Counting and packing Hamilton cycles in dense graphs and oriented graphs

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Abstract

We present a general method for counting and packing Hamilton cycles in dense graphs and oriented graphs, based on permanent estimates. We utilize this approach to prove several extremal results. In particular, we show that every nearly cn-regular oriented graph on n vertices with $c > 3/8$ contains $(cn/e)^n(1 + o(1))^n$ directed Hamilton cycles. This is an extension of a result of Cuckler, who settled an old conjecture of Thomassen about the number of Hamilton cycles in regular tournaments. We also prove that every graph G on n vertices of minimum degree at least $(1/2 + o(1))n$ contains at least $(1 - o(1))\text{reg}_{even}(G)/2$ edge-disjoint Hamilton cycles, where $\text{reg}_{even}(G)$ is the maximum *even* degree of a spanning regular subgraph of G. This establishes an approximate version of a conjecture of Kühn, Lapinskas and Osthus.

1 Introduction

A *Hamilton cycle* in a graph or a directed graph is a cycle passing through every vertex of the graph exactly once, and a graph is *Hamiltonian* if it contains a Hamilton cycle. Hamiltonicity is one of the most central notions in graph theory, and has been intensively studied by numerous researchers. Since the problem of determining Hamiltonicity of a graph is NP-complete it is important to find general sufficient conditions for Hamiltonicity and in the last 60 years many interesting results were obtained in this direction. Once Hamiltonicity is established it is very natural to strengthen such result by showing that a graph in question has many distinct or edge-disjoint Hamilton cycles.

In this paper we present a general approach for counting and packing Hamilton cycles in dense graphs and oriented graphs. This approach is based on the standard estimates for the permanent of a matrix (the famous Minc and Van der Waerden conjectures, established by Brégman [4], and by Egorychev [10] and by Falikman [11], respectively). In a nutshell, we use these permanent estimates to show that an r-factor in a given graph or digraph G on n vertices, where r is linear in n, contains many (edge-disjoint) 2-factors in the undirected case or 1-factors in the directed case, whose number of cycles is

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relatively small (much smaller than linear); then these factors are converted into many (edge-disjoint) Hamilton cycles using rotation-extension type techniques. Strictly speaking, the permanent-based approach to Hamiltonicity problems is not exactly new and has been used for the first time in [1] to bound the number of Hamilton paths in tournaments and in [13] to pack Hamilton cycles in pseudo-random graphs (see also [14], [19], [20], [21]). However, these prior papers worked in the setting of random or pseudo-random graphs, while the present contribution appears to be the first one where the permanent-based approach is applied in the general, extremal graph theoretic setting.

We employ our method to prove several new extremal results and to derive some known results in a conceptually different and easier way as well.

One of the first and probably most celebrated sufficient conditions for Hamiltonicity was established by Dirac [9] in 1952, who proved that every graph on n vertices, $n \geq 3$, with minimum degree at least $n/2$ is Hamiltonian. The complete bipartite graph $K_{m,m+1}$ shows that this theorem is best possible, i.e., the minimum degree condition cannot be improved. Later, Nash-Williams [27] proved that any *Dirac graph* (that is, a graph G on n vertices with minimum degree $\delta(G) \geq n/2$) has at least $\frac{5}{224}n$ edge-disjoint Hamilton cycles. He also asked [26, 27, 28] to improve this estimate. Clearly, $\lfloor (n+1)/4 \rfloor$ is a general upper bound on the number of edge-disjoint Hamilton cycles in a Dirac graph obtained by considering an $n/2$ regular graph, and originally Nash-Williams [26] believed that this is tight.

Babai (see also [26]) found a counterexample to this conjecture. Extending his ideas further, Nash-Williams gave an example of a graph on $n = 4k$ vertices with minimum degree $2k$ and with at most $\lfloor (n+4)/8 \rfloor$ edge-disjoint Hamilton cycles. He conjectured that this example is tight, i.e., any Dirac graph contains at least $\lfloor (n+4)/8 \rfloor$ edge-disjoint Hamilton cycles. Moreover, Nash-Williams pointed out that the example depends heavily on the graph being not regular. He thus also proposed the following conjecture which has become known as the "Nash-Williams Conjecture":

Conjecture 1.1 *Every d-regular Dirac graph contains $\lfloor d/2 \rfloor$ edge-disjoint Hamilton cycles.*

Recently, this conjecture was settled asymptotically by Christofides, Kühn and Osthus [5], who proved that any d-regular graph G on n vertices with $d \geq (1/2+\varepsilon)n$, contains at least $(1-\varepsilon)d/2$ edge-disjoint Hamilton cycles. For large graphs, Kühn and Osthus [24] further improved this to $\lfloor d/2 \rfloor$ edge-disjoint Hamilton cycles. Even more recently, after the first version of the present paper has been submitted, Csaba, Kühn, Lo, Osthus and Treglown [6] proved the exact version of the above conjecture for all large enough n.

For the non-regular case, Kühn, Lapinskas and Osthus [22] proved that if $\delta(G) \geq (1/2+\varepsilon)n$, then G contains at least $\mathrm{reg}_{even}(n, \delta(G))/2$ edge-disjoint Hamilton cycles where $\mathrm{reg}_{even}(n, \delta)$ is the largest *even* integer r such that every graph G on n vertices with minimum degree $\delta(G) = \delta$ must contain an r-regular spanning subgraph (an *r-factor*). As for a concrete G, the maximal *even* degree r of an r-factor of G, which we denote by $\mathrm{reg}_{even}(G)$, can be much larger than $\mathrm{reg}_{even}(n, \delta)$. Therefore, it is natural to look for bounds in terms of $\mathrm{reg}_{even}(G)$. In [24], Kühn and Osthus showed that any graph G with $\delta(G) \geq (2-\sqrt{2}+\varepsilon)n$ contains $\mathrm{reg}_{even}(G)/2$ edge-disjoint Hamilton cycles, and in [22], Kühn, Lapinskas and Osthus conjectured the following tight result.

Conjecture 1.2 *Suppose G is a Dirac graph. Then G contains at least* $\mathrm{reg}_{even}(G)/2$ *edge-disjoint Hamilton cycles.*

Answering an open problem from [22], in this paper we prove an approximate asymptotic version of this conjecture.

Theorem 1.3 *For every $\varepsilon > 0$ and a sufficiently large integer n the following holds. Every graph G on n vertices and with $\delta(G) \geq (1/2+\varepsilon)n$ contains at least* $(1-\varepsilon)\mathrm{reg}_{even}(G)/2$ *edge-disjoint Hamilton cycles.*

Given a graph G, let $h(G)$ denote the number of distinct Hamilton cycles in G. Strengthening Dirac's theorem Sárközy, Selkow and Szemerédi [31] proved that every Dirac graph G contains not only one but at least $c^n n!$ Hamilton cycles for some small positive constant c. They also conjectured that c can be improved to $1/2 - o(1)$. This has later been proven by Cuckler and Kahn [8]. In fact, Cuckler and Kahn proved a stronger result: every Dirac graph G on n vertices with minimum degree $\delta(G)$ has $h(G) \geq \left(\frac{\delta(G)}{e}\right)^n (1-o(1))^n$. The random graph $G(n,p)$ with $p > 1/2$ shows that this estimate is sharp (up to the $(1-o(1))^n$ factor). Indeed in this case with high probability $\delta(G(n,p)) = pn + o(n)$ and the expected number of Hamilton cycles is $p^n(n-1)! < (pn/e)^n$.

To illustrate our techniques we prove the following proposition which gives a lower bound on the number of Hamilton cycles in a dense graph G in terms of $\mathrm{reg}(G)$, where $\mathrm{reg}(G)$ is the maximal r for which G contains an r-factor. Although this bound is asymptotically tight for nearly regular graphs, it is weaker than the result of Cuckler and Kahn in general. On the other hand, since every Dirac graph contains an r-factor with r about $n/4$ (see [17]), our bound implies the result of Sárközy, Selkow and Szemerédi mentioned above.

Proposition 1.4 *Let G be a graph on n vertices with minimum degree $\delta(G) \geq n/2$. Then the number of Hamilton cycles in G is at least* $\left(\frac{\mathrm{reg}(G)}{e}\right)^n (1-o(1))^n$.

Proposition 1.4 implies that, given a dense regular graph G, the number of Hamilton cycles in G is asymptotically exactly (in exponential terms) what we expect in a random graph with the same edge density.

Corollary 1.5 *Let $c \geq 1/2$ and let G be a graph on n vertices which is cn-regular. Then*

$$h(G) = \left(\frac{cn}{e}\right)^n (1+o(1))^n.$$

Using a technical lemma from [5], in Section 2 we show that given an almost regular graph G on n vertices with $\delta(G) \geq n/2 + \varepsilon n$, G contains an r-factor with r very close to $\delta(G)$. Therefore, we conclude that if the minimum degree of G is at least $n/2 + \varepsilon n$, then condition (ii) in Corollary 1.5 can be relaxed to the requirement that G is "almost regular". Before stating it formally, we introduce the following notation: whenever we want to write that x lies in the interval between $a-b$ and $a+b$, we simply write $x \in (a \pm b)$.

