathcal{G}(n, c/n)) = \Omega(n)$.*

Fix an integer $k \geq 3$. The standard algorithm for finding the k-core of a graph G on n vertices is the following, called the k-peeling process: starting from $G_0 = G$, let $(G_t)_{t \geq 0}$ be the sequence of subgraphs of G, where G_{t+1} is obtained from G_t by deleting all edges incident with vertices of degree at most $k - 1$. Since $G_{t+1} \subseteq G_t$, this (deterministic) process stabilises after some finite time $T^*(G) \leq n$. If G is $(k-1)$-degenerate, the graph G_{T^*} is empty; otherwise, G_{T^*} is the k-core of G, plus, possibly, some isolated vertices. (We only delete edges, so G_t has n vertices for all $t \geq 0$, although some of them may become isolated along the process).

Remark. Two variations of this process are: (*i*) delete vertices instead of edges; (*ii*) process in every step only a single vertex of degree less than k.

Jiang, Mitzenmacher, and Thaler used a branching process argument in [17] (see also [10]) to show:

Theorem 1.3 ([17, Theorems 1 and 2]). *Fix $k \geq 3$ and $c < c_k$. Then WHP*

$$T^*(n, c) = \log_{k-1} \log n + \Theta(1).$$

We will use a related result of Achlioptas and Molloy [1, Section 8], that the graph is WHP already shattered into small fragments after a *constant* number of iterations:

Theorem 1.4 ([1]). *Fix $k \geq 3$ and $c < c_k$. There exists a constant $t^\dagger = t^\dagger(c)$ such that, in the k-peeling process on $G \sim \mathcal{G}(n, c/n)$, WHP all connected components of $G_{t^\dagger}$ have size $o(\log^3 n)$.*

Remark. The result was misstated in [1] with the exponent 3 missing. We believe the result deserves attention as it may be of independent interest. The proof, which we sketch for completeness, actually yields the somewhat better bound, $O\left(\log^3 n / \log^2 \log n\right)$, in Theorem 1.4, and consequently in Theorem 1.1. We use the term $o(\log^3 n)$ in both theorems for brevity.

In Section 2 we provide some notations and a few bounds we use throughout the paper. In Section 3, we prove Theorem 1.1, by showing how BREAKER can WHP limit the radius of MAKER's components (via Theorem 1.4) and hence limit their size to poly-logarithmic. In Section 4 we prove Theorem 1.2, by showing how MAKER can WHP build a tree of linear size within the $(b+2)$-core of $\mathcal{G}(n, c/n)$. This is done in a two-step strategy: MAKER carefully nurtures a small sapling, which she afterwards grows arbitrarily into a full-scale tree. Two ingredients of the proof of Theorem 1.2 are rather technical; we provide required background in Section 5 and the proofs themselves in Section 6.

4

2 Preliminaries

2.1 Notation

We use standard graph theory terminology, and in particular the following. For a given graph G we denote by $V(G)$ and $E(G)$ the set of its vertices and the set of its edges, respectively. The *excess* of G is defined as $\text{exc}(G) = |E(G)| - |V(G)|$. For two disjoint sets of vertices $A, B \subseteq V$ we denote by $E(A, B)$ the set of all edges $ab \in E$ with $a \in A$ and $b \in B$. For a subgraph $F \subset G$ we denote its *edge boundary* (with respect to G) by $\partial F = E(V(F), V(G) \setminus V(F))$. For a positive integer t, the *t-neighbourhood* of a vertex $v \in V(G)$, also referred to as the *ball* of radius t around v, is the subset of vertices of G whose distance to v is at most t.

Given a rooted tree T and an integer $k \geq 3$, a non-leaf vertex $v \in V(T)$ is called *k-light* if v has less than k children in T and *k-heavy* otherwise. A *tree path* of T is a sequence $(v_0, v_1, \ldots, v_j)$ of vertices of T such that v_{i-1} is the parent of v_i for all $i = 1, \ldots, j$. A tree path is called *k-light* if all its vertices are k-light. The *level* of any vertex $v \in V(T)$ is the length of the unique path in T between v and the root, and the *height* of T (assuming T is finite) is the maximum level of a vertex $v \in V(T)$.

For two integers $n > 0$ and $m \geq 0$, let $\mathcal{D}_{n,2m}$ denote the set of all nonnegative integer vectors $\vec{d} = (d_1, \ldots, d_n)$ such that $\sum d_i = 2m$. Each $\vec{d} \in \mathcal{D}_{n,2m}$ is called a *degree sequence*, and denote its maximum degree by $\Delta(\vec{d}) := \max_i d_i$.

At any point during the game, an unclaimed edge is called *free*. The act of claiming one free edge by one of the players is called a *step*. BREAKER's b successive steps (and MAKER's single step) are called a *move*. A *round* in the game consists of one move of the first player, followed by one move of the second player. Since being the first player is never a disadvantage, we will assume MAKER starts when proving Theorem 1.1, and assume BREAKER starts when proving Theorem 1.2.

Throughout the paper we use the well-known bound $\binom{n}{k} \leq n^k/k! \leq (en/k)^k$ for nonnegative integers n and k. Let $[n]_k = n(n-1)\cdots(n-k+1)$, and note that $[n]_k = n!/(n-k)!$ for $n \geq k$. For a positive integer m, let $(2m-1)!! = (2m-1)(2m-3)\cdots 3 \cdot 1$ and $(2m)!! = (2m)(2m-2)\cdots 4 \cdot 2 = 2^m \cdot m!$.

In this paper we make extensive use of the Poisson distribution. For an integer $j \geq 0$ and a real number $\lambda \geq 0$, let $\Psi_j(\lambda) = \Pr[\text{Poisson}(\lambda) = j] = e^{-\lambda}\lambda^j/j!$, let $\Psi_{\geq j}(\lambda) = \Pr[\text{Poisson}(\lambda) \geq j] = \sum_{i \geq j} \Psi_i(\lambda)$, and let $\Psi_{<j}(\lambda) = \Pr[\text{Poisson}(\lambda) < j] = 1 - \Psi_{\geq j}$. Note that for any real number $\lambda \geq 0$ and for any two integers $j \geq \ell \geq 0$ we have

$$[j]_\ell \Psi_j(\lambda) = e^{-\lambda}\lambda^j/(j-\ell)! = \lambda^\ell \Psi_{j-\ell}(\lambda). \tag{1}$$

Given a positive integer $\ell > 0$ and a real number $\lambda > 0$, let $Z_\ell(\lambda)$ denote an ℓ-truncated Poisson random variable, which is a $\text{Poisson}(\lambda)$ random variable conditioned on being at least ℓ. In other words, $\Pr[Z_\ell(\lambda) < \ell] = 0$ and $\Pr[Z_\ell(\lambda) = j] = \Psi_j(\lambda)/\Psi_{\geq \ell}(\lambda)$ for $j \geq \ell$.

Our proofs are asymptotic in nature and whenever necessary, we assume n is large enough. We omit floor and ceiling signs when these are not crucial. All logarithms in this paper, unless specified otherwise, are natural.

2.2 Local structure of $\mathcal{G}(n, c/n)$

We also provide several results about the typical local structure of $\mathcal{G}(n, c/n)$, which will be useful later. We begin with the following bound on the volume of balls in $\mathcal{G}(n, c/n)$.

Lemma 2.1 ([6, Lemma 1]). *Let $G \sim \mathcal{G}(n, c/n)$ for $c > 1$. Then WHP, for every vertex $v \in V(G)$ and for every $1 \le t \le n$, there are at most $2t^3 c^t \log n$ vertices in G within distance t of v.*

This lemma leads to the well known fact that even though G is not acyclic WHP for $c > 1$, we do not expect short cycles near a given vertex; in other words, local neighbourhoods in G are trees.

Claim 2.2. *The probability of having a cycle in the t-neighbourhood of a given vertex v in $G \sim \mathcal{G}(n, c/n)$ is $o(1)$ for $t = \lceil \alpha \log n \rceil$, where $\alpha = \alpha(c)$ is some positive constant.*

Proof. It suffices to prove this for $c > 1$. By Lemma 2.1, a breadth-first search from v discovers WHP at most $s = \lfloor 2t^3 c^t \log n \rfloor$ vertices in the t-neighbourhood of v; the probability of having either a back-edge or a side edge closing a cycle, in addition to the tree edges, among the first s vertices discovered, is bounded by

$$\binom{s}{2} c/n < s^2 c/n \le 4t^6 c^{1+2t} \log^2 n / n,$$

which is $o(1)$ for our choice of t and for sufficiently small α (which only depends on c). $\square$

Last, we show that sufficiently large connected subgraphs of G expand by only a constant factor WHP.

Claim 2.3. *Let $G \sim \mathcal{G}(n, c/n)$ for $c > 1$. Then WHP, every connected subset $U \subseteq V(G)$ of size $|U| \ge \log n$ has at most $4c |U|$ neighbours in G.*

Proof. Fix a set $U \subseteq V(G)$ of size $u = |U|$. It is connected (and thus contains at least one of the u^{u-2} possible spanning trees of U) with probability at most $u^{u-2} (c/n)^{u-1}$. The size of the external neighbourhood of U is distributed $\text{Bin}(n-u, 1 - (1 - c/n)^u)$, stochastically dominated by $\text{Bin}(n, cu/n)$, and thus by the Chernoff bound, it has more than $(1+\delta)cu$ neighbours with probability at most $\left(e^\delta / (1+\delta)^{1+\delta} \right)^{cu}$. Therefore, the probability that, for some $u \ge \log n$, there exists a connected set U of u vertices with more than $4cu$ neighbours is bounded by

$$\sum_{u=\lceil \log n \rceil}^{n} \binom{n}{u} u^{u-2} (c/n)^{u-1} \exp\left((3 - 4\ln 4)cu \right) \le \frac{n}{c} \sum_{u=\lceil \log n \rceil}^{n} u^{-2} \left(ce^{1-2c} \right)^u$$

$$< \frac{n}{c \log^2 n} \sum_{u=\lceil \log n \rceil}^{\infty} e^{-u} \le \frac{e}{c(e-1) \log^2 n} = o(1). \ \square$$

3 His Side

We now present a strategy for BREAKER to use in the $(1 : b)$ component game on $\mathcal{G}(n, c/n)$ for $c < c_{b+2}$, and show that it WHP restricts MAKER's connected components to poly-logarithmic size.

To facilitate the description of BREAKER's strategy, we introduce some terminology. Denote the *rank* of a vertex v with respect to the $(b+2)$-peeling process $(G_t)_{t \geq 0}$ of a graph $G = G_0$ by $\rho(v) = \inf \left\{ t \geq 0 \mid \deg_{G_t}(v) < b+2 \right\}$. When G is $(b+1)$-degenerate, $\rho(v) \leq T^*(G)$ is finite for all v. In particular, this holds WHP for $G \sim \mathcal{G}(n, c/n)$ with $c < c_{b+2}$.

An edge $uv \in E(G)$ is called *horizontal* if $\rho(u) = \rho(v)$ and *vertical* otherwise. A *horizontal component* (H-comp, for short) C is a connected component in the graph consisting of horizontal edges claimed by MAKER, and its *rank* $\rho(C)$ is the rank shared by all its vertices. A vertical edge e is *above* C if $V(C)$ contains e's endpoint of lesser rank. Finally, denote by $F(C)$ the set of free edges incident with $V(C)$ in G_t, where $t = \rho(C)$. We partition $F(C)$ to vertical and horizontal edges, writing $F(C) = F_V(C) \cup F_H(C)$.

BREAKER's Strategy $\mathcal{S}_B$: We assume MAKER is the first player, so each round consists of a move by MAKER and a response by BREAKER. For a particular round, let $e = uv$ be the edge claimed by MAKER on her move. Assume without loss of generality that $\rho(u) \leq \rho(v)$, and let C be the H-comp containing u in MAKER's graph after this move. BREAKER's strategy $\mathcal{S}_B$ is to claim in each of his b steps during this round:

- an arbitrary edge from $F_V(C)$, if $F_V(C)$ is nonempty; otherwise

- an arbitrary edge from $F_H(C)$, if $F_H(C)$ is nonempty; otherwise

- an arbitrary free edge.