Corollary 1.6 *For every $c > 1/2$ there exists $\varepsilon > 0$ such that for large enough integer n the following holds. Suppose that:*

(i) G is a graph on n vertices, and

(ii) $d(v) \in (c \pm \varepsilon)n$ for every $v \in V$.

Then $h(G) \in \left(\frac{(c\pm\varepsilon')n}{e}\right)^n$, where $\varepsilon'(\varepsilon) = \varepsilon'$ is a specific function of ε tending to 0 with ε.

An *oriented* graph G is a graph obtained by orienting the edges of a simple graph. That is, between every unordered pair of vertices $\{x, y\} \subseteq V(G)$ there exists at most one of the (oriented) edges xy or yx. Hamiltonicity problems in oriented graphs are usually much more challenging. Given an oriented graph G, let $\delta^+(G)$ and $\delta^-(G)$ denote the minimum *outdegree* and *indegree* of the vertices in G, respectively. We also use the notation $d^{\pm}(v) \in (a \pm b)$ for the statement that both $d^+(v)$ and $d^-(v)$ lie between $a - b$ to $a + b$. In addition, we set $\delta^{\pm}(G) = \min\{\delta^+(G), \delta^-(G)\}$ and refer to it as the *semi-degree* of G. In the late 70's Thomassen [34] raised the natural question of determining the minimum semi-degree that ensures the existence of a Hamilton cycle in an oriented graph G. Häggkvist [15] found a construction which gives a lower bound of $\frac{3n-4}{8} - 1$. The problem was resolved only recently by Keevash, Kühn and Osthus [18], who proved that every oriented graph G on n vertices with $\delta^{\pm}(G) \geq \frac{3n-4}{8}$ contains a Hamilton cycle.

Counting Hamilton cycles in tournaments is another very old problem which goes back some seventy years to one of the first applications of the probabilistic method by Szele [33]. He proved that there are tournaments on n vertices with at least $(n-1)!/2^n$ Hamilton cycles. Alon [1] showed that this result is nearly tight and every n vertex tournament has at most $O(n^{3/2}(n-1)!/2^n)$ Hamilton cycles. Thomassen [35] and later Friedgut and Kahn [12] conjectured that the randomness is unnecessary in Szele's result and that in fact every regular tournament contains at least $n^{(1-o(1))n}$ Hamilton cycles. This conjecture was solved by Cuckler [7] who proved that every regular tournament on n vertices contains at least $\frac{n!}{(2+o(1))^n}$ Hamilton cycles. The following theorem substantially extends Cuckler's result [7].

Theorem 1.7 *For every $c > 3/8$ and every $\eta > 0$ there exists a positive constant $\varepsilon := \varepsilon(c, \eta) > 0$ such that for every sufficiently large integer n the following holds. Suppose that:*

(i) G is an oriented graph on n vertices, and

(ii) $d^{\pm}(v) \in (c \pm \varepsilon)n$ for every $v \in V(G)$.

Then $h(G) \in \left(\frac{(c\pm\eta)n}{e}\right)^n$. In particular, if G is cn-regular, then $h(G) = \left(\frac{(c+o(1))n}{e}\right)^n$.

The bound on in/out-degrees in this theorem is tight. This follows from the construction of Häggkvist [15] (mentioned above), which shows that there are n-vertex oriented graphs with all in/outdegrees $(3/8 - o(1))n$ and no Hamilton cycles.

Definitions and notation: Our graph-theoretic notation is standard and follows that of [36]. For a graph G, let $V = V(G)$ and $E = E(G)$ denote its sets of vertices and edges, respectively. For subsets $U, W \subseteq V$, and for a vertex $v \in V$, we denote by $E_G(U)$ all the edges of G with both endpoints in U, by $E_G(U, W)$ all the edges of G with one endpoint in U and one endpoint in W, and by $E_G(v, U)$ all the edges with one endpoint being v and one endpoint in U. We write $N_G(v)$ for the neighborhood of v in G and $d_G(v)$ for its degree. For an oriented graph G we write uv for the edge directed from u to v. We denote by $N_G^+(v)$ and $N_G^-(v)$ the *outneighborhood* and *inneighborhood* of v, respectively, and write $d_G^+(v) = |N_G^+(v)|$ and $d_G^-(v) = |N_G^-(v)|$. We will omit the subscript G whenever there is no risk of confusion. We will denote the minimum outdegree by $\delta^+(G)$ and the minimum indegree by $\delta^-(G)$, and set $\delta^\pm(G) = \min\{\delta^+(G), \delta^-(G)\}$. Finally we write $a = (b \pm c)$ for $a \in (b - c, b + c)$.

For the sake of simplicity and clarity of presentation, and in order to shorten some of our proofs, no real effort has been made here to optimize the constants appearing in our results. We also omit floor and ceiling signs whenever these are not crucial. Most of our results are asymptotic in nature and whenever necessary we assume that the underlying parameter n is sufficiently large.

2 Tools

In this section we introduce the main tools to be used in the proofs of our results.

2.1 Probabilistic tools

We will need to employ bounds on large deviations of random variables. We will mostly use the following well-known bound on the lower and the upper tails of the Binomial distribution due to Chernoff (see [3], [16]).

Lemma 2.1 *If* $X \sim \mathrm{Bin}(n, p)$, *then*

- $\Pr(X < (1 - a)np) < e^{-a^2np/2}$ *for every* $a > 0$*;*
- $\Pr(X > (1 + a)np) < e^{-a^2np/3}$ *for every* $0 < a < 3/2$.

2.2 r-factors

One of the main ingredients in our results is the ability to find an r-factor in a graph with r as large as possible. The following theorem of Katerinis [17] shows that a dense graph contains a dense r-factor.

Theorem 2.2 *Let* r *be a positive integer and let* G *be a graph such that:*

(i) $r|V(G)|$ *is even, and*

(ii) $\delta(G) \geq |V(G)|/2$, *and*

(iii) $|V(G)| \geq 4r - 5$.

Then G contains an r-factor.

When a given graph G is almost regular, it turns out that G contains r-factors with r much closer to $\delta(G)$ than given by Theorem 2.2. The following lemma was proved by Christofides, Kühn and Osthus in [5].

Lemma 2.3 *(Theorem 12 in [5]) Let G be a graph on n vertices of minimum degree $\delta = \delta(G) \geq n/2$.*

(i) Let r be an even number such that $r \leq \frac{\delta+\sqrt{n(2\delta-n)}}{2}$. Then G contains an r-factor.

(ii) Let $0 < \xi < 1/9$ and suppose $(1/2 + \xi)n \leq \Delta(G) \leq \delta + \xi^2 n$. If r is an even number such that $r \leq \delta - \xi n$ and n is sufficiently large, then G contains an r-factor.

The result of Lemma 2.3 (ii) immediately implies the following useful corollary:

Corollary 2.4 *Let $1/2 < c \leq 1$ and let $0 < \varepsilon < 1/9$ be such that $c - \varepsilon - 3\sqrt{\varepsilon} \geq 1/2$. Then for every sufficiently large integer n the following holds. Suppose that:*

(i) G is a graph with $|V(G)| = n$, and

(ii) $d(v) = (c \pm \varepsilon)n$ for every $v \in V(G)$.

Then G contains an r-factor for every even $r \leq (c - \varepsilon')n$, where $\varepsilon' = 3\sqrt{\varepsilon} + \varepsilon$.

2.3 Permanent estimates

Let S_n be the set of all permutations of the set $[n]$. Given a permutation $\sigma \in S_n$, let $A(\sigma)$ be an $n \times n$ matrix which represents the permutation σ, that is, for every $1 \leq i, j \leq n$, $A(\sigma)_{ij} = 1$ if $\sigma(i) = j$ and 0 otherwise. Notice that for every $\sigma \in S_n$, in each row and each column of $A(\sigma)$ there is exactly one "1". Every permutation $\sigma \in S_n$ has a (unique up to the order of cycles) cyclic form. Given two $n \times n$ matrices A and B, we write $A \geq B$ in case that $A_{ij} \geq B_{ij}$ for every $1 \leq i, j \leq n$. The *permanent* of an $n \times n$ matrix A is defined as $per(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} A_{i\sigma(i)}$. Notice that in case A is a 0-1 matrix, every summand in the permanent is either 0 or 1, and the permanent of A counts the number of distinct permutations $\sigma \in S_n$ which are *contained* in A, that is, the number of σ's for which $A \geq A(\sigma)$. A 0-1 matrix A is called *r-regular* if it contains exactly r 1's in every row and in every column.

Using the following two well known permanent estimates, in the next subsection we prove that if A is any 0-1 αn-regular matrix, then most of the permutations which are contained in it have relatively few cycles in their cyclic form.

We state first an upper bound for the permanent. This bound was conjectured by Minc and has been proven by Brégman [4].

Theorem 2.5 *Let A be an $n \times n$ matrix of 0-1 with t ones altogether. Then $per(A) \leq \Pi_{i=1}^{n}(r_i!)^{1/r_i}$, where r_i are integers satisfying $\sum_{i=1}^{n} r_i = t$ and are as equal as possible.*

A square matrix A of nonnegative real numbers is called doubly stochastic if each row and column of A sum to 1. The following lower bound is also known as the Van der Waerden conjecture and has been proven by Egorychev [10] and by Falikman [11]:

Theorem 2.6 *Let A be an $n \times n$ doubly stochastic matrix. Then $per(A) \geq \frac{n!}{n^n}$.*

2.4 2-factors with few cycles

Motivated by ideas from [2, 13, 21], in this subsection we prove that for every sufficiently large integer n, in every r-regular, 0-1, $n \times n$ matrix A, most of the permutations contained in A have relatively few cycles in their cyclic form, provided that r is linear in n. For a positive integer r and a graph G, we define a $(\leq r)$-factor to be any spanning subgraph H of G for which each connected components of H is s-regular for some $s \leq r$. We conclude that in every dense r-regular graph G, most of the (≤ 2)-factors do not contain too many cycles (we consider a single edge as a cycle too). We also prove that in case r is even, G contains such a 2-factor with all cycles of length at least 3. These lemmas are crucial since one of the main ingredients of our proofs is the ability to find "enough" 2-factors with only few cycles and then to turn them into Hamilton cycles.

Lemma 2.7 *Let $\alpha > 0$ be a constant and let n be a positive integer. Suppose that:*

(i) A is an $n \times n$ matrix, and

(ii) all entries of A are 0 or 1, and

(iii) A is αn-regular.

Then the number of permutations $\sigma \in S_n$ for which $A \geq A(\sigma)$ and such that there are at most $s^* := \sqrt{n \ln n}$ cycles in their cyclic form, is $(1 + o(1))^n \left(\frac{\alpha n}{e}\right)^n$.

Note that in case A is the adjacency matrix of a graph G, every permutation $\sigma \in S_n$ for which $A \geq A(\sigma)$ corresponds to a (≤ 2)-factor with exactly the same number of cycles as in the cyclic form of σ (we consider a single edge as a cycle too); and every (≤ 2)-factor F of G corresponds to at most 2^s permutations, where s is the number of cycles in F (each cycle can be oriented in at most two ways). Therefore, the following is an immediate corollary of Lemma 2.7:

Corollary 2.8 *Let $\alpha > 0$ be a constant and let n be a positive integer. Suppose that:*

(i) G is a graph on n vertices, and

(ii) G is αn-regular.

Then the number of (≤ 2)-factors of G with at most $s^* := \sqrt{n \ln n}$ cycles is $(1+o(1))^n \left(\frac{\alpha n}{e}\right)^n$.

Proof of Lemma 2.7. Given a 0-1 matrix of order $n \times n$, let $S(A) = \{\sigma \in S_n : A \geq A(\sigma)\}$ be the set of all permutations contained in A, and let $f(A,k)$ be the number of permutations $\sigma \in S(A)$ with exactly k cycles. Notice that $f(A) := \sum_k f(A,k) = |S(A)|$. Given an integer $1 \leq t \leq n$ we also define

$$\phi(A,t) := \max\{f(A') : A' \text{ is a } t \times t \text{ submatrix of } A\}.$$

For the upper bound, using Theorem 2.5 and the fact that $(k/e)^k \leq k! \leq k(k/e)^k$ we conclude that

$$per(A) \leq ((\alpha n)!)^{n/(\alpha n)} = (1+o(1))^n \left(\frac{\alpha n}{e}\right)^n.$$

Now, note that

$$per(A) = \sum_{s=1}^{n} f(A,s).$$

Applying Theorem 2.6 to the doubly stochastic matrix $\frac{1}{\alpha n}A$, we obtain

$$\sum_{s=1}^{n} f(A,s) = per(A) \geq n!\alpha^n \geq \left(\frac{\alpha n}{e}\right)^n.$$

In order to complete the proof we need to show that $\sum_{s>s^*} f(A,s) = o\left(\left(\frac{\alpha n}{e}\right)^n\right)$. Let $s > \sqrt{n \ln n}$, we wish to estimate $f(A,s)$ from above. Given a permutation $\sigma \in S(A)$ with s cycles, there must be at least $\frac{1}{2}\sqrt{n \ln n}$ cycles, each of which is of length at most $2\sqrt{n/\ln n}$. Therefore, by the pigeonhole principle we get that there must be a cycle length $\ell := \ell(\sigma) \leq 2\sqrt{n/\ln n}$ which appears at least $j = \frac{\ln n}{4}$ times in σ. The number of permutations in $S(A)$ which contain at least j cycles of fixed length ℓ is at most:

$$\binom{n}{j} \prod_{i=1}^{j} (\alpha n)^{\ell-1} \cdot \phi(A, n-j\ell) \leq \left(\frac{en}{j}\right)^j (\alpha n)^{j\ell-j} \cdot \phi(A, n-j\ell). \tag{2.1}$$

Indeed, first we fix j cycles of length ℓ. To do so we choose j elements, $x_1, \ldots, x_j$, one for each such a cycle. This can be done in $\binom{n}{j}$ ways. Since A is αn-regular, for each $1 \leq i \leq j$, there are at most $(\alpha n)^{\ell-1}$ options to close a cycle of length ℓ which contains x_i. Given these j cycles of total length $j\ell$, there are at most $\phi(A, n-j\ell)$ ways to extend it to a cyclic form of a permutation $\sigma \in S(A)$.

Next we estimate $\phi(A, n-j\ell)$. Let $t = j\ell$ and let A_1 be an arbitrary $(n-t) \times (n-t)$ submatrix of A. By switching order of some rows and columns, we can assume that $A = \begin{pmatrix} A_1 & B \\ C & A_2 \end{pmatrix}$, where A_2, B and C are $t \times t$, $(n-t) \times t$ and $t \times (n-t)$ submatrices of A, respectively. Given a 0-1 matrix M, let $g(M) = \mathbf{1}^T M \mathbf{1}$ be the number of 1's in M ($\mathbf{1}$ is a vector with all entries equal 1). Since $g(A_2) \leq t^2$ and since A is αn-regular, it follows that $g(B) \geq \alpha n t - t^2$. Therefore, we conclude that

$g(A_1) = \alpha n(n-t) - g(B) \le \alpha n(n-t) - (\alpha n t - t^2)$ and that the average number of 1's in a row or a column of A_1 is

$$\frac{g(A_1)}{n-t} \le \alpha n - \frac{t(\alpha n - t)}{n-t} =: d_1.$$

Note that $\alpha(n-t) \le d_1 \le \alpha n$. Now, by Brégman's Theorem 2.5 we get that

$$per(A_1) \le (d_1!)^{\frac{n-t}{d_1}} \le \left(\left(\frac{d_1}{e}\right)^{d_1} d_1\right)^{\frac{n-t}{d_1}} \le \left(\frac{\alpha n - \frac{t(\alpha n - t)}{n-t}}{e}\right)^{n-t} (\alpha n)^{1/\alpha}$$

$$\le \left(\frac{\alpha n}{e}\right)^{n-t}\left(1 - \frac{t(\alpha n - t)}{\alpha n(n-t)}\right)^{n-t} (\alpha n)^{1/\alpha} \le \left(\frac{\alpha n}{e}\right)^{n-t} e^{-t+t^2/(\alpha n)} (\alpha n)^{1/\alpha}.$$

Hence, we conclude that

$$\phi(A, n-t) \le \left(\frac{\alpha n}{e}\right)^{n-t} e^{-t+t^2/(\alpha n)} (\alpha n)^{1/\alpha}.$$

Now, plugging it into the estimate (2.1) and recalling that $\ell \le 2\sqrt{n/\ln n}$, $j = \frac{\ln n}{4}$ and $t = j\ell \le \frac{1}{2}\sqrt{n \ln n}$, we have

$$\begin{aligned}
f(A,s) &\le \sum_{\ell \le 2\sqrt{n/\ln n}} \left(\frac{en}{j}\right)^j (\alpha n)^{t-j} \phi(A, n-t) \\
&\le \sum_{\ell \le 2\sqrt{n/\ln n}} \left(\frac{en}{j}\right)^j (\alpha n)^{t-j} \left(\frac{\alpha n}{e}\right)^{n-t} e^{-t+t^2/(\alpha n)} (\alpha n)^{1/\alpha} \\
&\le \left(\frac{\alpha n}{e}\right)^n \sum_{\ell \le 2\sqrt{n/\ln n}} \left(\frac{en}{j}\right)^j (\alpha n)^{-j} e^{t^2/(\alpha n)} (\alpha n)^{1/\alpha} \\
&\le \left(\frac{\alpha n}{e}\right)^n 2\sqrt{n/\ln n} \left(\frac{e/\alpha}{j}\right)^j e^{O(\ln n)} (\alpha n)^{1/\alpha} \\
&\le \left(\frac{\alpha n}{e}\right)^n 2\sqrt{n/\ln n}\, n^{-\Omega(\ln \ln n)}\, n^{O(1)} (\alpha n)^{1/\alpha} \\
&= \left(\frac{\alpha n}{e}\right)^n \cdot o\left(\frac{1}{n}\right).
\end{aligned}$$

This clearly implies that $\sum_{s > s^*} f(A,s) = o\left(\left(\frac{\alpha n}{e}\right)^n\right)$ and completes the proof. $\square$

In the following lemma we prove that given a dense r-regular graph G, if r is even, then G contains a 2-factor with not too many components.

Lemma 2.9 *Let $\alpha > 0$ be a constant and let n be sufficiently large integer. Suppose that:*

(i) *αn is even, and*

(ii) *G is a graph n vertices, and*

(iii) *G is αn-regular.*

Then G contains a 2-factor with at most $\sqrt{n \ln n}$ components.