It is clear that BREAKER can follow $\mathcal{S}_B$ throughout the game. The following claim characterises the horizontal components allowed by $\mathcal{S}_B$:

Claim 3.1. *At the beginning of every round, every H-comp C satisfies exactly one of the following:*

(i) MAKER *claimed exactly one edge above C and $F(C)$ is empty.*

(ii) MAKER *claimed no edges above C and $|F(C)| \leq \max\{0, b+2-|V(C)|\}$.*

Proof. We prove this by induction on the number of rounds. Before the game starts, (ii) holds for every H-comp C since no edges have been claimed, $|V(C)| = 1$ and $|F(C)| \leq b+1$ by definition of $(b+2)$-rank. Assume the claim holds at the beginning of round $r \geq 0$; we show it holds at the end of that round. Let $e = uv$ be the edge claimed by MAKER on her rth move, and assume WLOG that $\rho(u) \leq \rho(v)$. Denote by C_u and C_v, respectively, the H-comps containing u and v at the beginning of round r, and by C the H-comp containing u after her move. It remains to show that after BREAKER's move C satisfies either (i) or (ii) since no other H-comp is affected by MAKER's move, and BREAKER's move cannot disrupt an H-comp which was already satisfying (i) or (ii) from doing so. We distinguish between two cases:

Case 1. If e is vertical, observe that $C = C_u$. Since we had $e \in F(C)$ before MAKER's move, e must be the first edge MAKER claimed above C by assumption. Before BREAKER's move we have $|F(C)| \leq b$ (as e is already claimed), so he claims all of $F(C)$ according to $\mathcal{S}_B$.

Case 2. If e is horizontal, $C = C_u \cup C_v$. Since both $F(C_u)$ and $F(C_v)$ contained e at the beginning of the round, both satisfied (ii) and thus MAKER claimed no edges above C.

Moreover, before MAKER's move we had $0 < |F(C_u)| \le b + 2 - |V(C_u)|$, and similarly for v, thus before BREAKER's move we have

$$|F(C)| = |(F(C_u) \setminus \{e\}) \cup (F(C_v) \setminus \{e\})| \le 2(b+1) - |V(C)|$$

and after his move, according to $\mathcal{S}_B$, either $F(C)$ is empty or $|F(C)| \le b + 2 - |V(C)|$. □

Claim 3.1 yields the following two corollaries. In the second one, we use the fact that the graph is a tree since each contracted vertex can have at most one edge going 'up'.

Corollary 3.2. *Throughout the game* $|V(C)| \le 2(b+1)$ *for every H-comp* C.

Proof. Indeed, no free edges are incident with H-comps of size $b+2$ or more by Claim 3.1, and thus the largest H-comp possibly achievable by MAKER is obtained by claiming a free horizontal edge between two H-comps of size $b+1$ each. □

Corollary 3.3. *Given a connected component* Γ *of* MAKER*'s graph, if we contract every H-comp in* Γ *to a single vertex, we get a tree* T_Γ *of height at most* $\rho(C_0)$, *where* C_0 *is the H-comp in* Γ *of maximal rank (that is, the H-comp corresponding to the root of* T_Γ).

At this point we can already establish a weaker version of Theorem 1.1. Indeed, using the above corollaries, Lemma 2.1 and Theorem 1.3, we obtain the following poly-logarithmic bound on the size of MAKER's connected components.

Proposition 3.4. *For* $c < c_{b+2}$, $G \sim \mathcal{G}(n, c/n)$ *is WHP such that by playing according to* $\mathcal{S}_B$, BREAKER *ensures that all connected components in* MAKER*'s graph are of size* $O\left(\log^{4b+5} n\right)$.

Proof. Fix a connected component Γ in MAKER's graph. The height of T_Γ from Corollary 3.3 is bounded by the stabilization time, and we have WHP $T^* = T^*(n, c) \le \log_{b+1} \log n + O(1)$ by Theorem 1.3. Moreover, the distance between two vertices in the same H-comp is at most $2b+1$ by Corollary 3.2, so Γ is contained in a ball of radius $t = 2(b+1)T^*$ around any vertex $v \in V(\Gamma)$ of maximal rank. Applying Lemma 2.1 yields the bound

$$2t^3 c^t \log n = O\left(\log^3 \log n \cdot c^{2(b+1)\log_{b+1}\log n} \cdot \log n\right) = O\left(\log^{1+2(b+1)\log_{b+1} c} n \cdot \log^3 \log n\right).$$

Note that $c < c_{b+2} < 2(b+1)$ and thus

$$1 + 2(b+1)\log_{b+1} c < 1 + 2(b+1)\left(1 + \log_{b+1} 2\right) \le 4b + 5,$$

establishing the bound on the size of C. □

Proposition 3.4 gives a nonuniform poly-logarithmic bound (i.e., the exponent depends on b, even when computed precisely). Using Theorem 1.4 and Claim 2.3, we improve the bound to a uniform $o\left(\log^3 n\right)$ for any value of b, and establish Theorem 1.1, via the following proposition:

Proposition 3.5. *For* $c < c_{b+2}$, $G \sim \mathcal{G}(n, c/n)$ *is WHP such that if* BREAKER *plays according to* $\mathcal{S}_B$, *all connected components in* MAKER*'s graph are of size* $o\left(\log^3 n\right)$.

Proof. Fix a connected component Γ in MAKER's graph. If $|V(\Gamma)| < \log n$, we are done; otherwise, let $U' \subseteq V(\Gamma)$ be the set of vertices of rank at least $t^\dagger$, where $t^\dagger = t^\dagger(c)$ is the constant from Theorem 1.4, and let U be the vertex set of a connected $\Gamma' \subseteq \Gamma$ of minimal size such that $U' \subseteq U$ and $|U| \geq \log n$. By Theorem 1.4, $|U| = o\left(\log^3 n\right)$. Furthermore, Γ is contained in a ball of radius $t = 2(b+1)t^\dagger$ around U, thus t applications of Claim 2.3 yield the desired bound

$$|V(\Gamma)| \leq (4c)^t |U| = O(|U|) = o\left(\log^3 n\right). \qquad \square$$

Remark. A slightly refined strategy by BREAKER would be to claim edges from $F_H(C)$ before edges from $F_V(C)$ if $|F_V(C)| > b$. It is not very hard to see that by this he makes sure that throughout the game all H-comps except the "root" ones are of size at most two. This improves the exponent in Proposition 3.4 to $3 + \log_{b+1} 4$, which approaches 3 from above as b increases. However, for all $b \geq 1$ this is still inferior to the bound given in Proposition 3.5 (which does not benefit from the modified strategy). We thus chose to present the simpler strategy.

4   Her Side

It is quite easy to prove a weaker version of Theorem 1.2: that WHP $s_b^*(\mathcal{G}(n, c/n)) = \Omega(n)$ when $c > c_{b+3}$. Indeed, in this case $\mathcal{K}'$, the $(b+3)$-core of $\mathcal{G}(n, c/n)$, is WHP of linear size and a very good edge expander. It follows that the naive tree-building strategy within $\mathcal{K}'$ (i.e., the strategy that builds a single tree $T \subset \mathcal{K}'$ by repeatedly claiming an arbitrary free edge of ∂T, starting from an arbitrary vertex of $\mathcal{K}'$) is successful for MAKER, since as long as T is sublinear in size, i.e., has $o(n)$ vertices, we have $|\partial T| > b|V(T)|$ and BREAKER cannot claim all the boundary of T. This is the same strategy MAKER uses in [15] when playing on a random $(b+3)$-regular graph.

For $c_{b+2} < c < c_{b+3}$, however, the above strategy applied to $\mathcal{K}$, the $(b+2)$-core of $G \sim \mathcal{G}(n, c/n)$, is prone to being shut down by BREAKER in the early stages of the game despite the minimum degree in $\mathcal{K}$ being $b+2$. Thus in the proof of Theorem 1.2, MAKER uses a tree-building strategy that begins in a more refined manner. In the tree $T \subset \mathcal{K}$ MAKER builds, she makes sure to have many heavy vertices (i.e., of degree at least $b+3$ in $\mathcal{K}$). Only after securing sufficiently many such vertices does she continue by growing T naively.

Recall that for a rooted tree, a non-leaf vertex is k-light if it has less than k children, and it is k-heavy otherwise, and that a tree path (a descending path towards the leaves) is k-light if it consists solely of k-light vertices. Note that since leaves are not considered light, any tree path terminating in a leaf is not k-light by definition (although permitting leaves to be light would require only a trivial modification of our arguments).

For simplicity, in the remainder of the paper k stands for $b+2$, and $\mathcal{K}$ is the k-core of G. The proof of Theorem 1.2 contains three main ingredients, stated in three lemmas. The first two lemmas require the following definition.

Definition. An (N, L)-tree is a balanced tree (i.e., all of its leaves are at the same level) of height N with no $(k-1)$-light vertices and no k-light tree paths of length L.

Lemma 4.1. *In the $(1 : k-2)$ MAKER–BREAKER game played on the edge set of an (N, L)-tree T^*, MAKER has a strategy to build a subtree $T \subset T^*$ with at least α^N vertices that are k-heavy in T^*, for some constant $\alpha = \alpha(k, L) > 1$. Moreover, using this strategy MAKER's graph is a tree throughout the game, until T is achieved.*

We assume the hypothesis of Theorem 1.2, i.e., $c > c_k$, in the following two lemmas.

Lemma 4.2. $\mathcal{K}$ *contains WHP an* (N, L)-*tree for* $N = \log^2 \log n$ *and some constant* $L = L(k, c)$.

Lemma 4.3. *There exists some* $\varepsilon = \varepsilon(k, c) > 0$ *such that WHP any subgraph* $C \subset \mathcal{K}$ *with minimum degree at least* 2, $\text{exc}(C) \geq \log^4 n$ *and* $|\partial C| \leq (k - 2)|V(C)|$ *satisfies* $|V(C)| \geq \varepsilon n$.

Lemma 4.1 is proved in Section 4.1, while the proofs of Lemmas 4.2 and 4.3 are postponed to Sections 6.2 and 6.3, respectively.

Next we show how these lemmas imply Theorem 1.2, but first we need the following easy claim.

Claim 4.4. *Let* F *be an induced subgraph of* $\mathcal{K}$ *with non-negative excess such that* $|\partial F| \leq b|V(F)|$, *and let* C *be the* 2-*core of* F *(note that* C *is not empty since* $\text{exc}(F) \geq 0$ *and thus* F *contains a cycle). Then* $\text{exc}(C) \geq \text{exc}(F)$ *and* $|\partial C| \leq b|V(C)|$.

Proof. In order to obtain C from F we use a sequential vertex-deleting variant of the 2-peeling process. Starting from $F_0 = F$, as long as the minimum degree of F_t is less than two, we obtain F_{t+1} from F_t by deleting one arbitrary vertex $v \in V(F_t)$ of degree $\deg_{F_t}(v) \leq 1$. Regardless of the order of deletions, this process terminates with the 2-core $C = F_{T^*}$. We prove that $\text{exc}(F_t) \geq \text{exc}(F)$ and $|\partial F_t| \leq b|V(F_t)|$ for all $t \geq 0$ by induction.

For $t = 0$ there is nothing to prove; assuming it holds for t, let $v \in V(F_t)$ be the vertex selected for deletion, whose degree in F_t is 0 or 1. In F_{t+1} there is one less vertex (i.e., v) and at most one less edge than F_t, hence $\text{exc}(F_{t+1}) \geq \text{exc}(F_t) \geq \text{exc}(F)$; furthermore

$$\partial F_{t+1} = \partial F_t \cup E(v, V(F_{t+1})) \setminus E(v, V(\mathcal{K} \setminus F_t))$$

so

$$|\partial F_{t+1}| = |\partial F_t| + \deg_{F_t}(v) - \left(\deg_{\mathcal{K}}(v) - \deg_{F_t}(v)\right) \leq b|V(F_t)| + 1 - (b + 1) = b|V(F_{t+1})|. \quad \square$$

Proof of Theorem 1.2. By Lemma 4.2 MAKER can WHP locate a $\left(\log^2 \log n, L\right)$-tree T^* in $\mathcal{K}$. She first builds a subtree $T \subset T^*$ with at least $\alpha^{\log^2 \log n} \gg 2\log^4 n$ vertices of degree at least $b + 3$ in $\mathcal{K}$, which she can do by Lemma 4.1. She then proceeds to build T naively by claiming in every move an arbitrary free edge in ∂T as long as possible. MAKER can no longer proceed with her strategy only when BREAKER has claimed the entire boundary of T, and in particular $|\partial T| \leq b|V(T)|$. When this happens, consider the subgraph $F \subset \mathcal{K}$ induced by $V(T)$. Clearly $V(F) = V(T)$ and thus $\partial F = \partial T$. Now F satisfies

$$2|E(F)| = \sum_{v \in V(F)} \deg_F(v) = \sum_{v \in V(F)} \deg_{\mathcal{K}}(v) - |\partial F|$$
$$\geq (b + 2)|V(T)| + 2\log^4 n - b|V(T)| = 2\left(|V(F)| + \log^4 n\right),$$

meaning $\text{exc}(F) \geq \log^4 n$. By Claim 4.4 the 2-core C of F has large excess and small boundary, namely $\text{exc}(C) \geq \text{exc}(F) \geq \log^4 n$ and $|\partial C| \leq b|V(C)|$. Finally, by Lemma 4.3, WHP C has at least εn vertices and thus

$$|V(T)| = |V(F)| \geq |V(C)| \geq \varepsilon n,$$

establishing the theorem. $\square$

4.1 Proof of Lemma 4.1: accumulating heavy vertices

An (N, L)-tree is *simple* if its root is k-light, and all k-heavy vertices in the tree have exactly k children. Clearly, any (N, L)-tree contains a simple (N, L)-tree as a subtree, so we may assume without loss of generality that T^* is simple. Note that w, the root of T^*, has degree $k - 1$, every other k-light vertex but the leaves has degree k, and all k-heavy vertices of T^*, to which we refer from now on simply as heavy, have degree $k + 1$. In this subsection $l(v)$ denotes the level of any vertex $v \in V(T^*)$; in particular, $l(w) = 0$ and $l(v) = N$ for every leaf v. Whenever we refer to an edge $uv \in E(T^*)$ we assume $l(u) < l(v)$, and define the level of uv to be $l(v)$.