Proof. Since αn is even, G has an Eulerian orientation $\overrightarrow{E}$ (if G is not connected, then find such an orientation for every connected component). Assume that $V(G) = [n]$ and let A be an $n \times n$ matrix of 0 and 1s such that $A_{ij} = 1$ if and only if $(i, j) \in \overrightarrow{E}$. Note that A is an $(\alpha n/2)$-regular $n \times n$ matrix, and therefore, by Lemma 2.7 we conclude that there exists a permutation $\sigma \in S_n$ such that $A \geq A(\sigma)$ and with at most $\sqrt{n \ln n}$ cycles in its cyclic form. Since every such permutation defines a (≤ 2)-factor of G, and since each cycle is built by out-edges of the orientation $\overrightarrow{E}$, we conclude that the shortest possible such cycle is of length at least 3. □

2.5 Rotations

The most useful tool in turning a path P into a Hamilton cycle is the Pósa rotation-extension technique (see [30]). Motivated by this technique, in this section we establish tools for turning a path into a Hamilton cycle under certain assumptions suitable for proving our main results.

First we need the following notation. Given a path $P = v_0 v_1 \dots v_k$ in a graph G and a vertex $v_i \in V(P)$, define $v_i^+ = v_{i+1}$ and $v_i^- = v_{i-1}$ ($v_0^- = v_k$ and $v_k^+ = v_0$). For a subset $I \subseteq V(P)$, we define $I^+ = \{v^+ : v \in I\}$ and $I^- = \{v^- : v \in I\}$.

Now, given a dense graph and a path in it, the following lemma enables us to obtain a longer path with only few rotations.

Lemma 2.10 *Let G be a graph on n vertices and with $\delta(G) \geq n/2$. Let P_0 be a path in G. Then there exist two vertices $a, b \in P_0$ and a path P^* in G connecting a to b so that:*

(i) $V(P^*) = V(P_0)$.

(ii) $|E(P_0) \Delta E(P^*)| \leq 4$.

(iii) *$ab \in E(G)$ and the cycle obtained by adding this edge to P^* is a Hamilton cycle, or G contains an edge between $\{a, b\}$ and $V(G) \setminus V(P^*)$.*

Proof. Let $P_0 = v_0 \dots v_k$ be a path in G. If there exists an edge $v_0 v \in E(G)$ or $v_k v \in E(G)$ for some $v \notin V(P_0)$, then by setting $P_0 = P^*$, $a = v_0$ and $b = v_k$ we are done. Assume then that there is no such edge. In particular, it means that $N(v_0) \cup N(v_k) \subseteq V(P_0)$. First, we claim that there must be a vertex $v \in N(v_0)^-$ such that $v v_k \in E(G)$. Otherwise, we have that $N(v_k) \subseteq (V(P_0) \setminus \{v_k\}) \setminus N(v_0)^-$.

Since $\delta(G) \geq n/2$ and since $|V(P_0) \setminus \{v_k\}| \leq n-1$ we conclude that $|(V(P_0) \setminus \{v_k\}) \setminus N(v_0)^-| < n/2$ which is clearly a contradiction.

Let $v_i \in N(v_0)^-$ be such vertex with $v_i v_k \in E(G)$. Notice that $C = v_0 v_1 \dots v_i v_k v_{k-1} \dots v_{i+1} v_0$ is a cycle on the vertex set $V(P_0)$, obtained be deleting one edge from P and adding two new edges. If C is a Hamilton cycle then we are done. Otherwise, since G is a connected graph (this follows easily from $\delta(G) \geq n/2$), there exist two vertices $v \in V(C)$ and $u \in V(G) \setminus V(C)$ such that $vu \in E(G)$. By deleting an edge vw from C and by denoting $a = v$ and $b = w$ we get the desired path. $\square$

In the following lemma we prove that every dense graph G contains a subgraph H with some nice pseudorandom properties for which $\mathrm{reg}_{even}(G)$ and $\mathrm{reg}_{even}(G - H)$ are relatively close to each other. We will use this subgraph H to form edge disjoint Hamilton cycles from a set of edge disjoint 2-factors. This is crucial for the proof of Theorem 1.3. Before stating the lemma we introduce the following notation which will be used in its proof and in later sections. An r-factor of an oriented graph is a spanning subgraph with all in- and out-degrees equal to r.

Lemma 2.11 *For every $0 < \varepsilon < 1/4$ and $0 < \alpha < \varepsilon^2$, there exist $\beta > 0$ and $n_0 := n_0(\varepsilon, \alpha)$ such that for every $n \geq n_0$ the following holds. Suppose that:*

(*i*) *G is a graph on n vertices, and*

(*ii*) *$\delta(G) \geq (1/2 + \varepsilon)n$.*

Then G contains a subgraph $H \subset G$ with the following properties:

($P1$) *$G' = G - E(H)$ is r-regular and r is an even integer which satisfies $r \geq (1 - \varepsilon/2)\mathrm{reg}_{even}(G)$;*

($P2$) *$\delta(H) \geq \varepsilon n/8$;*

($P3$) *for every subset $S \subset V(G), |S| = \alpha n$ and for every subset $E' \subset E(H)$ of size $|E'| \leq \beta n^2$, we have $|N_{H-E'}(S) \setminus S| \geq (1/2 + \varepsilon/4)n$;*

($P4$) *$H - E'$ is a connected graph for every $E' \subset E(H)$ such that $\delta(H - E') \geq \alpha n$ and $|E'| \leq \beta n^2$.*

Proof of Lemma 2.11. Let R be a $\mathrm{reg}_{even}(G)$-factor of G and observe by Theorem 2.2 that $\mathrm{reg}_{even}(G) \geq n/4$. Since $\mathrm{reg}_{even}(G)$ is even, we can find an Eulerian orientation $\overrightarrow{E}$ and obtain a $\mathrm{reg}_{even}(G)/2$-regular oriented graph $\overrightarrow{R} = (V(G), \overrightarrow{E})$. Now, choose a collection $\mathcal{F}$ of $t := \varepsilon n/16 \leq \varepsilon \cdot \mathrm{reg}_{even}(G)/4$ edge-disjoint random 1-factors from $\overrightarrow{R}$ as follows. Let $\overrightarrow{R}_0 := \overrightarrow{R}$, and for $i := 1, \dots, t$ do: let F_i be a 1-factor of $\overrightarrow{R}_{i-1}$ chosen uniformly at random among all such 1-factors, and let $\overrightarrow{R}_i := \overrightarrow{R}_{i-1} - F_i$ (the existence of such factors follows immediately from the fact that $\overrightarrow{R}_{i-1}$ is regular and Hall's Marriage Theorem). Delete the orientations of edges in every $F \in \mathcal{F}$ and let H be the graph spanned by all of these edges (that is, $\cup_{F \in \mathcal{F}} E(F)$) and the edges of $G - R$. We prove that with high probability, H satisfies all the properties stated in the theorem.

Properties $(P1)$ and $(P2)$ follow immediately from the definition of H and from Theorem 2.2.

For proving $(P3)$, it is enough to prove that for every two disjoint subsets $S, T \subseteq V(G)$ of size $|S| = \alpha n$ and $|T| \geq \frac{(1-\varepsilon)n}{2}$, we have $|E_H(S,T)| \geq \beta n^2$. Property $(P3)$ thus follows immediately using the fact that $|S| = \alpha n \leq \varepsilon^2 n \leq \varepsilon n/4$. Indeed, given a subset $S \subset V(G)$ for which $|S| = \alpha n$, the number of edges (in H) between S to every subset of size $(1/2 - \varepsilon/2)n$ is $\Theta(n^2)$. Therefore, for some small constant $\beta > 0$, by removing at most βn^2 edges one cannot delete all the edges between two such sets. It follows that $|N_{H-E'}(S) \setminus S| \geq (1/2 + \varepsilon/2 - \alpha)n \geq (1/2 + \varepsilon/4)n$ as required.

To this end, let $S, T \subseteq V(G)$ be two disjoint subsets for which $|S| = \alpha n$ and $|T| = \frac{(1-\varepsilon)n}{2}$. Since $\delta(G) \geq (1/2 + \varepsilon)n$, it follows that $d(v,T) \geq \varepsilon n/2$ for every $v \in S$. Therefore, $|E_G(S,T)| \geq |S|\varepsilon n/2 = \frac{\varepsilon \cdot \alpha}{2} n^2$. Now, let β be a fixed constant smaller than $\frac{\varepsilon \cdot \alpha}{4}$ (to be determined later), and note that if $|E_{G-R}(S,T)| \geq \frac{\varepsilon \cdot \alpha}{4} n^2$, then we are done. Otherwise, we have $|E_R(S,T)| \geq \frac{\varepsilon \cdot \alpha}{4} n^2$. We wish to bound from above the probability that for two such subsets S and T, the 2-factors in H use at most βn^2 edges from $E_R(S,T)$. For this end, consider $\overrightarrow{R}$ again and let A be an $n \times n$, 0/1 matrix for which $(A)_{ij} = 1$ if and only if $ij \in \overrightarrow{E}$. Since A is $\text{reg}_{even}(G)/2$-regular, by Theorem 2.6 we conclude that

$$per(A) \geq \left(\frac{\text{reg}_{even}(G)}{2e} \right)^n.$$

Now, note that if A' is a matrix which is obtained from A by deleting $cn^2/2$ many 1's (where $c > 0$ is some positive constant), then by Theorem 2.5 we have

$$per(A') \leq (1 + o(1))^n \left(\frac{\text{reg}_{even}(G) - cn}{2e} \right)^n.$$

Now, picking a 1-factor F of $\overrightarrow{R}$ at random, the probability that for some fixed subset $E_0 \subseteq E_R(S,T)$ of size at most βn^2 the 1-factor F does not use any edges from from $E_R(S,T) \setminus E_0$ is bounded from above by $\frac{per(A')}{per(A)}$, where $c = 2\varepsilon\alpha/4 - 2\beta$ (recall that $|E_R(S,T)| \geq \varepsilon \alpha n^2/4$). Note that when we remove a 1-factor from $\overrightarrow{R}$, the new graph remains regular (the in- and out-degrees decrease by exactly 1). Therefore, while choosing the $(i+1)$st factor F_{i+1}, using the fact that R_i is $(\text{reg}_{even}(G)/2 - i)$-regular and the estimation on $per(A')$ and $per(A)$ mentioned above, we obtain that the probability for not touching edges in $E_R(S,T) \setminus E_0$ is upper bounded by

$$(1 + o(1))^n \left(\frac{\text{reg}_{even}(G) - cn - 2i}{\text{reg}_{even}(G) - 2i} \right)^n.$$

All in all, we conclude that for some $0 \leq \delta < 1$, the probability for the existence of such a set $E_0 \subseteq E_R(S,T)$ of size βn^2 for which none of the 1-factors in $\mathcal{F}$ uses edges from $E_R(S,T) \setminus E_0$ is at most

$$(1 + o(1))^{nt} \binom{n^2}{\beta n^2} \cdot \prod_{i=1}^{t} \left(\frac{\text{reg}_{even}(G) - cn - 2i}{\text{reg}_{even}(G) - 2i} \right)^n \leq (1 + o(1))^{nt} \left(\frac{e}{\beta} \right)^{\beta n^2} \delta^{nt} = \delta^{\Theta(n^2)}.$$

Indeed, recall that $t = \varepsilon n/16$ and that by Theorem 2.2 we have (say) $\text{reg}_{even}(G) \geq n/5$, and therefore, if we require that $\beta < \frac{\varepsilon \alpha}{8}$, then for example $\delta = 1 - \frac{5\varepsilon\alpha}{4}$ is such that $\frac{\text{reg}_{even}(G) - cn - 2i}{\text{reg}_{even}(G) - 2i} \leq \delta$ holds for every $i \leq t$. All in all, for a small enough β we have $\left(\frac{\varepsilon}{\beta}\right)^{\beta} \delta^{\varepsilon/16} = \delta^{\Theta(1)}$ and the last equality holds. Now, by applying the union bound we get that the probability for having two such sets is at most $4^n \cdot \delta^{\Theta(n^2)} = o(1)$.

For $(P4)$, note that from the minimum degree condition we have that every component of $H - E'$ has size at least αn. Now, by $(P3)$ we have that every connected component is in fact of size more than $n/2$ even after deleting at most βn^2 many edges. This completes the proof. $\square$

In the next lemma, using some ideas from [32], we prove that in a graph with good expansion properties, every non-Hamilton path can be extended by changing only a few edges.

Lemma 2.12 *For every $0 < \varepsilon < 1/200$ and a sufficiently large integer n the following holds. Suppose that:*

(1) H is a graph on n vertices, and

(2) $\delta(H) \geq \varepsilon n/8$, and

(3) $|N_H(S) \setminus S| > (1/2 + \varepsilon/4)n$ for every subset $S \subset V(H)$ of size $|S| = \varepsilon^3 n$.

Then for every path P with $V(P) \subseteq V(H)$ (P does not necessarily need to be a subgraph of H), there exist a pair of vertices a, b and a path P^* in $H \cup P$ connecting these vertices so that:

(i) $V(P^*) = V(P)$, and

(ii) $|E(P) \Delta E(P^*)| \leq 8$, and

(iii) $ab \in E(H)$ and the cycle obtained by adding this edge is a Hamilton cycle, or $H \cup P$ contains an edge between $\{a, b\}$ and $V(H) \setminus V(P^*)$.

Proof. Let $P = v_0 v_1 \ldots v_k$ be a path. We distinguish between three cases:

Case I: There exists $v \in V(H) \setminus V(P)$ for which $v_0 v \in E(H)$ or $v_k v \in E(H)$. In this case, by denoting $P^* = P$, $a = v_0$ and $b = v_k$, we are done.

Case II: $v_0 v_k \in E(H)$. Let C be the cycle obtained by adding the edge $v_0 v_k$ to P. If C is a Hamilton cycle then we are done. Otherwise, since H is connected (immediate from properties (2) and (3)), we can find $v \in V(C)$ and $u \in V(H) \setminus V(C)$ for which $vu \in E(H)$. Now, let P^* be the path obtained from C by deleting the edge vv^+, $a = v$, $b = v^+$ and we are done.

Case III: $N_H(v_0) \cup N_H(v_k) \subseteq V(P)$ and $v_0 v_k \notin E(H)$. Let $t = \lceil 10/\varepsilon \rceil$ and let $I_1, \ldots, I_t$ be a partition of P into t intervals of length at most $|P|/t \leq \varepsilon n/10$ each. Note that, since $t = \lceil 10/\varepsilon \rceil$ and since $\varepsilon < 1/200$, we can find I_i for which $|N_H(v_0) \cap I_i| \geq (\varepsilon n/8)/t \geq \varepsilon^2 n/81$. Similarly there exists an interval I_j which contains at least $\varepsilon^2 n/81$ neighbors of v_k. If $i \neq j$ then set $I = I_i$ and $J = I_j$. Otherwise, divide I_i into two intervals such that each of them contains at least $\varepsilon^2 n/162$ neighbors of v_0. Clearly one of them contains at least $\varepsilon^2 n/162$ neighbors of v_k. Hence, we obtain two disjoint intervals I and J of P such that $|I|, |J| \leq \varepsilon n/10$ and for which $|N_H(v_0) \cap I|, |N_H(v_k) \cap J| \geq \varepsilon^2 n/160 \geq \varepsilon^3 n$.

Now, assume that the interval I is to the left of the interval J according to the orientation of P (the case where I is to the right of J is similar). Let $i_1 = \min\{i : v_i \in I\}$ and define $L := \{v_0, \ldots, v_{i_1-1}\}$ to be the set of all vertices of P which are to the left of I. For $i_2 = \max\{i : v_i \in I\}$ and $i_3 = \min\{i : v_i \in J\}$, set $M := \{v_{i_2+1}, \ldots, v_{i_3-1}\}$ to be the set of all vertices between I and J. Similarly, set $R := \{v_{i_4+1}, \ldots, v_k\}$ to be the set of all vertices which are to the right of J in P (where $i_4 = \max\{i : v_i \in J\}$). We prove that by a sequence of at most four additions and at most three deletions of edges we can turn P into a cycle C on $V(P)$, and then the result follows exactly as described in Case II (deleting at most one more edge). Let $I_0 \subseteq N_H(v_0) \cap I$ and $J_0 \subseteq N_H(v_k) \cap J$ be two subsets of size exactly $\varepsilon^3 n$. Let

$$N := \left(N_H(I_0^-)^+ \cap L\right) \cup \left(N_H(I_0^-)^- \cap M\right) \cup \left(N_H(I_0^-)^+ \cap R\right).$$

Then, by Property (3) we have $|N| \geq (1/2 + \varepsilon/4)n - |I| - |J| > n/2$ and also $|N_H(J_0^+)| > n/2$. Therefore we conclude that $N \cap N_H(J_0^+) \neq \emptyset$ and need to consider only the following three scenarios:

(a) $\left(N_H(I_0^-)^+ \cap L\right) \cap N_H(J_0^+) \neq \emptyset$. Let $v^+ \in \left(N_H(I_0^-)^+ \cap L\right)$ and $u^+ \in J_0^+$ be such that $v^+u^+ \in E(H)$, and let $w \in I_0$ be such that $w^-v \in E(H)$. Then we have the following cycle

$$C = v^+ \ldots w^-v \ldots v_0 w \ldots u v_k \ldots u^+ v^+.$$

(b) $\left(N_H(I_0^-)^- \cap M\right) \cap N_H(J_0^+) \neq \emptyset$. Let $v^- \in (N_H(I_0^-))^- \cap M$ and $u^+ \in J_0^+$ be such that $v^-u^+ \in E(H)$, and let $w \in I_0$ be such that $w^-v \in E(G)$. In this case the cycle is

$$C = v \ldots u v_k \ldots u^+ v^- \ldots w v_0 \ldots w^- v.$$

(c) $\left(N_H(I_0^-)^+ \cap R\right) \cap N_H(J_0^+) \neq \emptyset$. Let $v^+ \in \left(N_H(I_0^-)^+ \cap R\right)$ and $u^+ \in J_0^+$ be such that $v^+u^+ \in E(H)$, and let $w \in I_0$ be such that $w^-v \in E(H)$. We obtain the following cycle

$$C = v_0 \ldots w^- v \ldots u^+ v^+ \ldots v_k u \ldots w v_0.$$

This completes the proof. □

2.6 Oriented graphs

In this subsection we establish tools needed in the proof of Theorem 1.7 which deals with counting the number of Hamilton cycles in oriented graphs. We start with the following notion of a *robust* expander due to Kühn, Osthus and Treglown [25]:

Definition 2.13 *Let G be an oriented graph of order n and let $S \subseteq V(G)$. The ν-*robust outneighborhood* $RN^+_{\nu,G}(S)$ of S is the set of vertices with at least νn inneighbors in S. The graph G is called a* robust (ν,τ)-outexpander *if $|RN^+_{\nu,G}(S)| \geq |S| + \nu n$ for every $S \subseteq V(G)$ with $\tau n \leq |S| \leq (1-\tau)n$.*

The following fact is an immediate consequence of the definition of a robust (ν,τ)-outexpander.