We now describe MAKER's strategy $\mathcal{S}_M$. Throughout the game, MAKER's graph is a single tree $T = T^{(r)} \subset T^*$, where $T^{(r)}$ denotes her tree after the rth round. Initially, $T^{(0)}$ consists of w solely. In each move, as long as there exist free edges of level smaller than N in ∂T, MAKER enlarges her tree by claiming one of these edges, selecting an arbitrary edge of minimum level. The game stops when MAKER cannot proceed with this strategy, i.e., when all edges in ∂T have already been claimed by BREAKER (note that by definition none of them could have been previously claimed by MAKER), except perhaps for some edges in level N.

Given $\mathcal{S}_M$, we may assume that BREAKER only claims edges from ∂T as well. Indeed, suppose that BREAKER, according to his strategy, wishes to claim an edge $uv \notin \partial T$ in one of his steps. He can claim instead the (unique) edge $u'v' \in \partial T$ such that v' is an ancestor of v. By $\mathcal{S}_M$, this prevents MAKER from claiming any edge in the subtree of T^* rooted at v', and particularly the edge uv. If the edge $u'v'$ was already claimed by BREAKER, then he claims an arbitrary free edge from ∂T (if none of those exists the game ends anyway). It is evident that BREAKER cannot be harmed from this modification of his strategy, and our assumption is justified.

We now assume that MAKER plays second and follows $\mathcal{S}_M$ (it is trivial to see that she can do so), and show that it is a winning strategy for her, that is, we show that when the game is stopped, her tree T contains sufficiently many heavy vertices. Before doing so we need some additional terminology. First, since both players only claim edges from ∂T by assumption, we naturally redefine free edges to be unclaimed edges from ∂T (instead of all unclaimed edges in T^*) for the remainder of this subsection. For $j = 1, 2, \ldots, N-1$, level j is *complete* when no free edges remain in level j or less. An edge $uv \in E(T)$ *survives* in level j for $j \geq l(v)$ if T contains a level j descendant of v. Finally, we define $C = C(k, L) = 1 + k^{L+1}$.

Claim 4.5. For $r \geq 0$ let h_r denote the number of heavy vertices in $T^{(r)}$, and for $r \geq 1$ let f_r denote the number of free edges before MAKER's rth move. Then $f_{r+1} = h_r + 1$ for every $r \geq 0$.

Proof. Since T^* is simple, and since MAKER never claims an edge in level N, it follows that

$$\left| \partial T^{(r)} \right| = \sum_{v \in V\left(T^{(r)} \right)} \deg_{T^*}(v) - 2 \left| E\left(T^{(r)} \right) \right| = (k-1) + rk + h_r - 2r.$$

In each of his first $r + 1$ moves BREAKER claims $b = k - 2$ edges, each of them incident with some $v \in V(T)$, so $f_{r+1} = \left| \partial T^{(r)} \right| - (r+1)(k-2) = h_r + 1$. $\qquad \square$

An immediate corollary of Claim 4.5 is that at the end of the game T is of height $N - 1$, and level $N - 1$ is complete. Indeed, since there exist free edges before each of MAKER's moves, the game stops only when all free edges are in level N. Having a free edge in level N means that T must have reached level $N - 1$.

Claim 4.6. *Let s and $j < N - L$ be two integers, and assume that there are s free edges in level j right before* MAKER *claims her first edge in level j. Then at least s/C of them will be claimed by* MAKER *and survive in level $j + L$.*

Proof. Consider first the situation right before MAKER claims her first edge in level j, and denote the s free edges in this level by $u_1 v_1, \ldots, u_s v_s$ (the parents u_i are not necessarily distinct, but the children v_i are). For each $1 \le i \le s$, let $T_i \subset T^*$ be the subtree rooted at u_i consisting of the edge $u_i v_i$ and the subtree of T^* rooted at v_i of height L. Note that all free edges are in level j at this point by $\mathcal{S}_{\mathrm{M}}$. By the strategies of MAKER and BREAKER, it follows that all edges in each T_i are still unclaimed, and that exactly one of them, namely $u_i v_i$, is considered free in our new terminology.

Now let us examine the game when level $j + L$ is complete. Denote by $M \subseteq \{1, 2, \ldots, s\}$ the set of indices of the edges that survived in level $j + L$ of T, and by B its complement, so $|M| + |B| = s$. We need to show that $|M| > s/C$.

Let m_i and b_i denote the number of steps that were played in T_i by MAKER and BREAKER, respectively. Recalling that T^* is simple, for every $i = 1, 2, \ldots, s$, we have the trivial bound

$$m_i \le |E(T_i)| \le \sum_{t=0}^{L} k^t = \frac{k^{L+1} - 1}{k - 1} < \frac{k^{L+1}}{k - 2}.$$

Now let $i \in B$. Since MAKER did not reach level L in T_i, i.e., did not reach any of its leaves, it follows by the assumption on BREAKER's strategy that every step MAKER made in T_i increased the number of free edges in this tree by at least $k - 2$. Since no free edges remain in T_i, and there was one free edge there at the beginning of the analysis, it follows that $b_i \ge m_i(k - 2) + 1$.

During this analysis, which begins with MAKER's move, she only plays in $T_1, \ldots, T_s$. Thus $\sum_{i=1}^{s} b_i \le (k - 2) \sum_{i=1}^{s} m_i$. The left hand side can be bounded from below by

$$\sum_{i=1}^{s} b_i \ge \sum_{i \in B} b_i \ge \sum_{i \in B} (m_i(k - 2) + 1) = (k - 2) \sum_{i \in B} m_i + |B|,$$

while the right hand side can be bounded from above by

$$(k - 2) \sum_{i=1}^{s} m_i = (k - 2) \left(\sum_{i \in B} m_i + \sum_{i \in M} m_i \right) < (k - 2) \sum_{i \in B} m_i + |M| k^{L+1}.$$

Putting it all together, we get $|B| < |M| k^{L+1}$, which implies $s = |M| + |B| < C|M|$, thus the proof is complete. $\qquad \square$

Claim 4.7. *For every $0 \le i \le \lfloor N/(C(L+1)) \rfloor$, when level $iC(L + 1)$ is complete, T contains at least $(2^i - 1)C$ heavy vertices.*

Proof. We prove by induction on i. The claim holds trivially for $i = 0$. Assuming it holds for i, we show that it holds for $i + 1$ as well. Let $0 \le j < C$ and write $J_j^i = (iC + j)(L + 1)$. By the induction hypothesis and by Claim 4.5, right before MAKER claims her first edge in level $J_j^i + 1 > iC(L + 1)$ there are at least $(2^i - 1)C + 1$ free edges, all of them in level $J_j^i + 1$ by $\mathcal{S}_M$. By Claim 4.6, at least 2^i of these free edges will be claimed by MAKER and survive in level J_{j+1}^i, resulting in at least 2^i vertex disjoint tree paths of length L in T, each of them containing at least one heavy vertex by the property of T^*. It follows that T contains at least $2^i C$ heavy vertices between levels $J_0^i + 1$ and J_0^{i+1}. By the induction hypothesis T also contains at least $(2^i - 1)C$ heavy vertices until level J_0^i, and the claim holds. $\qquad \square$

The game ends when level $N-1$ is complete, and T then contains by Claim 4.7 at least $(2^{\lfloor (N-1)/(C(L+1)) \rfloor} - 1)C > \alpha^N$ heavy vertices for an appropriate $\alpha = \alpha(k, L) > 1$, establishing Lemma 4.1.

5 Technical Background

In this section we describe the technical background required for Section 6.

5.1 The configuration model

We begin this section with a description of the so-called *configuration model*, introduced by Bollobás [4], which is extremely useful for generating random graphs with a given degree sequence.

Given two positive integers n and m, fix an arbitrary degree sequence $\vec{d} \in \mathcal{D}_{n,2m}$. Let $V = \{v_1, \ldots, v_n\}$ be a set of vertices, where each v_i is incident with d_i labelled half-edges. Let W denote the set of all half-edges, and let F be a partition of W into m pairs; such a partition, which may also be viewed as a perfect matching of the half-edges, is called a *configuration*. Note that there are exactly $(2m-1)!!$ configurations for $\vec{d}$. By forming an edge from any two half-edges which belong to some pair in F, we obtain a multigraph $H = H(F)$ on the vertex set V, such that $d_H(v_i) = d_i$.

Let $\Omega^*(n, \vec{d}) = \{H(F) \mid F \text{ is a configuration for } \vec{d}\}$ be the set of all multigraphs on n labelled vertices with degree sequence $\vec{d}$, and let $\mathcal{G}^*(n, \vec{d})$ be the probability space of $\Omega^*(n, \vec{d})$ when F is chosen uniformly at random from all possible configurations for $\vec{d}$. Let $\Omega(n, \vec{d})$ be the set of all simple graphs in $\Omega^*(n, \vec{d})$, and let $\mathcal{G}(n, \vec{d})$ be the uniform distribution over $\Omega(n, \vec{d})$. We will make use of the following theorem, due to Frieze and Karoński [9].

Theorem 5.1 ([9, Theorem 10.3]). *Let* $\vec{d} = (d_1, \ldots, d_n)$ *and assume that* $\Delta(\vec{d}) \leq n^{1/6}$ *and* $\sum_{i=1}^{n}[d_i]_2 = \Omega(n)$. *Then for any multigraph property* $\mathcal{P}$

$$\Pr_{G \sim \mathcal{G}(n,\vec{d})}[G \in \mathcal{P}] \leq (1 + o(1))e^{\lambda(\lambda+1)} \cdot \Pr_{G^* \sim \mathcal{G}^*(n,\vec{d})}[G^* \in \mathcal{P}],$$

where $\lambda = \lambda(\vec{d}) = \frac{1}{2}\sum_{i=1}^{n}[d_i]_2 / \sum_{i=1}^{n} d_i$.

5.2 Exploring $\mathcal{G}(n, c/n)$ via a Poisson branching process

The main ingredient in the proof of Theorem 1.4 is coupling local behavior in $\mathcal{G}(n, c/n)$ with an appropriate branching process. We describe the coupling quickly, borrowing much of the notation from [25]. We also refer the reader to [7, 16]. Recall that $\Psi_j(\lambda)$, $\Psi_{\geq j}(\lambda)$ and $\Psi_{<j}(\lambda)$ denote the probabilities of a Poisson(λ) random variable being equal to j, at least j, or less than j, respectively.

Let X_c be a Galton–Watson branching process that starts with a single particle x_0 in generation zero, where the number of children of each particle is an independent Poisson(c) random variable. Define a sequence $B_0 \supseteq B_1 \supseteq \cdots$ of events: for an integer $t \geq 0$, let $B_t = B_t(c)$ be the event that X_c contains a complete $(k-1)$-ary tree of height t rooted at x_0, and let $B = \bigcap_{t \geq 0} B_t = \lim_{t \to \infty} B_t$ be the event that X_c contains an infinite $(k-1)$-ary tree rooted at x_0. Denote the probability of B_t by β_t and the probability of B by $\beta = \lim_{t \to \infty} \beta_t$. Then $\beta_0 = 1$. Also, each particle in the first generation of X_c has probability β_t of having property B_t, independently, so the number of such particles is distributed Poisson($c\beta_t$). Thus, $\beta_{t+1} = \Psi_{\geq k-1}(c\beta_t)$.

Since β_t is decreasing in t, this suggests that $\beta = \beta(c)$ should be the maximum solution to the equation $x = \Psi_{\geq k-1}(cx)$. (This is true, though we do not need to prove it here.)