Fact 2.14 *For every $\nu, \nu' > 0$ such that $\nu' < \nu$, and for every sufficiently large integer n the following holds. Suppose that:*

(i) G is an oriented graph on n vertices, and

(ii) G is a robust (ν, τ)-outexpander.

Then every graph G' which is obtained from G by adding a new vertex (does not matter how) is a robust (ν', τ)-outexpander.

The following theorem shows that given a robust outexpander G which is almost regular, G contains an r-factor with almost the same degree as the degrees of G. Before stating the theorem we remark that the constants in the hierarchies used to state our results are chosen from the largest to the smallest. More precisely, whenever we write something like $0 < 1/n \ll \nu \ll \tau \ll \alpha < 1$ (where n is the order of the graph or digraph), then this means that there are non-decreasing functions $f : (0, 1] \to (0, 1]$, $g : (0, 1] \to (0, 1]$ and $h : (0, 1] \to (0, 1]$ such that the result holds for all $0 < \nu, \tau, \alpha < 1$ and all positive integers n with $\tau \le f(\alpha)$, $\nu \le g(\tau)$ and $1/n \le h(\nu)$. We will not calculate these functions explicitly.

Theorem 2.15 *For every $\alpha > 0$ there exists $\tau > 0$ such that for all $\nu \le \tau$ and $\eta > 0$ there exist $n_0 := n_0(\alpha, \nu, \tau, \eta)$ and $\gamma := \gamma(\alpha, \nu, \tau, \eta) > 0$ such that the following holds. Suppose that G is an oriented graph with $|V(G)| = n \ge n_0$ satisfying:*

(i) $d^{\pm}(v) \in (\alpha \pm \gamma)n$ for every $v \in V(G)$, and

(ii) G is a robust (ν, τ)-outexpander.

Then G contains an $(\alpha - \eta)n$-factor.

In order to prove Theorem 2.15 we need the following lemma from [23].

Lemma 2.16 (Lemma 5.2 in [23]) *Suppose that $0 < 1/n \ll \varepsilon \ll \nu \le \tau \ll \alpha < 1$ and that $1/n \ll \xi \le \nu^2/3$. Let G be a digraph on n vertices with $\delta^{\pm}(G) \ge \alpha n$ which is a robust (ν, τ)-outexpander. For every vertex x of G, let $n_x^+, n_x^- \in \mathbb{N}$ be such that $(1 - \varepsilon)\xi n \le n_x^+, n_x^- \le (1 + \varepsilon)\xi n$ and such that $\sum_{x \in V(G)} n_x^+ = \sum_{x \in V(G)} n_x^-$. Then G contains a spanning subdigraph G' such that $d^+_{G'}(x) = n_x^+$ and $d^-_{G'}(x) = n_x^-$ for every $x \in V(G)$.*

Proof of Theorem 2.15 The proof is quite similar to the first paragraph of the proof of Corollary 1.2 in [29]. For the convenience of the reader we will add it here.

Since in a digraph G, whenever G contains an r-factor it also contains an $(r-1)$-factor, it is enough to prove the statement for some $\eta > 0$. Next, since we are going to use Lemma 2.16, it will be convenient to introduce the following parameters. In order to call Lemma 2.16 with the corresponding parameters we require that $0 < \varepsilon \ll \nu \le \tau \ll \alpha < 1$ and $\xi \le \nu^2/3$. By definition, this means that there exist non-decreasing functions $f, g : (0, 1] \to (0, 1]$ such that the conclusion of Lemma 2.16 holds for all constant $0 < \varepsilon, \nu, \tau, \alpha < 1$ for which $\tau \le f(\alpha)$, $\nu \le \tau$, $\varepsilon \le g(\nu)$.

Now, given α we choose τ so that $\tau \le f(\alpha/2)$. Next, choose $\eta > 0$ so that $\eta \le \min(g(\nu), \sqrt{\alpha/2}, \nu^2/3)$, set $\gamma := \eta^2$. Since f is non-decreasing, one obtains $\tau \le f(\alpha - \gamma)$ and we therefore have $\eta \ll \nu \le \tau \ll \alpha - \gamma < 1$. Next, for each $x \in V(G)$ let

$$n_x^+ := d_G^+(x) - (\alpha - \eta)n \text{ and } n_x^- := d_G^-(x) - (\alpha - \eta)n.$$

Note that $(\eta - \eta^2)n \le n_x^+, n_x^- \le (\eta + \eta^2)n$ for every $x \in V(G)$.

Applying Lemma 2.16 to G with $\xi = \varepsilon = \eta$ and $\alpha - \gamma$ in place of α, we obtain a subdigraph G' for which $d_{G'}^+(x) = n_x^+$, $d_{G'}^-(x) = n_x^-$, and therefore the graph $G'' = G - G'$ is an $(\alpha - \eta)n$-regular digraph on n vertices. This completes the proof. □

The following technical lemma is one of the main ingredients in the proof of Theorem 1.7. We use it to turn a directed path of length $n - o(n)$ into a directed Hamilton cycle:

Lemma 2.17 *For every $\alpha > 3/8$ and a sufficiently large integer n the following holds. Suppose that:*

(i) G is an oriented graph on n vertices, and

(ii) $\delta^{\pm}(G) \ge \alpha n$.

Then for every two disjoint subsets $A, B \subseteq V(G)$ with $|A| = |B| = \alpha n/2$, G contains a Hamilton path which starts inside A and ends inside B.

Before proving Lemma 2.17 we need the following two results which are stated below. The first lemma, due to Kühn and Osthus [23], asserts that a dense oriented graph is also a robustly expanding graph.

Lemma 2.18 (Lemma 13.1 [23]) *Let $0 < 1/n \ll \nu \ll \tau \le \varepsilon/2 \le 1$ and suppose that G is an oriented graph on n vertices with $\delta^+(G) + \delta^-(G) + \delta(G) \ge 3n/2 + \varepsilon n$ (where $\delta(G) := \min_{x \in V(G)}(d_G^+(x) + d_G^-(x))$). Then G is a robust (ν, τ)-outexpander.*

The following theorem states that if a graph G is a robust outexpander with a linear minimum degree, then G contains a Hamilton cycle.

Theorem 2.19 (Theorem 16 [25]) *Let $1/n \ll \nu \le \tau \ll \eta < 1$, and let G be a digraph on n vertices with $\delta^{\pm}(G) \ge \eta n$ which is a robust (ν, τ)-outexpander. Then G contains a Hamilton cycle.*

Now we are ready to prove Lemma 2.17.

Proof of Lemma 2.17. Let $\alpha > 3/8$ and let G be an oriented graph on n vertices with $\delta^{\pm}(G) \ge \alpha n$. Let $A, B \subseteq V(G)$ be two disjoint subsets of size $|A| = |B| = \alpha n/2$. We wish to show that G contains a

Hamilton path which starts inside A and ends inside B. First, notice that since $\delta^-(G)+\delta^+(G)+\delta(G) \geq 3n/2+\varepsilon n$ (for some small positive constant ε), by Lemma 2.18 we get that for every choice of constants $0 < 1/n \ll \nu \ll \tau \leq \varepsilon/2$, G is a robust (ν, τ)-outexpander. Second, by adding a new vertex x to $V(G)$ in such a way that $N^+(x) = A$ and $N^-(x) = B$, by Fact 2.14 we obtain a new graph G' which is a robust $(\nu/2, \tau)$-outexpander. Third, by applying Theorem 2.19 to G' (applied with $\eta = \alpha/2$), we conclude that G' is Hamiltonian. Last, let C be a Hamilton cycle in G', by deleting x we obtain the desired Hamilton path in G. □

The following lemma enables us to pick a subgraph of an oriented graph which inherits some properties of the base graph.

Lemma 2.20 *For every $c > 0$, for every $0 < \varepsilon < c/2$, and for every sufficiently large integer n the following holds. Suppose that:*

(i) G is an oriented graph with $|V(G)| = n$, and

(ii) $d^{\pm}(v) = (c \pm \varepsilon)n$ for every $v \in V(G)$.

Then there exists a subset $V_0 \subseteq V(G)$ of size $n^{2/3}$ for which the following property holds:

$|N^+_G(v) \cap V_0| \in (c \pm 2\varepsilon)|V_0|$ and $|N^-_G(v) \cap V_0| \in (c \pm 2\varepsilon)|V_0|$ for every $v \in V(G)$ $(*)$.

Proof. Let $V_0 \subseteq V(G)$ be a subset of size $|V_0| = n^{2/3}$, chosen uniformly at random among all such subsets. We prove that V_0 w.h.p satisfies Property $(*)$.

For this aim, let $v \in V(G)$ be an arbitrary vertex. Since $|N^+_G(v) \cap V_0| \sim HG(n, n^{2/3}, d^+(v))$ and since $d^+(v) \in (c \pm \varepsilon)n$, by Chernoff's inequality (Lemma 2.1 is also valid for the hypergeometric distribution, see [16]) we have that $\Pr(|N^+_G(v) \cap V_0| \geq (c + 2\varepsilon)|V_0|) \leq e^{-anp}$, for $p = n^{-1/3}$ and for some positive constant $a = a(\varepsilon)$. Applying the union bound we get that

$$\Pr\left(\exists v \in V(G) \text{ such that } |N^+_G(v) \cap V_0| \geq (c + 2\varepsilon)|V_0|\right) \leq ne^{-anp} = ne^{-an^{2/3}} = o(1).$$

In a similar way we prove it for $|N^-_G(v) \cap V_0|$. This completes the proof. □

Last, we need the following simple fact:

Fact 2.21 *Let G be an oriented graph with $|V(G)| = n$ and $\delta^{\pm}(G) \geq 3n/8$. Then, the directed diameter of G is at most 4.*

Proof. Let $x, y \in V(G)$. We wish to prove that there exists a path P of length at most 4 which is oriented from x to y. Let $A \subseteq N^+_G(x)$ and $B \subseteq N^-_G(y)$ two subsets of size $|A| = |B| = 3n/8$. If $A \cap B \neq \emptyset$ then we are done. Otherwise, let $a \in A$ be a vertex for which $d^+(a, A) \leq |A|/2$ (there must be such a vertex since $\sum_{z \in A} d^+(z, A) \leq \binom{|A|}{2}$), and let $b \in B$ be a vertex for which $d^-(b, B) \leq |B|/2$. The result will follow by proving that $N^+_G(a) \cap B \neq \emptyset$, $N^+_G(a) \cap N^-_G(b) \neq \emptyset$ or $N^-_G(b) \cap A \neq \emptyset$. Indeed, otherwise we get that $|V(G)| = n \geq 2 + |A| + |B| + |A|/2 + |B|/2 \geq 3n/8 + 3n/8 + 3n/16 + 3n/16 > n$, which is a contradiction. □

3 Counting Hamilton cycles in undirected graphs

In this section we prove Proposition 1.4 and Corollaries 1.5 and 1.6.

Proof of Proposition 1.4. Let $H \subseteq G$ be a reg(G)-factor of G. By Theorem 2.2 we have that $\text{reg}(G) = \Theta(n)$. Therefore, we can apply Corollary 2.8 and conclude that $\sum_{s\le s^*} f(H,s) \geq \left(\frac{\text{reg}(G)}{e}\right)^n (1-o(1))^n$ (where $s^* = \sqrt{n\ln n}$ and $f(H,s)$ counts the number of (≤ 2)-factors of H with exactly s cycles).

Now, working in G, given a (≤ 2)-factor F with $s \leq s^*$ cycles, by repeatedly applying Lemma 2.10 we can turn F into a Hamilton cycle of G by adding and removing at most $O(s)$ edges in the following way: let C be a non-Hamilton cycle in F. If we can find vertex $v \in V(C)$ and a vertex $u \in V(G)\backslash V(C)$ for which $vu \in E(G)$, then by deleting the edge vv^+ from C (and doing nothing in case C is a cycle of length two) we get a path P which can be extended by the edge vu. Connecting it to a cycle C' which contains u (C' can be just an edge) we obtain a longer path P'. Repeat this argument as long as we can. If there are no edges between the endpoints of the current path P' and the other cycles from F, then we can use Lemma 2.10 in order to turn P' either into a Hamilton cycle (and then we are done) or into a path P^* for which $V(P^*) = V(P)$ and for which there exists an edge between one of its endpoints to $V(G) \setminus V(P^*)$. This can be done within 4 edge replacements and we then extend the path using such an edge. Note that in each such step we invest at most 4 edge replacements in order to decrease the number of cycles by 1, and unless the current cycle is a Hamilton cycle, we can always merge two cycles. Therefore, after $O(s)$ edge-replacements we get a Hamilton cycle.

In order to complete the proof, note that given a Hamilton cycle C in G, by replacing at most k edges we can get at most $\binom{n}{k}(2k)^{2k}$ 2-factors in H (choose k edges of C to delete, obtain at most k paths which need to be turned into a 2-factor by connecting endpoints of paths; for each endpoint we have at most $2k$ choices of other endpoints to connect it to). Therefore, for some positive constant D we have that $\sum_{s\le s^*} f(G,s) \leq h(G)\cdot s^*\binom{n}{Ds^*}(2Ds^*)^{2Ds^*} \leq h(G)n^{O(s^*)}$. This implies that

$$h(G) \quad \geq \quad (1-o(1))^n \left(\frac{\text{reg}(G)}{e}\right)^n n^{-O(s^*)} = (1-o(1))^n \left(\frac{\text{reg}(G)}{e}\right)^n,$$

and completes the proof of Proposition 1.4. □

Corollary 1.5 follows easily from Proposition 1.4.

Proof of Corollary 1.5. Let A be the adjacency matrix of G. Then A is an $n \times n$ matrix with all entries 0's and 1's which is cn-regular (the number of 1's in each row/column is exactly cn). Since G is cn-regular, it follows that $\text{reg}(G) = cn$. Therefore, since $cn \geq n/2$, by Proposition 1.4 we have

$$h(G) \geq \left(\frac{cn}{e}\right)^n (1-o(1))^n.$$

For the upper bound, note that since the number of Hamilton cycles in G, $h(G)$, is at most the number of (≤ 2)-factors in G, which is the permanent of A, using Theorem 2.5 we get that

$$h(G) \leq per(A) \leq ((cn)!)^{1/c} = (1+o(1))^n \left(\frac{cn}{e}\right)^n.$$

This completes the proof. $\square$

The proof of Corollary 1.6 follows quite immediately from the previous proof and Corollary 2.4.

Proof of Corollary 1.6. Let $c > 1/2$, let $0 < \varepsilon < 1/9$ be such that $c - \varepsilon - \sqrt{\varepsilon} > 1/2$, and let G be a graph satisfies the assumptions of the corollary. For the upper bound on $h(G)$, a similar calculation as in the proof of Corollary 1.5 will do the work. For the lower bound, note that by applying Corollary 2.4 to G, one can find a subgraph $G' \subseteq G$ which is $(c - \varepsilon')n$ regular, where $\varepsilon' = \varepsilon + \sqrt{\varepsilon}$. Apply now Propsition 1.4 to G' gives the lower bound. $\square$

4 Counting Hamilton cycles in oriented graphs

In this section we prove Theorem 1.7.

Proof of Theorem 1.7. Let $c > 3/8$ and let $\eta > 0$. Let $\varepsilon_0 > 0$ be a sufficiently small constant which satisfies $4(c - \varepsilon_0)n' > 3n'/2 + \varepsilon_0 n$ for each $n' \geq 0.9n$ (the existence of such ε_0 follows from the fact that $c > 3/8$ and that n is sufficiently large).

Next, note that for a given directed graph G on $n' \geq \ell$ vertices with $\delta^{\pm}(G) \geq (c - \varepsilon_0)n'$, and for each choice of ν, τ satisfying $0 < 1/n' \ll \nu \ll \tau \leq \varepsilon_0/2 \leq 1$, since $\delta^+(G) + \delta^-(G) + \delta(G) \geq 4(c - \varepsilon_0)n' > 3n'/2 + \varepsilon_0 n$, it follows by Lemma 2.18 that G is a robust (ν, τ)-expander. Now, let τ be a constant obtained by applying Theorem 2.15 with $\alpha = c$ and η, and let $\nu \ll \tau$ (recall that $0 < 1/n \ll \nu \ll \tau \ll \alpha < 1$). We obtain a positive constant γ and a positive integer n_0 for which the following holds: for every oriented graph G with $|V(G)| = n' \geq n_0$, if $d^{\pm}(v) \in (c \pm \gamma)n'$ for every $v \in V(G)$, then G contains a $(c - \eta)n'$-factor.

Now, let G be an oriented graph on n vertices, where n is such that $n' := n - n^{2/3} \geq n_0$. Moreover, assume that in G we have $d^{\pm}(v) \in (c \pm \varepsilon)n$ for every $v \in V(G)$, where $\varepsilon = \min\{\gamma/3, \varepsilon_0/3\}$. By applying Lemma 2.20 to G we find a subset $V_0 \subset V(G)$ of size $|V_0| = n^{2/3}$ for which $|N_G^+(v) \cap V_0| \in (c \pm 2\varepsilon)|V_0|$ and $|N_G^-(v) \cap V_0| \in (c \pm 2\varepsilon)|V_0|$ for every vertex $v \in V(G)$. Let $G_1 = G[V_0]$ and $G_2 = G[V(G) \setminus V_0]$ denote the two subgraphs induced by V_0 and $V(G) \setminus V_0$, respectively. Note that since $n' := |V(G_2)| = n - n^{2/3}$ and since $\varepsilon \leq \gamma/3$, it follows that $d^{\pm}_{G_2}(v) \in (c \pm \gamma)n'$ holds for each $v \in V(G_2)$. In addition, since $\varepsilon \leq \varepsilon_0/3$, it follows that $d^{\pm}_{G_2}(v) \in (c \pm \varepsilon_0)n'$ holds for each $v \in V(G_2)$, and therefore, using Lemma 2.18 we conclude that G_2 is a robust (ν, τ)-expander. Therefore, by applying Theorem 2.15 to G_2 we conclude that G_2 contains a $(c - \eta)n'$-factor H.