Recall c_k, defined in Section 1 as the threshold for the appearance of a nonempty k-core in $\mathcal{G}(n, c/n)$. In view of Claim 2.2, the local neighbourhood of any vertex v is usually a tree, a relation that has often been used in arguments about the giant component of $\mathcal{G}(n, c/n)$. Assuming that the k-core is linear in size, it follows that for a typical vertex v in the k-core, for any fixed t, the t-neighbourhood of v will contain complete $(k-1)$-ary trees of height t rooted at each of k different neighbours of v (such that the trees do not contain v). We would expect such trees to exist with positive probability at any neighbour of v whenever $\beta(c) > 0$ in the corresponding branching process X_c. When that happens, there should be a positive probability that v has at least k such neighbours. One can then hope that, at this point, the branches join up into a giant k-core, which suggests that

$$c_k = \inf \{c : \beta(c) > 0\}. \tag{2}$$

This was proved rigorously in [23, Theorems 1 and 2].

Given a graph G with degree sequence $(d_1, \ldots, d_n)$, the *degree histogram* of G is the sequence $D = (D_0, D_1, \ldots)$ such that $D_j = |i : d_i = j|$ for all j.

Let us describe the likely degree histogram D^t of the graph G_t obtained after t steps of the peeling process. For $t = 0$ the binomial degree distribution of $G_0 \sim \mathcal{G}(n, c/n)$ is asymptotically $\mathrm{Poisson}(c)$. In particular, it is easy to see (e.g., via the second moment method) that WHP

$$D_j^0(n) = \begin{cases} (1 + o(1)) \Psi_j(c) n, & j = o(\log n / \log\log n); \\ \Theta(\Psi_j(c) n), & j = \Theta(\log n / \log\log n); \\ 0, & j = \omega(\log n / \log\log n). \end{cases} \tag{3}$$

Now, for integers $j \geq 0$ and $t > 0$ let

$$\delta_j^t = \Psi_j(c\beta_t) \Psi_{\geq k-j}(c\beta_{t-1} - c\beta_t). \tag{4}$$

For the degree histogram after a fixed number of steps of the k-peeling process, the following was proved by Achlioptas and Molloy as a special case (2-uniform hypergraphs) of [1, Lemma 42]. Here, they treated the vertex deletion version of the peeling process.

Claim 5.2. *For fixed $t \geq 0$ and $j \geq 1$ we have WHP $D_j^t(n) = (1 + o(1)) \delta_j^t n$.*

For completeness, we show how to derive this result if we assume, for reasons as discussed above, that the neighbourhood of a vertex v is generated by the branching process X_c. Let us say a vertex *survives* t steps of the peeling process if it has degree at least 1 after t steps. A vertex v with degree r after $t-1$ steps will have degree $j \geq 1$ after t steps if and only if $r \geq k$ and exactly j of those r neighbours survive step t. Each of those j neighbours is the root vertex of a complete $(k-1)$-ary tree of height t, which occurs with probability β_t independently for each of them. The other $r - j \geq k - j$ neighbours have such a tree of height $t-1$ but not t, which occurs with probability $\beta_{t-1} - \beta_t$. Since the number of neighbours of v is distributed as $\mathrm{Poisson}(c)$, the numbers of neighbours of these two types are distributed as $\mathrm{Poisson}(c\beta_t)$ and $\mathrm{Poisson}(c\beta_{t-1} - c\beta_t)$ respectively. As $r \geq k - j$, this implies that v has degree $j \geq 1$ with probability δ_j^t.

By linearity of expectation, we have $\mathbb{E}[D_j^t(n)] = (1 + o(1)) \delta_j^t n$. A similar calculation for a pair of vertices u, v shows that $\mathrm{Var}[D_j^t(n)] = o(\mathbb{E}[D_j^t(n)]^2)$. Since $\mathbb{E}[D_j^t(n)] \to \infty$, this establishes the sharp concentration of $D_j^t(n)$ by Chebyshev's inequality.

Remark. For a vertex of degree 0 in G_t we have two options. If the vertex had degree at least k in G_{t-1}, then all of its neighbours must be 'peeled' in the next step and the above analysis applies. Otherwise, less than k of its neighbours survived $t-1$ peeling steps, which caused the vertex itself to be 'peeled'. Such an argument leads to the conclusion that WHP

$$D_0^t(n) = (1 + o(1)) \left(\delta_0^t + \Psi_{<k}(c\beta_{t-1}) \right) n. \tag{5}$$

The same conclusion can be shown to follow Claim 5.2 using $\sum_j D_j^t = n$ and the fact that the degree of a random vertex in $\mathcal{G}(n, c/n)$ is bounded in probability.

5.3 Degree histogram of the k-core

Considering the peeling process of Section 5.2, the limit of the formula in Claim 5.2 for large t is suggestive of the asymptotics of the degree histogram of the k-core $\mathcal{K}$ of $G \sim \mathcal{G}(n, c/n)$, but it is difficult to make this approach rigorous. We quote [7] for a convenient source of the following two lemmas.

With β defined as in Section 5.2, we define $\mu_c = c\beta$. We let $\hat{n}$ denote the number of vertices in $\mathcal{K}$, and $\hat{m}$ the number of edges.

Lemma 5.3. *The number of vertices in $\mathcal{K}$ WHP satisfies*

$$\hat{n} = (1 + o(1)) \Psi_{\geq k}(\mu_c) n, \tag{6}$$

and the number of edges in $\mathcal{K}$ WHP satisfies

$$\hat{m} = (1 + o(1)) \frac{1}{2} \mu_c \Psi_{\geq k-1}(\mu_c) n. \tag{7}$$

Lemma 5.4. *WHP the degree distribution in $\mathcal{K}$ is asymptotically that of $Z_k(\mu_c)$, i.e., the k-truncated Poisson with parameter μ_c. Equivalently, the number of vertices of degree j is 0 for $j < k$, whilst for fixed $j \geq k$ it is*

$$(1 + o(1)) \Pr[Z_k(\mu_c) = j]\hat{n}. \tag{8}$$

We will also find an alternative expression for the average degree useful. For every $\mu > 0$ and $j \geq k$ we have

$$\frac{\Pr[Z_k(\mu) = j]}{\Pr[Z_{k-1}(\mu) = j-1]} = \frac{\Psi_j(\mu)/\Psi_{\geq k}(\mu)}{\Psi_{j-1}(\mu)/\Psi_{\geq k-1}(\mu)}$$
$$= \frac{e^{-\mu}\mu^j/j!}{e^{-\mu}\mu^{j-1}/(j-1)!} \cdot \frac{\Psi_{\geq k-1}(\mu)}{\Psi_{\geq k}(\mu)} = \frac{\mu}{j} \cdot \frac{\Psi_{\geq k-1}(\mu)}{\Psi_{\geq k}(\mu)},$$

so, denoting the average degree in $\mathcal{K}$ by $\hat{d} := 2\hat{m}/\hat{n}$, for every $j \geq k$ we have WHP

$$\hat{d} = \frac{2\hat{m}}{\hat{n}} = (1 + o(1)) \frac{\mu_c \Psi_{\geq k-1}(\mu_c) n}{\Psi_{\geq k}(\mu_c) n} = (1 + o(1)) \frac{j \Pr[Z_k(\mu_c) = j]}{\Pr[Z_{k-1}(\mu_c) = j-1]}. \tag{9}$$

There is one more fact that we will need about the joint degree distribution of vertices in the k-core. Let $\mathcal{G}(n, m)$ denote the Erdős–Rényi random graph model, which is the uniform distribution over all graphs with n vertices and m edges. For a fixed k, let $\mathcal{K}(n, m, k)$ be the probability space

of graphs with minimum degree at least k, distributed as the k-core of $\mathcal{G}(n, m)$, and similarly define $\mathcal{K}(n, c, k)$ for $\mathcal{G}(n, c/n)$. When defining these graphs, we only consider the vertices in the k-core, so the vertex numbering is compressed into the range $[1, \hat{n}]$, where $\hat{n} \leq n$ is the (random) number of vertices in the k-core, while maintaining their order from the original graph.

For the $\mathcal{K}(n, m, k)$ case, a restatement of [7, Corollary 1] gives the following approximation to the distribution of its degree sequence. Let $\mathcal{M}(\hat{n}, \hat{m})$ denote the probability space of sequences $(X_1, \ldots, X_{\hat{n}})$ summing to $2\hat{m}$ with the multinomial distribution, that is, for every $(d_1, \ldots, d_{\hat{n}}) \in \mathcal{D}_{\hat{n}, 2\hat{m}}$, the probability that $X_i = d_i$ for all i is $(2\hat{m})!/(\hat{n}^{2\hat{m}} \prod d_i!)$. Let $\mathcal{M}(\hat{n}, \hat{m}, k)$ denote the same probability space, but conditioned upon $X_i \geq k$ for all i.

Proposition 5.5. *Let $k \geq 3$ and $c > c_k$ be fixed, and $m = (1 + o(1))cn/2$. Let H_n be any event in the probability space defined by the random vector distributed as the degree sequence of $\mathcal{K}(n, m, k)$. Suppose that whenever $\hat{n}$ and $\hat{m}$ have the asymptotic behaviour given in (6) and (7) respectively, it follows that for a given sequence P_n we have*

$$\Pr_{\mathcal{M}(\hat{n}, \hat{m}, k)}(H_n) < P_n.$$

Then $\Pr(H_n) = O(P_n) + o(1)$.

Next, let $\mathcal{P}(\hat{n}, \hat{m}, k)$ denote the probability space of sequences consisting of $\hat{n}$ independent copies of $Z_k(\lambda)$, where λ is chosen so that

$$\mathbb{E}\left[Z_k(\lambda)\right] = \frac{2\hat{m}}{\hat{n}}. \tag{10}$$

As observed in the proof of [7, Lemma 1], the probability that the sum of $\hat{n}$ copies of $Z_k(\lambda)$ is $2\hat{m}$ is $\Omega(1/\sqrt{\hat{n}})$. It follows that we may replace $\Pr_{\mathcal{M}(\hat{n}, \hat{m}, k)}$ by $\Pr_{\mathcal{P}(\hat{n}, \hat{m}, k)}$ in the above proposition, as long as we replace $O(P_n)$ by $O(\sqrt{n} \cdot P_n)$. We may also replace $\mathcal{K}(n, m, k)$ by $\mathcal{K}(n, c, k)$ where $c = 2m/n$, using the well-known strong connection between $\mathcal{G}(n, m)$ and $\mathcal{G}(n, p)$ in this case. That is, we have the following.

Theorem 5.6. *Let $k \geq 3$ and $c > c_k$ be fixed, and let H_n be any event in the probability space defined by the random vector distributed as the degree sequence of $\mathcal{K}(n, c, k)$. Suppose that whenever $\hat{n}$ and $\hat{m}$ have the asymptotic behaviour given in (6) and (7) respectively, it follows that*

$$\Pr_{\mathcal{P}(\hat{n}, \hat{m}, k)}(H_n) < P_n.$$

Then $\Pr(H_n) = O(\sqrt{n} \cdot P_n) + o(1)$.

6 Technical Proofs

In this section we sketch the proof of Theorem 1.4 (in Section 6.1), and prove Lemma 4.2 (in Section 6.2) and Lemma 4.3 (in Section 6.3). We first provide some useful minor results.

Let $\hat{n}$ and $\hat{m}$ denote the number of vertices and edges in $\mathcal{K}$, respectively. Since in the proof of Lemma 4.3 we work with fixed degree sequences, we wish to characterise a set of sequences which contains the degree sequences typical for $\mathcal{K}$, and is in particular compliant with the typical asymptotic values of $\hat{n}$ and $\hat{m}$, as well as the typical degree histogram of $\mathcal{K}$, stated in Section 5.3. Note that the following definition technically and tacitly applies to a fixed sequence of degree sequences $\vec{d}$, one for each n, since it describes an asymptotic property of the degree sequence.

Definition 6.1. *A degree sequence $\vec{d} \in \mathcal{D}_{\hat{n},2\hat{m}}$ is* proper *(with respect to the underlying parameters n,k and c) if $\hat{n}$ and $\hat{m}$ satisfy (6) and (7) respectively, $\Delta(\vec{d}) \leq \log n$, $\sum [d_i]_2 = \Theta(n)$, and the degree distribution follows the asymptotics in (8). In this case, we have for each fixed integer $j \geq 0$ that*

$$
|\{i \mid d_i = j\}| = \begin{cases} 0, & j < k; \\ (1 + o(1)) \, \Psi_j(\mu_c) \, n & j \geq k. \end{cases}
$$

Given the degree sequence of $\mathcal{K}$, we perform some of the analysis in the proof of Lemma 4.3 by using the configuration model. In order to make the results we obtain via this model applicable, we need the following immediate corollary of Theorem 5.1.

Corollary 6.2. *Let $f(n)$ be any function satisfying $f(n) \to \infty$. Then for n sufficiently large*

$$
\Pr_{G \sim \mathcal{G}(n,\vec{d})} [G \in \mathcal{P}] \leq f(n) \cdot \Pr_{G^* \sim \mathcal{G}^*(n,\vec{d})} [G^* \in \mathcal{P}]
$$

for any proper degree sequence $\vec{d} \in \mathcal{D}_{\hat{n},2\hat{m}}$ and any multigraph property $\mathcal{P}$.

To show that we may restrict to proper degree sequences, and other kinds to be defined below, we first show that the upper tail of the sum of squared degrees is negligible in $\mathcal{G}(n,p)$, as follows.

Lemma 6.3. *Let $(D_0, D_1, \ldots)$ be the degree histogram of $G \in \mathcal{G}(n,c/n)$. Then WHP*

(a) *$D_j = 0$ for all $j \geq \log n$;*

(b) *for every $\epsilon > 0$ there exists an integer j_0 such that $\sum_{j \geq j_0} [j]_2 D_j / n < \epsilon$.*

Proof. Part (a) is well-known and follows from the third case in (3). Part (b) also follows by standard methods, for instance as follows. First, note that we may assume that (a) holds and hence restrict the summation to $j < \log n$. Standard computations show for such j that $\mathbb{E}[D_j] < c^j n / j!$ and the variance of D_j is $O(n \log^2 n)$. Chebyshev's inequality, together with the union bound, then implies that WHP $D_j < \mathbb{E}[D_j] + n^{3/4}$ for all $j < \log n$. The result now follows, given the above bound on $\mathbb{E}[D_j]$. $\qquad \square$

Note that if G satisfies Part (b) of the lemma, then $\sum [d_i]_2 = O(n)$. Indeed, let j_0 such that $\sum_{j \geq j_0} [j]_2 D_j / n < 1$. Then all vertices of G of degree at least j_0 contribute at most n to $\sum [d_i]_2$, while all other vertices contribute at most $[j_0]_2 n$. Since the bounds on degree counts of $G \in \mathcal{G}(n,c/n)$ are also bounds for its core $\mathcal{K}$, an immediate consequence of Lemmas 5.3, 5.4 and 6.3 is the following.

Corollary 6.4. *The degree sequence of the core $\mathcal{K}$ is WHP proper.*

The following claim, related to moments of the Poisson distribution, is used in the proof of Theorem 1.4.

Claim 6.5. *For real numbers $\mu \geq \lambda \geq 0$ and integers $k \geq \ell \geq 0$ we have*

$$
\sum_{j=0}^{\infty} [j]_\ell \, \Psi_j(\lambda) \, \Psi_{\geq k-j}(\mu - \lambda) = \lambda^\ell \Psi_{\geq k-\ell}(\mu).
$$

Proof. First we prove the claim for $\ell = 0$, that is

$$\sum_{j=0}^{\infty} \Psi_j(\lambda) \, \Psi_{\geq k-j}(\mu - \lambda) = \Psi_{\geq k}(\mu). \tag{11}$$

Let $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu - \lambda)$ be independent. Then $X + Y \sim \text{Poisson}(\mu)$ and thus

$$\sum_{j=0}^{\infty} \Psi_j(\lambda) \, \Psi_{\geq k-j}(\mu - \lambda) = \sum_{j=0}^{\infty} \Pr[X = j] \Pr[Y \geq k - j] = \sum_{j=0}^{\infty} \Pr[X = j \wedge Y \geq k - j]$$
$$= \Pr[X + Y \geq k] = \Psi_{\geq k}(\mu).$$

Now, using the fact that $[j]_\ell = 0$ for every $0 \leq j < \ell$ we get that the claim holds for all ℓ:

$$\sum_{j=0}^{\infty} [j]_\ell \, \Psi_j(\lambda) \, \Psi_{\geq k-j}(\mu - \lambda) = \sum_{j=\ell}^{\infty} [j]_\ell \, \Psi_j(\lambda) \, \Psi_{\geq k-j}(\mu - \lambda)$$
$$\overset{(1)}{=} \lambda^\ell \sum_{j=\ell}^{\infty} \Psi_{j-\ell}(\lambda) \, \Psi_{\geq k-j}(\mu - \lambda)$$
$$= \lambda^\ell \sum_{j=0}^{\infty} \Psi_j(\lambda) \, \Psi_{\geq (k-\ell)-j}(\mu - \lambda)$$
$$\overset{(11)}{=} \lambda^\ell \Psi_{\geq k-\ell}(\mu). \qquad \square$$

The following lemma, bounding from above the probability of a truncated Poisson random variable achieving its minimum, is used in the proofs of Lemmas 4.2 and 4.3.

Lemma 6.6. *For $k \geq 3$ and $c > c_k$ let*

$$\delta = \delta(k, c) = \frac{1}{2}\left(1 - \frac{c_k}{c}\right) \frac{k-2}{k-1}.$$

Then

$$\Pr[Z_{k-1}(\mu_c) = k - 1] < \frac{1 - 2\delta}{k-1}.$$

Proof. Let

$$F(\mu) := \frac{\Psi_{\geq k-1}(\mu)}{\Psi_{k-1}(\mu)} = \frac{1}{\Pr[Z_{k-1}(\mu) = k-1]},$$

so we need to show that $k - 1 < (1 - 2\delta) F(\mu_c)$.

Let $h(\mu) = \mu / \Psi_{\geq k-1}(\mu)$, defined for all $\mu > 0$. Recall that $\beta(c)$ was determined to be the maximum solution of $x = \Psi_{\geq k-1}(cx)$, which enables us to express c_k as in (2). In terms of h, we can define μ_c as the maximum solution of $h(\mu) = c$, which exists if and only if $\beta(c) > 0$. We can therefore express c_k again in the following way:

$$c_k = \inf\{c \mid \exists \mu > 0 \; h(\mu) = c\} = \inf\{h(\mu) \mid \mu > 0\}.$$

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Clearly h is differentiable and its derivative is

$$
\begin{aligned}
h'(\mu) &= \frac{1}{\Psi_{\geq k-1}(\mu)} \left(1 - \mu \frac{(\Psi_{\geq k-1})'(\mu)}{\Psi_{\geq k-1}(\mu)} \right) \\
&= \frac{1}{\Psi_{\geq k-1}(\mu)} \left(1 - \mu \frac{\Psi_{k-2}(\mu)}{\Psi_{\geq k-1}(\mu)} \right) \\
&= \frac{1}{\Psi_{\geq k-1}(\mu)} \left(1 - (k-1) \frac{\Psi_{k-1}(\mu)}{\Psi_{\geq k-1}(\mu)} \right) \\
&= \frac{1}{\Psi_{\geq k-1}(\mu)} \left(1 - \frac{k-1}{F(\mu)} \right) \\
&< \frac{1}{\Psi_{\geq k-1}(\mu)}.
\end{aligned}
\tag{12}
$$

Note that $h'(\mu)$ is positive if and only if $F(\mu) > k - 1$, and since F is an increasing function approaching 1^+ and ∞ as μ approaches 0^+ and ∞, respectively, the infimum c_k of h is actually its minimum, attained at a unique point μ_{c_k}. In particular, h is increasing for $\mu > \mu_{c_k}$ and since $h'(\mu_{c_k}) = 0$ we have

$$
F(\mu_{c_k}) = k - 1.
\tag{13}
$$

By the mean value theorem (applied to h) there exists some $\tilde{\mu} \in (\mu_{c_k}, \mu_c)$ such that

$$
\begin{aligned}
\frac{c - c_k}{\mu_c - \mu_{c_k}} &= \frac{h(\mu_c) - h(\mu_{c_k})}{\mu_c - \mu_{c_k}} = h'(\tilde{\mu}) \\
&\stackrel{(12)}{<} \frac{1}{\Psi_{\geq k-1}(\tilde{\mu})} < \frac{1}{\Psi_{\geq k-1}(\mu_{c_k})} = \frac{h(\mu_{c_k})}{\mu_{c_k}} = \frac{c_k}{\mu_{c_k}},
\end{aligned}
$$

where the second inequality is due to the monotonicity of $\Psi_{\geq k-1}$ in μ. Rearranging, we get

$$
\mu_{c_k} < (c_k/c) \, \mu_c.
\tag{14}
$$

Recall that F is an increasing function of μ, so

$$
\begin{aligned}
1 - 2\delta &= 1 - \left(1 - \frac{c_k}{c} \right) \left(1 - \frac{1}{k-1} \right) \\
&\stackrel{(13)}{=} 1 - \left(1 - \frac{c_k}{c} \right) \left(1 - \frac{1}{F(\mu_{c_k})} \right) \\
&> 1 - \left(1 - \frac{c_k}{c} \right) \left(1 - \frac{1}{F(\mu_c)} \right) \\
&= \frac{1 + (c_k/c) \, (F(\mu_c) - 1)}{F(\mu_c)}.
\end{aligned}
\tag{15}
$$

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Finally, observe that for every $\mu > 0$

$$\begin{aligned}
F(\mu) - 1 &= \frac{\Psi_{\geq k-1}(\mu) - \Psi_{k-1}(\mu)}{\Psi_{k-1}(\mu)} = \frac{\Psi_{\geq k}(\mu)}{\Psi_{k-1}(\mu)} \\
&= \sum_{j=k}^{\infty} e^{-\mu} \frac{\mu^j}{j!} \Big/ \left[e^{-\mu} \frac{\mu^{k-1}}{(k-1)!} \right] \\
&= (k-1)! \mu^{1-k} \sum_{j=1}^{\infty} \frac{\mu^{j+k-1}}{(j+k-1)!} \\
&= \sum_{j=1}^{\infty} \frac{\mu^j}{[j+k-1]_j},
\end{aligned}$$

and thus $F(\alpha\mu) - 1 < \alpha(F(\mu) - 1)$ for every $0 < \alpha < 1$, implying

$$k - 1 \overset{(13)}{=} F\left(\mu_{c_k}\right) \overset{(14)}{<} F\left(\left(c_k/c\right)\mu_c\right) < 1 + \left(c_k/c\right)\left(F\left(\mu_c\right) - 1\right) \overset{(15)}{<} \left(1 - 2\delta\right) F\left(\mu_c\right),$$

which establishes the lemma. $\qquad\square$

6.1 Sketch of proof of Theorem 1.4

Recall that degree histograms were defined in Section 5.2. An *asymptotic degree histogram* is a sequence $D = (D_0, D_1, \ldots)$ of functions $D_j : \mathbb{N} \to \mathbb{N}$ such that $\sum_{j=0}^{\infty} D_j(n) = n$ and $\sum_{j=0}^{\infty} j D_j(n)$ is even for all n. For a given asymptotic degree histogram D, denote by $\Omega(n, D)$ the set of all simple graphs on n vertices with degree histogram $(D_0(n), D_1(n), \ldots)$. If $\Omega(n, D) \neq \varnothing$ for all $n \geq 1$, D is *feasible*; in this case, let $\mathcal{G}(n, D)$ be the uniform distribution over $\Omega(n, D)$. A feasible asymptotic degree histogram D is *sparse* if $\sum_{j=0}^{\infty} j D_j(n)/n = \kappa_D + o(1)$ for some constant κ_D, called the asymptotic edge density of $\mathcal{G}(n, D)$; D is *well-behaved* if:

1. There exist constants $\{\delta_j\}_{j=0}^{\infty}$ such that $\lim_{n\to\infty} D_j(n)/n = \delta_j$ for all fixed $j \geq 0$.

2. $\{j(j-2)D_j(n)/n\}_{j=0}^{\infty}$ tends uniformly to $\{j(j-2)\delta_j\}_{j=0}^{\infty}$.

3. $\lim_{n\to\infty} \sum_{j=0}^{\infty} j(j-2)D_j(n)/n$ exists, and the sum uniformly approaches the limit

$$Q_D := \sum_{j=0}^{\infty} j(j-2)\delta_j.$$

Molloy and Reed [21] showed that the sign of Q_D WHP determines the existence of a giant component in $\mathcal{G}(n, D)$:

Lemma 6.7 ([21, Theorem 1]). *Let D be a feasible well-behaved sparse asymptotic degree histogram, and let $\Delta(n) = \max\{j \in \mathbb{N} \mid D_j(n) > 0\}$.*

1. *If $\Delta(n) = o\left(n^{1/4}\right)$ and $Q_D > 0$ then WHP $\mathcal{G}(n, D)$ has a linear-size connected component;*

2. *If $\Delta(n) = o\left(n^{1/8}\right)$ and $Q_D < 0$ then the size of the largest connected component in $\mathcal{G}(n, D)$ is WHP $O\left(\Delta^2(n)\log n\right)$.*

Implicitly, for fixed t (which we will choose later), we regard G_t as a sequence of random graphs, one for each n, so their degree histograms D^t determine an asymptotic degree histogram that is feasible by definition. It is not hard to verify that this is WHP well-behaved, using Claim 5.2 for bounded degrees and Lemma 6.3 for the unbounded degrees. (We omit details.) It is clear that each graph with a given degree histogram is equally likely to appear after t steps of the peeling process. Thus, we can view G_t as drawn from $\mathcal{G}(n, D^t)$.

Applying Claim 6.5 with $\lambda = c\beta_t$, $\mu = c\beta_{t-1}$ and $\ell = 1$, we get that D^t is also sparse, with asymptotic edge density

$$\kappa_t := \kappa_{D^t} = \sum_{j=0}^{\infty} j\delta_j^t = \sum_{j=0}^{\infty} j\Psi_j(c\beta_t)\Psi_{\geq k-j}(c\beta_{t-1} - c\beta_t) = c\beta_t\Psi_{\geq k-1}(c\beta_{t-1}) = c\beta_t^2.$$

We now bound the parameter $Q_t := Q_{D^t}$, by another application of Claim 6.5 with $\lambda = c\beta_t$ and $\mu = c\beta_{t-1}$, but this time with $\ell = 2$. We get

$$\sum_{j=0}^{\infty} [j]_2\,\delta_j^t = \sum_{j=0}^{\infty} [j]_2\,\Psi_j(c\beta_t)\Psi_{\geq k-j}(c\beta_{t-1} - c\beta_t) = (c\beta_t)^2\,\Psi_{\geq k-2}(c\beta_{t-1}) = c\kappa_t\Psi_{\geq k-2}(c\beta_{t-1}),$$

hence

$$
\begin{aligned}
Q_t &= \sum_{j=0}^{\infty} j(j-2)\,\delta_j^t = \sum_{j=0}^{\infty} ([j]_2 - j)\,\delta_j^t \\
&= c\kappa_t\Psi_{\geq k-2}(c\beta_{t-1}) - \kappa_t \\
&= (c\Psi_{\geq k-2}(c\beta_{t-1}) - 1)\,\kappa_t.
\end{aligned}
$$

The decreasing sequence $(\beta_t)_{t\geq 0}$ converges to $\beta = 0$ in the subcritical regime, and $x \mapsto c\Psi_{\geq k-2}(cx)$ is a continuous function, so

$$\lim_{t\to\infty} c\Psi_{\geq k-2}(c\beta_t) = c\Psi_{\geq k-2}(c\beta) = 0.$$

In particular, there exists some constant $t^\dagger$ such that $c\Psi_{\geq k-2}(c\beta_{t^\dagger-1}) < 1$, implying $Q_{t^\dagger} < 0$. Moreover, $\Delta(G_{t^\dagger}) \leq \Delta(G_0) = O(\log n/\log\log n)$ by (3), and the second part of Lemma 6.7 then implies Theorem 1.4.

Remark. By [17, Lemma 6] we have $t^\dagger = \Theta(1/\sqrt{c_k - c})$.

6.2 Proof of Lemma 4.2: the existence of a $\left(\log^2\log n, L\right)$-tree in $\mathcal{K}$

We prove Lemma 4.2 for $L := \lceil\log_{1-\delta}(\delta^2/(2\mu_c))\rceil$, where $\delta < 1/2$ is the constant from Lemma 6.6. Throughout this subsection we let $N = \lceil\log^2\log n\rceil$ and $p_j = \Pr[Z_{k-1}(\mu_c) = j-1]$. In addition, we set $C = (1-\delta)^2/(1-2\delta) > 1$, and let d_0 be the minimal integer satisfying $1 - \sum_{i\leq d_0} p_i/C \leq \delta^2/(2\mu_c)$. Note that L, C and d_0 are all constants depending only on k and c.

We consider an exploration process in the k-core $\mathcal{K}$, attempting to reveal an (N, L)-tree in it, but instead of analysing the exploration process on $\mathcal{K}$ itself, we condition on it having a proper degree sequence $\vec{d}$ (see Definition 6.1) and apply the exploration process to the configuration model for the sequence $\vec{d}$. In view of Lemma 5.3 and Corollary 6.4, Theorem 5.1 implies that it is enough

to show that the multigraph of this configuration model WHP contains an (N, L)-tree, where the convergence implicit in WHP is uniform over all proper degree sequences $\vec{d}$.

The exploration starts with an arbitrary vertex v_0 in this configuration model, and explores its $(2N)$-neighbourhood in DFS manner. In each exploration step, an unmatched half-edge, say x, incident with an exposed vertex at distance at most $2N - 1$ from v_0, is matched to some other half-edge, say y, chosen uniformly at random from the set of all unmatched half-edges. We refer to the vertex containing y as the *next encountered vertex*. The selection of x in each step is arbitrary among those incident with the vertex currently being treated by the DFS algorithm. Initially, v_0 is the only exposed vertex, and whenever a new half-edge y is chosen, its incident vertex (i.e. the next encountered vertex) becomes exposed. Unless that vertex was already exposed, the new edge and vertex are added to the growing DFS tree.

Let T denote the tree resulting from the exploration described above. The root of T is v_0, and all other vertices of T have distance at most $2N$ from v_0 in T. We assign each vertex in T a *type* from $\{0, 1, \ldots, L + 1\}$ in the following manner. The type assignment for a vertex u is performed at the point when the DFS algorithm has finished fully exploring the subtree of T below u and looks to move back to the parent of u (or terminate, if $u = v_0$). First, if any back-edge has been encountered up to this point in the exploration process, u is assigned type $L + 1$. If no back-edge has been encountered, the following rules are applied. If u is a leaf, i.e. in level $2N$, its type is set to 0. Otherwise, u is in level $i < 2N - 1$, and all its children have been assigned types already; denote by $S(u)$ the set of its children of type less than L. Note that $d_T(u) = d_{\mathcal{K}}(u)$ in this case, so we can omit the subscript, and set

$$\text{type}\,(u) = \begin{cases} 0, & |S\,(u)| \geq k \quad \text{ and } d(u) \leq d_0; \\ 1 + \max\{\text{type}\,(v) \mid v \in S\,(u)\}, & |S\,(u)| = k - 1 \text{ and } d(u) \leq d_0; \\ L, & |S\,(u)| < k - 1 \quad \text{ or } d(u) > d_0. \end{cases}$$

For $v \in V(T)$, let $T(v)$ denote the subtree of T consisting of v and all its descendants. Given the types of vertices as defined above, let $T^*(v)$ denote the result of removing from $T(v)$ all subtrees rooted at vertices of type L or $L + 1$. Then for every vertex u in $T^*(v)$, the number of vertices in a longest k-light tree path originating at u is exactly $\text{type}(u) < L$. If $\text{type}(u) = 0$ it simply means that no such paths exist as u itself is not k-light. In particular, we have the following.

Observation 6.8. Let $v \in V(T)$ at level i. If $\text{type}(v) < L$ then $T^*(v)$ is a $(2N - i, L)$-tree.

Let us now take a closer look at the process of matching half-edges. When the first random half-edge (called y in the above description) is chosen, the probability that its incident vertex u has degree j is weighted by a multiplicative factor of j (so-called "degree-biased" selection). Hence, for any fixed $j \geq k$ and a proper degree sequence, we have by (8) that the probability that u has degree j is

$$(1 + o(1))j \Pr[Z_k\,(\mu_c) = j]\hat{n}/2\hat{m} \overset{(9)}{=} (1 + o(1))p_j. \tag{16}$$

As the exploration carries on, the degree sequence of the unmatched half-edges does not represent the degrees of the vertices any more, but their "remaining" degrees, that is, the number of unmatched half-edges incident with each vertex at that point. Of course, this distinction only applies to the exposed vertices. Additionally, this degree sequence contains values smaller than k. To handle these subtleties, we define a new class of sequences, closely related to proper sequences.

For a constant $\eta > 1$, a degree sequence $\vec{d} \in \mathcal{D}_{\hat{n}, 2\hat{m}}$ is η-*normal* (with respect to n, k, c and d_0), if $n \geq \hat{n} \geq \Psi_{\geq k}\,(\mu_c)\,n/2$ and $\hat{m} \geq \mu_c\Psi_{\geq k-1}\,(\mu_c)\,n/4$, its maximum degree is at most $\log n$,

and for every $j \leq d_0$ the degree-biased probability that a half-edge selected uniformly at random belongs to a vertex of (remaining) degree j is between p_j/η and ηp_j. Note that in particular, for every $j \leq d_0$, the number of 'j' entries in any η-normal sequence is at least $2\hat{m}p_j/(j\eta) = \Omega(n)$. It is immediate to see that for any fixed $\eta > 1$, every proper sequence is η-normal for n sufficiently large.

We are finally ready for the main part of the proof. Recall the definition of C from the beginning of this subsection, and let $C' = (C + 1)/2 > 1$.

Claim 6.9. *Assume that $\vec{d}$ is a C'-normal sequence and consider any moment during the exploration process when a random half-edge is about to be chosen. Conditional upon the exploration so far, the probability that the next encountered vertex will eventually have type L is at most δ/μ_c.*

We first show why Claim 6.9 implies Lemma 4.2. Assume $\vec{d}$ is C'-normal and consider any moment at which an unmatched half-edge incident with a vertex at level $N-1$ in T is about to be treated, i.e., the next encountered vertex will belong to level N unless already exposed. At this point the types of all vertices so far encountered at level N have been assigned. We may therefore apply Claim 6.9 to deduce that, conditional on the labels of the previously encountered vertices in level N, the probability that the next one receives type L is at most δ/μ_c. By coupling this process with a sequence of independent Bernoulli trials each with parameter δ/μ_c, we conclude that, for any $t > 0$, the probability that at least t vertices are encountered at level N and all are given type L is at most $(\delta/\mu_c)^t$.

Since $\vec{d}$ is C'-normal, all the degrees are at most $\log n$, and so the number of vertices reached in the exploration is at most $(\log n)^{2N} = \exp\left(O(\log^3 \log n)\right) = o(n^{1/3})$. Consequently, since there are $\Theta(n)$ half-edges altogether, each step of the exploration process chooses an exposed vertex with probability $o(n^{-2/3} \log n)$. Thus, the probability that at least one of the $o(n^{1/3} \log n)$ steps encounters a back-edge is $o(1)$. Since all vertices of $\mathcal{K}$ have degree at least k, it follows that WHP, T has at least $(k-1)^N = \omega(1)$ vertices at level N. From the previous paragraph, the probability that these are all assigned type L is $o(1)$. As there are WHP no back-edges, this implies that WHP some vertex v at level N receives a type less than L. By Observation 6.8, this event implies that $T^*(v)$ is an (N, L)-tree.

One can easily check that the convergence in the above WHP statements is uniform over all C'-normal degree sequences $\vec{d}$. Since WHP $\mathcal{K}$ has a C'-normal degree sequence, Lemma 4.2 follows, and it only remains to prove the claim.

Proof of Claim 6.9. For the given degree sequence $\vec{d} = (d_1, \ldots, d_{\hat{n}}) \in \mathcal{D}_{\hat{n}, 2\hat{m}}$, let $R(\vec{d})$ be the set of all sequences $\vec{d}' = (d_1', \ldots, d_{\hat{n}}') \in \mathcal{D}_{\hat{n}, 2\hat{m}'}$, such that $d_i \geq d_i'$ for all i, and $m - m' \leq n^{1/3} \log n$. By the arguments above, at every step of the exploration, the degree sequence of the unmatched half-edges in $\mathcal{K}$ is some element of $R(\vec{d})$.

Now let $\vec{d}' \in R(\vec{d})$, and for every $j \leq d_0$, let n_j and n_j' denote the number of vertices of degree j in $\vec{d}$ and in $\vec{d}'$, respectively. Since $n_j = \Theta(n)$, and since $d_i \neq d_i'$ for $O(n^{1/3} \log n)$ coordinates, we have $n_j' = (1 + o(1))n_j$. Similarly, $m' = (1 + o(1))m$, and thus $jn_j'/(2m') = (1 + o(1))jn_j/(2m)$. In short, for every $\vec{d}' \in R(\vec{d})$ and $j \leq d_0$, the probability of choosing a half-edge belonging to a vertex of (remaining) degree j, asymptotically equals the probability of the same event for $\vec{d}$. Since, in addition, the probability that a randomly selected half-edge belongs to an already exposed vertex is $o(1)$, and since $C > C'$, we can state the following.

Observation 6.10. At every step of the exploration, conditioning upon the exploration steps taken previously, the probability that the next encountered vertex is unexposed and has degree $j \leq d_0$ is bounded between p_j/C and Cp_j.

To complement this observation, consider any moment during the exploration, let S denote the exploration sequence up to that point, and let $p_{>d_0}(S)$ denote the probability that the next encountered vertex will have degree larger than d_0, conditional on S. Let $p_{>d_0}$ denote the maximum of $p_{>d_0}(S)$, taken over all possible (partial) exploration sequences S. Then by Observation 6.10 and the definition of d_0 (at the beginning of this subsection), we have $p_{>d_0} \leq \delta^2/(2\mu_c)$.

Before proceeding with the proof of the claim we introduce more terminology. We say a vertex v of T has height h if it is in level $2N - h$, i.e. $T(v)$ has height h. The height of an unmatched half-edge belonging to an exposed vertex is the same as the height of that vertex.

Similarly to the definition of $p_{>d_0}$, for every $1 \leq h \leq 2N$ and $0 \leq i \leq L$, let $P_{i,h}$ denote the maximum, over all possible exploration sequences S up to any step in which a half-edge at height h is being matched, of the probability that the next encountered vertex will be assigned type i, conditional on S. Note that this next encountered vertex is at height $h - 1$ unless it was already exposed.

We prove the claim by showing that the following hold for every $1 \leq h \leq 2N$:

(a) $P_{L,h} \leq \delta/\mu_c$;

(b) $P_{i,h} \leq (1 - \delta)^i$ for every $0 \leq i < L$.

Recall that if a back-edge occurs before assigning the type of a vertex u, it is given type $L + 1$, and that otherwise, and if u is also not a leaf, then $d_T(u) = d_{\mathcal{K}}(u)$ and we simply refer to the degree of u with no specification.

Observe that (b) holds trivially for $i = 0$ (for every h), so from now on we assume for simplicity $i > 0$. Since the only positive type a leaf can be assigned is $L + 1$, there is nothing to prove for $h = 1$. We now prove (a) and (b) for $h > 1$ by induction on h, beginning with $h = 2$.

Assume that no back-edge has yet occurred when a vertex u at height 1 is assigned its type. Then every child of u is a leaf of type 0, and there are therefore exactly three options. If $d(u) > d_0$ then type$(u) = L$; if $d(u) = k$, i.e., u has $k - 1$ children in T, then type$(u) = 1$; otherwise, type$(u) = 0$. So when a half-edge at height 2 is being matched, the probability that the next encountered vertex will eventually have type 1 or L is bounded from above by the probability that the vertex will have degree k or larger than d_0, respectively. We therefore immediately obtain (a) since

$$P_{L,2} \leq p_{>d_0} \leq \delta^2/(2\mu_c) \leq \delta/\mu_c.$$

As for (b), we only have to show that $P_{1,2} \leq 1 - \delta$, and this is true since by the above argument, Observation 6.10 and Lemma 6.6 we have

$$P_{1,2} \leq Cp_k < C\frac{1 - 2\delta}{k - 1} = \frac{(1 - \delta)^2}{k - 1} < 1 - \delta.$$

Assume now that (a) and (b) hold for $1 < h - 1 < 2N$; we prove (a) and (b) for h.

Observe that if a vertex v at height $h - 1$ with degree j is assigned type $0 < i < L$ (implying in particular that $j \leq d_0$ by definition of type), then no back-edge has yet been encountered, exactly $j - k$ of v's $j - 1$ children must be assigned type L, and the maximum type among its other $k - 1$ children must be exactly $i - 1$.

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Now consider any moment when a half-edge at height h is matched, and let w denote the next encountered vertex. By Observation 6.10, the probability that w will be an unexposed vertex with degree j is at most Cp_j. Since the exploration is DFS, the bounds $P_{i,h-1}$ (for any $0 \le i \le L$) can be applied to each of the children of w consecutively and conditionally. Thus at any given step in which a half-edge at height h is being matched, the probability that the next encountered vertex will have degree j and type $0 < i < L$, conditional on the previous history of the process, can be bounded from above by

$$f_{i,h}(j) := Cp_j \binom{j-1}{j-k} (P_{L,h-1})^{j-k}(k-1)P_{i-1,h-1}.$$

In the same manner, consider the event that a vertex v at height $h-1$ with degree $j \le d_0$ is assigned type L. (Recall that a vertex will also be assigned type L if it has degree larger than d_0, which happens with probability at most $p_{>d_0}$.) Then no back-edge has yet occurred, and either exactly $j-k$ of v's children are of type L and there is at least one child of type $L-1$, or there are at least $j-k+1$ children of v of type L. So at any moment in which a half-edge at height h is being matched, we can bound from above the probability that the next encountered vertex will have degree $j \le d_0$ and be assigned type L, conditional on the previous history of the process, by

$$f_h(j) := Cp_j \binom{j-1}{j-k} (P_{L,h-1})^{j-k}(k-1)(P_{L,h-1} + P_{L-1,h-1}).$$

Next, we observe that for every $0 < i < L$ and for every $k \le j < d_0$ the following holds:

$$\frac{f_{i,h}(j+1)}{f_{i,h}(j)} = \frac{f_h(j+1)}{f_h(j)} = \frac{p_{j+1}}{p_j} \cdot \frac{\binom{j}{j-k+1}}{\binom{j-1}{j-k}} \cdot P_{L,h-1} = \frac{\mu_c}{j} \cdot \frac{j}{j-k+1} \cdot P_{L,h-1} \le \delta, \qquad (17)$$

where the last inequality follows from the induction hypothesis. We now use Lemma 6.6 and the induction hypothesis to get

$$f_{i,h}(k) = Cp_k(k-1)P_{i-1,h-1} < C(1-2\delta)\,(1-\delta)^{i-1} = (1-\delta)^{i+1},$$

which together with (17) yields

$$P_{i,h} \le \sum_{j=k}^{d_0} f_{i,h}(j) \le \sum_{j=k}^{d_0} f_{i,h}(k)\delta^{j-k} \; < \; \frac{f_{i,h}(k)}{1-\delta} < (1-\delta)^i,$$

establishing (b). To prove (a), we similarly use

$$f_h(k) = Cp_k(k-1)(P_{L,h-1} + P_{L-1,h-1}) < (1-\delta)^2 \left(\frac{\delta}{\mu_c} + (1-\delta)^{L-1} \right)$$

and (17) to obtain

$$\sum_{j=k}^{d_0} f_h(j) \; \le \; \sum_{j=k}^{d_0} f_h(k)\delta^{j-k} \; < \; \frac{f_h(k)}{1-\delta}$$

$$< \; (1-\delta)\frac{\delta}{\mu_c} + (1-\delta)^L = \frac{\delta}{\mu_c} - \left(\frac{\delta^2}{\mu_c} - (1-\delta)^L \right) \le \frac{\delta}{\mu_c} - \frac{\delta^2}{2\mu_c},$$

where the last inequality holds by the choice of L. We conclude that (a) holds by using the bound on $p_{>d_0}$ and the fact that

$$P_{L,h} \le p_{>d_0} + \sum_{j=k}^{d_0} f_h(j). \qquad \square$$

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6.3 Proof of Lemma 4.3: high excess, small boundary 2-cores are WHP linear

Throughout this subsection we denote by $\hat{n}$ and $\hat{m}$ the number of vertices and edges, respectively, in $\mathcal{K}$, the k-core of G, and denote by $\hat{d} = 2\hat{m}/\hat{n}$ the average degree in $\mathcal{K}$. For a subgraph $G' \subset \mathcal{K}$ we write $t = |V(G')|$, $s = |E(G')|$ and $r = \text{exc}(G') = s - t$. We refer to any graph of minimum degree at least 2 as a *2-core*. A 2-core $C \subset \mathcal{K}$ is *bad* if C has large excess $\text{exc}(C) \geq \log^4 n$, small boundary $|\partial C| \leq (k-2)V(C)$, and small size $|V(C)| < \varepsilon n$. Lemma 4.3 claims that WHP $\mathcal{K}$ has no bad 2-cores for some constant $\varepsilon = \varepsilon(k,c) > 0$. We will consider bad 2-cores in $\mathcal{K}$ with exactly t vertices and s edges for pairs $(t,s) \in \mathcal{I}$, where

$$\mathcal{I} = \left\{ (t,s) \mid t < \varepsilon n \ \text{ and } \ t + \log^4 n \leq s \leq \binom{t}{2} \right\}.$$

Observe that in particular $\log^4 n < \binom{t}{2} < t^2$, and so from now on we assume $t > \log^2 n$.

Recall the constant δ from Lemma 6.6, and let $\delta_1 < 1/e$ be constant sufficiently small to ensure that $(1 - \delta/2)\delta_1^{-4\delta_1} < 1 - \delta/4$. Such a δ_1 exists since $x^{-x} \to 1^+$ as $x \to 0^+$. In the proof we separate potential bad 2-cores into two classes: dense (i.e., with excess $r \geq \delta_1 t$) and sparse (i.e., with excess $r \leq \delta_1 t$), and partition $\mathcal{I}$ into $\mathcal{I} = \mathcal{I}_{\text{dense}} \cup \mathcal{I}_{\text{sparse}}$ accordingly.

We are now finally ready to define ε. Since the discussion is restricted to the k-core $\mathcal{K}$, which is WHP of linear size $\hat{n} = (1 + o(1))\Psi_{\geq k}(\mu_c)n$, it will be convenient to define a constant

$$\varepsilon_1 = \min\left\{ (e^2 c)^{-1-1/\delta_1}, \frac{\delta}{2-\delta}, \frac{1}{1 + 2e^4 k^6} \right\}$$

and set $\varepsilon = \varepsilon_1 \Psi_{\geq k}(\mu_c)/2$.

6.3.1 Dense 2-cores: $r \geq \delta_1 t$

We show that WHP, not only $\mathcal{K}$ does not contain dense bad 2-cores, but G does not contain *any* dense subgraphs of relevant size, without further restrictions. Clearly, it therefore suffices to only consider the case $r = \delta_1 t$. So, for $\log^2 n < t < \varepsilon n$, let N_t denote the expected number of subgraphs of G with t vertices and $(1 + \delta_1)t$ edges. Then

$$
\begin{aligned}
N_t &\leq \binom{n}{t}\binom{\binom{t}{2}}{(1+\delta_1)t}\left(\frac{c}{n}\right)^{(1+\delta_1)t} \\
&\leq \left(\frac{en}{t}\right)^t \left(\frac{et^2/2}{(1+\delta_1)t}\right)^{(1+\delta_1)t}\left(\frac{c}{n}\right)^{(1+\delta_1)t} \\
&\leq \left(\frac{t}{n}\right)^{\delta_1 t}\left(\frac{e^2 c}{2}\right)^{(1+\delta_1)t} \\
&\leq \left(\varepsilon\left(e^2 c\right)^{1+1/\delta_1}\right)^{\delta_1 t} 2^{-t} \\
&\leq 2^{-t},
\end{aligned}
$$

where the last inequality is due to the fact that $\varepsilon < \varepsilon_1$ and the definition of ε_1. Summing over t, we get

$$\sum_{t=\log^2 n}^{\varepsilon n} N_t \leq \sum_{t=\log^2 n}^{\varepsilon n} 2^{-t} \leq n 2^{-\log^2 n} = o(1),$$

implying that WHP G does not contain any subgraph with t vertices and s edges for any $(t,s) \in \mathcal{I}_{\text{dense}}$.

6.3.2 Sparse 2-cores: $r \leq \delta_1 t$

By using Corollary 6.4, we will be able to restrict to k-cores with proper degree sequences. Of course, "the degree sequence is proper" is not an event but an asymptotic statement. Strictly, what Corollary 6.4 means is that there is a concrete specification of the asymptotic bounds in the definition of "proper" such that such bounds hold WHP for the degree sequence of the k-core. When we refer to the event that the sequence is proper below, we mean the event that a set of such bounds hold.

In view of Theorem 5.6, we consider a sequence $\vec{d}$ of $\hat{n}$ independent copies of $Z_k(\lambda)$ where λ is determined by (10). By Lemma 5.3 we only need to consider $\hat{n} = (1 + o(1)) \Psi_{\geq k} (\mu_c) n$ and $\hat{m} = (1 + o(1)) \mu_c \Psi_{\geq k-1} (\mu_c) n/2$, and consequently, the estimation of $\hat{d}$ given in (9) holds. Since $\mathbb{E}[Z_k(\lambda)] = \lambda \Psi_{\geq k-1}(\lambda)/\Psi_{\geq k}(\lambda)$, it follows by the definition of λ and Lemma 5.3 that $\lambda = (1 + o(1))\mu_c$.

Let A_n denote the event that (i) $\vec{d}$ is a proper sequence, and (ii) a random k-core $\mathcal{K}$ with degree sequence $\vec{d}$ has probability at least $1/n$ of containing a 2-core with parameters $(t,s) \in \mathcal{I}_{\text{sparse}}$. Implicitly, this event is contained in the event that the sum of components of $\vec{d}$ is even. Note that the restriction $t < \varepsilon n$ and the definition of ε imply that $t < \varepsilon_1 \hat{n}$.

In this subsection we make use of two types of degree sequences; proper degree sequences $\vec{d} \in \mathcal{D}_{\hat{n},2\hat{m}}$ for $\mathcal{K}$, and degree sequences $\vec{d} \in \mathbb{N}^t$ for subsets of $V(\mathcal{K})$ of size t. In order to distinguish between these two types we write either $\vec{d}_{(\hat{n})}$ or $\vec{d}_{(t)}$, respectively. When referring to a subset $U \subset V(\mathcal{K})$ of size t, we use $u_1, \ldots, u_t$ to denote its vertices, even when this is not written explicitly.

Since estimating the expected number of sparse 2-cores involves some tedious calculations, we make them in several steps. We begin with a few bounds which are given without context at this moment and will be useful later. First, for a given $\vec{d}_{(t)} \in \mathbb{N}^t$, by using simple combinatorial identities and by letting $r = s - t$ and $h'_i = h_i - 2$ for every i, we have

$$
\begin{aligned}
\sum_{\substack{h_1,\ldots,h_t \geq 2 \\ \sum h_i = 2s}} \prod_{i=1}^t \binom{d_i}{h_i} &= \sum_{\substack{h_1,\ldots,h_t \geq 2 \\ \sum h_i = 2s}} \prod_{i=1}^t \binom{d_i - 2}{h_i - 2} \frac{[d_i]_2}{[h_i]_2} \\
&\leq \sum_{\substack{h'_1,\ldots,h'_t \geq 0 \\ \sum h'_i = 2r}} \prod_{i=1}^t \binom{d_i - 2}{h'_i} [d_i]_2/2 \\
&= 2^{-t} \binom{\sum_{i=1}^t (d_i - 2)}{2r} \prod_{i=1}^t [d_i]_2 \, .
\end{aligned}
\tag{18}
$$

Second, for $j \leq 2r$ let $\widetilde{\mathcal{D}}_j = \left\{ \vec{d}_{(t)} \in \mathcal{D}_{t,kt+j} \mid \forall i \, d_i \geq k \right\}$, and note that $\left| \widetilde{\mathcal{D}}_j \right| = \binom{j+t-1}{t-1}$ and that

$$
\sum_{j=0}^{2r} \binom{j+t-1}{t-1} = \binom{2r+t}{t} \leq \binom{2t}{2r} \leq \left(\frac{et}{r} \right)^{2r} .
\tag{19}
$$

Furthermore, at least $t - j$ entries in every $\vec{d}_{(t)} \in \widetilde{\mathcal{D}}_j$ equal k, and $\prod_{i=1}^t [d_i]_2$ is maximised when the entries in $\vec{d}_{(t)} \in \widetilde{\mathcal{D}}_j$ are as equal as possible; that is, when j of them equal $k + 1$. Recall that

the distribution of a single component of $\vec{d}$ is $Z_k(\lambda)$, abbreviated to Z for the remainder of this subsection. Since $\lambda = (1 + o(1))\mu_c$ we may estimate the probabilities in the distribution of Z asymptotically by using $Z_k(\mu_c)$. By Lemma 6.6, for n sufficiently large we have

$$
\begin{aligned}
[k]_2 \Pr[Z = k] &\overset{(9)}{=} (1 + o(1)) [k]_2 \frac{\hat{d}}{k} \Pr[Z_{k-1}(\mu_c) = k - 1] \\
&\leq (1 + o(1))(1 - 2\delta)\hat{d} < (1 - \delta)\hat{d}. \tag{20}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sum_{\substack{d_1,\ldots,d_t \geq k \\ \sum d_i \leq kt + 2r}} \prod_{i=1}^{t} [d_i]_2 \Pr[Z = d_i] &= \sum_{j=0}^{2r} \sum_{\vec{d}_{(t)} \in \widetilde{\mathcal{D}}_j} \prod_{i=1}^{t} [d_i]_2 \Pr[Z = d_i] \\
&\leq \sum_{j=0}^{2r} \left|\widetilde{\mathcal{D}}_j\right| ([k+1]_2)^j ([k]_2)^{t-j} \Pr[Z = k]^{t-j} \\
&\overset{(20)}{\leq} ([k+1]_2)^{2r} \sum_{j=0}^{2r} \left|\widetilde{\mathcal{D}}_j\right| \left((1-\delta)\hat{d}\right)^{t-j} \\
&\leq \left((1-\delta)\hat{d}\right)^t (2k^2)^{2r} \sum_{j=0}^{2r} \binom{j+t-1}{t-1} \\
&\overset{(19)}{\leq} \left((1-\delta)\hat{d}\right)^t \left(\frac{2ek^2 t}{r}\right)^{2r}. \tag{21}
\end{aligned}
$$

We now wish to bound the probability that a given set of vertices induces a sparse bad 2-core in a random k-core $\mathcal{K}$ with a given degree sequence, and this is where the configuration model becomes useful. For a given degree sequence $\vec{d}_{(\hat{n})} \in \mathcal{D}_{\hat{n}, 2\hat{m}}$ and for positive integers $t < \hat{n}$ and $s < \hat{m}$, let us count how many configurations F for $\vec{d}_{(\hat{n})}$ yield a multigraph $H = H(F)$ such that $H[U]$ is a 2-core with s edges, where $U = \{u_1, \ldots, u_t\}$ is the set of the first t vertices in the sequence. First we have to choose a degree sequence for $H[U]$, that is, choose $\vec{h} = \{h_1, \ldots, h_t\} \in \mathcal{D}_{t, 2s}$ such that $d_{H[U]}(u_i) = h_i \geq 2$ for every i. Given $\vec{h}$, there are $\prod_{i=1}^{t} \binom{d_i}{h_i}$ possibilities to determine for each u_i which h_i of its d_i half-edges go inside U (while the rest go outside). Finally, there are $(2s - 1)!!$ configurations for $H[U]$, and $(2\hat{m} - 2s - 1)!!$ configurations for the rest of H. It follows that if F is chosen uniformly at random from all $(2\hat{m} - 1)!!$ possible configurations for $\vec{d}_{(\hat{n})}$ then the probability of $H[U]$ being a 2-core with s edges is at most

$$
\sum_{\substack{h_1,\ldots,h_t \geq 2 \\ \sum h_i = 2s}} \prod_{i=1}^{t} \binom{d_i}{h_i} \frac{(2s-1)!!(2\hat{m} - 2s - 1)!!}{(2\hat{m}-1)!!} \overset{(18)}{\leq} \frac{2^s s!}{(2\hat{m}-2s)^s} \cdot 2^{-t} \binom{\sum_{i=1}^{t}(d_i - 2)}{2r} \prod_{i=1}^{t} [d_i]_2. \tag{22}
$$

Note that this bound does not depend on $d_{t+1}, \ldots, d_{\hat{n}}$.

Now, for integers t and s, an arbitrary subset $U \subset V(\mathcal{K})$ of size t, and a degree sequence $\vec{d}_{(t)} \in \mathbb{N}^t$, denote by $P(U, \vec{d}_{(t)}, s)$ the probability that $\mathcal{K}[U]$ is a bad 2-core with s edges, where $\mathcal{K}$ is as in the definition of the event A_n but conditioned upon satisfying $d_{\mathcal{K}}(u_i) = d_i$ for every $1 \leq i \leq t$. By the condition on the boundary of bad 2-cores, we only need to consider sequences $\vec{d}_{(t)}$ such that

$\sum_{i=1}^{t} d_i \le (k-2)t + 2s = kt + 2r$, and in particular $\sum_{i=1}^{t}(d_i - 2) \le kt$, since $r < t$. Recall that $2\hat{m} = \hat{d}\hat{n}$ and note that for sparse 2-cores we have

$$2s = 2(t + r) \le 3t \le kt \le \hat{d}\varepsilon_1\hat{n}. \tag{23}$$

Using (22), and Corollary 6.2 with 2^r standing for $f(n)$, we can bound $P(U, \vec{d}_{(t)}, s)$ from above by

$$
\begin{aligned}
P(U, \vec{d}_{(t)}, s) &\le 2^r \frac{2^s s!}{(2\hat{m} - 2s)^s} \cdot 2^{-t}\binom{\sum_{i=1}^{t}(d_i - 2)}{2r} \prod_{i=1}^{t}[d_i]_2 \\
&\overset{(23)}{\le} \frac{2^{2r} s!}{\left(\hat{d}\hat{n} - \hat{d}\varepsilon_1\hat{n}\right)^s}\binom{kt}{2r}\prod_{i=1}^{t}[d_i]_2 \\
&\le \frac{2^{2r} s!}{\left((1-\varepsilon_1)\hat{d}\hat{n}\right)^s}\left(\frac{ekt}{2r}\right)^{2r}\prod_{i=1}^{t}[d_i]_2 \\
&= \frac{s!}{\left((1-\varepsilon_1)\hat{d}\hat{n}\right)^s}\left(\frac{ekt}{r}\right)^{2r}\prod_{i=1}^{t}[d_i]_2 \ . \tag{24}
\end{aligned}
$$

For the random sequence $\vec{d}$ of independent truncated Poissons under consideration, define $A_n(t, s)$ to be the expected number of bad 2-cores, that have t vertices and s edges, in the random k-core $\mathcal{K}$ (in the case the sequence is proper and has even sum — otherwise treat the number as 0). It then follows from above that for any given pair $(t, s) \in \mathcal{I}_{\text{sparse}}$ we have

$$
\begin{aligned}
A_n(t, s) &\le \sum_{\substack{U \subset V(\mathcal{K}) \\ |U| = t}} \sum_{\substack{d_1, \dots, d_t \ge k \\ \sum d_i \le kt + 2r}} \Pr\left[\deg_\mathcal{K}(u_i) = d_i \text{ for } i = 1, 2, \dots, t\right] P(U, \vec{d}_{(t)}, s) \\
&\overset{(24)}{\le} \binom{\hat{n}}{t} \sum_{\substack{d_1, \dots, d_t \ge k \\ \sum d_i \le kt + 2r}} \left(\prod_{i=1}^{t}\Pr\left[Z = d_i\right]\right) \cdot \frac{s!}{\left((1-\varepsilon_1)\hat{d}\hat{n}\right)^s}\left(\frac{ekt}{r}\right)^{2r}\prod_{i=1}^{t}[d_i]_2 \\
&\le \frac{\hat{n}^t}{t!}\frac{s!}{\left((1-\varepsilon_1)\hat{d}\hat{n}\right)^s}\left(\frac{ekt}{r}\right)^{2r}\sum_{\substack{d_1, \dots, d_t \ge k \\ \sum d_i \le kt + 2r}}\prod_{i=1}^{t}[d_i]_2 \Pr\left[Z = d_i\right] \\
&\overset{(21)}{\le} \frac{[s]_r}{\left((1-\varepsilon_1)\hat{d}\right)^s\hat{n}^r}\left(\frac{ekt}{r}\right)^{2r}\left((1-\delta)\hat{d}\right)^t\left(\frac{2ek^2t}{r}\right)^{2r} \\
&\le \left(\frac{1-\delta}{1-\varepsilon_1}\right)^t\left(\frac{t}{r}\right)^{4r}\left(\frac{4se^4k^6}{(1-\varepsilon_1)\hat{d}\hat{n}}\right)^r \\
&\overset{(23)}{\le} \left(\frac{1-\delta}{1-\varepsilon_1}\left(\frac{t}{r}\right)^{4r/t}\right)^t\left(\frac{\varepsilon_1 2e^4k^6}{1-\varepsilon_1}\right)^r .
\end{aligned}
$$

By definition of ε_1 we have $\frac{1-\delta}{1-\varepsilon_1} \le 1 - \delta/2$ and $\varepsilon_1 2e^4k^6 \le 1 - \varepsilon_1$. Using the fact that $x \mapsto x^{-x}$ is increasing for $0 < x < 1/e$, and by definition of δ_1, for every $(t, s) \in \mathcal{I}_{\text{sparse}}$ we obtain

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$$A_n(t,s) \leq \left(\frac{1-\delta}{1-\varepsilon_1} \left(\frac{r}{t} \right)^{-4r/t} \right)^t \leq \left(\left(1 - \frac{\delta}{2} \right) \delta_1^{-4\delta_1} \right)^t \leq \left(1 - \frac{\delta}{4} \right)^t.$$

Recalling that $t > \log^2 n$, and observing that $|\mathcal{I}| < n^3$, we get

$$\sum_{(t,s)\in\mathcal{I}_{\text{sparse}}} A_n(t,s) \leq \sum_{(t,s)\in\mathcal{I}_{\text{sparse}}} (1-\delta/4)^t \leq |\mathcal{I}| \, (1-\delta/4)^{\log^2 n} = O(1/n^2).$$

The probability of the event A_n is hence $O(1/n)$. Thus, by Theorem 5.6, if $\vec{d}$ is distributed as the degree sequence of $\mathcal{K}(n,c,k)$, the probability that a random core with degree sequence $\vec{d}$ has a sparse 2-core and is proper is $o(1)$. That is, WHP $\mathcal{K}(n,c,k)$ does not have both a proper degree sequence and a sparse 2-core. Recalling that it has proper degree sequence WHP, Lemma 4.3 follows.

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