Next, assume that $V(G_2) = [n']$ and let A be an $n' \times n'$ matrix with all entries 0's and 1's for which $A_{ij} = 1$ if and only if $ij \in E(H)$. A is clearly $(c - \eta)n'$-regular and recall that $(c - \eta)n' = (1 - o(1))(c - \eta)n$. Therefore, by Lemma 2.7 it follows that there are at least $\left(\frac{(c-\eta)n}{e}\right)^n (1 - o(1))^n$ permutations $\sigma \in S_{n'}$ such that $A \geq A(\sigma)$ and such that σ contains at most $s^* := \sqrt{n \ln n}$ cycles in its cyclic form. Note that every such permutation corresponds to a 1-factor of G_2 with at most s^* many cycles, and therefore, since all the degrees in V_0 are larger than $\frac{3}{8}|V_0|$ we obtain that G_1 contains a

Hamilton cycle (using [18]) and we have that

$$\sum_{s\leq s^*+1} f(G,s) \geq \sum_{s\leq s^*} f(G_2,s) \geq \left(\frac{(c-\eta)n}{e}\right)^n (1-o(1))^n,$$

where $f(G,s)$ denote the number of 1-factors of G with exactly s cycles.

Now, given a 1-factor F of G_2, we wish to turn it into a Hamilton cycle of G by changing at most $O(n^{2/3})$ edges. This can be done as follows: Let C be a cycle in F. Since G_2 is strongly connected (follows for example from Fact 2.21) we can find a vertex $v \in V(C)$ and a vertex $u \in V(G_2)\setminus V(C)$ for which $vu \in E(G)$. Deleting the edge vv^+ from C we get a path Q which can be extended to a longer path Q' by adding the edge vu and all edges of the cycle C' in F including u apart from u^-u. Let x and y be the endpoints of the current path Q' (from x to y). Using the subgraph G_1, we can close Q' into a cycle, using at most 6 additional edges. Indeed, by Lemma 2.20 x has an in-neighbor and y has an out-neighbor in V_0 and by Fact 2.21 y can be connected to x (in G_1) by a directed path of length at most 4. Delete from G_1 the edges and vertices we used to close Q'. Update F by replacing C and C' by the newly created cycle. Repeat this argument until we have a cycle C with $V(G_2) \subseteq V(C)$. Note that during this process we constantly change G_1 and G_2 (we use vertices of G_1 in order to connect vertices from G_2 and then move them into G_2 and repeat until a Hamiltonian cycle is obtained). So far, we have invested $O(s^*)$ edge replacements and have deleted at most $O(s^*) = o(|V_0|)$ vertices from G_1. Hence, G_1 (minus all the edges/vertices deleted so far) still satisfies (i) and (ii) of Lemma 2.17 with respect to some $\alpha > 3/8$. Deleting an arbitrary edge vu from C, we obtain a path P with v, u as its endpoints. Next, choose disjoint sets $A \subset N^+_G(v) \cap V_0$ and $B \subset N^-_G(u) \cap V_0$, each of size at least $(c-\eta)|V_0|/2$. Using Lemma 2.20, and applying Lemma 2.17 with respect to $A = N^+_G(v) \cap V_0$ and $B = N^-_G(u) \cap V_0$ we obtain a Hamilton path P' of G_1 which starts inside A and ends inside B. This path together with P forms a Hamilton cycle of G. Note that this cycle was obtained from F by changing $O(n^{2/3})$ edges and vertices.

In order to complete the proof, we need to show that by performing this transformation we do not get the same Hamilton cycle too many times. For this aim we first note that given a Hamilton cycle C in G, by replacing at most k edges we can get at most $\binom{n}{k}(2k)^{2k}$ 1-factors. Indeed, we need to choose k edges of C to delete, we obtain at most k paths which need to be turned into a 1-factor by connecting their endpoints; for each endpoint we have at most $2k$ choices of other endpoints to connect it to. Therefore, since in the whole process we changed $O(n^{2/3})$ edges, for some positive constant D we have that $\sum_{s\leq s^*} f(G,s) \leq h(G)\cdot s^*\binom{n}{Dn^{2/3}}(2Dn^{2/3})^{2Dn^{2/3}} \leq h(G)n^{O(n^{2/3})}$. This implies that

$$h(G) \quad \geq \quad \left(\frac{(c-\eta)n}{e}\right)^n (1-o(1))^n n^{-O(n^{2/3})} = (1-o(1))^n \left(\frac{(c-\eta)n}{e}\right)^n,$$

which proves the lower bound on $h(G)$.

For the upper bound, note that since the number of Hamilton cycles in G, $h(G)$, is at most the number of 1-factors in G, using Theorem 2.5 and the fact that $d^{\pm}(v) \in (c\pm\eta)n$ for every $v \in V(G)$, we get that

$$h(G) \leq \# \text{ of 1-factors} = per(A) \leq (((c+\eta)n)!)^{1/(c+\eta)} = (1+o(1))^n \left(\frac{(c+\eta)n}{e}\right)^n.$$

This completes the proof. $\square$

5 Packing Hamilton cycles in undirected graphs

In this section we prove Theorem 1.3.

Proof of Theorem 1.3. Let $\varepsilon > 0$ and let G be a graph with minimum degree $\delta(G) \geq (1/2 + \varepsilon)n$. Let $\varepsilon' < \min\{\varepsilon, 1/160\}$ be a positive constant, let $H \subset G$ be an auxiliary subgraph of G obtained by applying Lemma 2.11 to G with ε' and $\alpha = (\varepsilon')^3$, and let $G' = G - H$. Recall that by $(P1)$ of Lemma 2.11, G' is r-regular for some even integer r which satisfies

$$r \geq (1 - \varepsilon'/2)\text{reg}_{even}(G) \geq (1 - \varepsilon/2)\text{reg}_{even}(G).$$

Since $(1 - \varepsilon/2)^2 \geq 1 - \varepsilon$, the result will then follow by proving that G contains at least $(1 - \varepsilon/2)r/2$ edge disjoint Hamilton cycles.

To this end we first note that since $\delta(G) > n/2$, it follows from Theorem 2.2 that $r = \Theta(n)$. Therefore, we can use Lemma 2.9 repeatedly (starting with $\alpha = r/n$ and until the last time we have $\alpha \geq \varepsilon r/(2n)$) in order to find $m = (1 - \varepsilon/2)r/2$ edge-disjoint 2-factors of G', $\{F_1, \ldots, F_m\}$, each of them containing at most $s^* = \sqrt{n \ln n}$ cycles, each of which of length at least 3. Note that by removing such a factor from an r'-regular graph, the obtained graph is $(r' - 2)$-regular, and therefore one can apply Lemma 2.9 over and over. Now, we wish to turn each of the F_i's into a Hamilton cycle H_i, using the edges of $G \setminus (H_1 \cup \ldots \cup H_{i-1} \cup F_{i+1} \cup \ldots \cup F_m)$. For this goal, we make an extensive use of Lemma 2.12 and the properties of the auxiliary graph H.

Assume inductively that we have built edge-disjoint Hamilton cycles $H_1, \ldots, H_{i-1}$, which are edge disjoint from $F_i, \ldots, F_m$, and that the current graph $G_i = G \setminus (H_1 \cup \ldots \cup H_{i-1} \cup F_i \cup \ldots \cup F_m)$ satisfies *(2)* and *(3)* of Lemma 2.12 with ε'. Moreover, assume that each of the H_j's has been created from F_j by replacing $O(s^*)$ edges. Note that for $i = 0$, since H is a subgraph of G_0, it follows that G_0 satisfies *(2)* and *(3)* of Lemma 2.12. Now, starting with F_i, using the fact that G_i satisfies *(2)* and *(3)* of Lemma 2.12 (the induction hypothesis), by repeatedly applying this lemma, one can turn F_i into a Hamilton cycle by using $O(s^*)$ edge replacements. This is done in a similar way as in the proof of Proposition 1.4. Now, note that during the procedure, every edge that we delete from F_i is added back to G_i and therefore the minimum degree of G_i remains the same and therefore G_i satisfies *(2)* of Lemma 2.12. Since this procedure takes $O(s^*)$ edge replacements each time and since there are $\Theta(n)$ factors to work on, the total number of edges deleted (or replaced) from G_0 (and in particular, from H) is at most $O(ns^*) = o(n^2)$. Thus, since H satisfies $(P3)$ and $(P4)$ of Lemma 2.11, using the fact that n is sufficiently large, the graph G_i also satisfies *(3)* of Lemma 2.12, which therefore can be further applied. This completes the proof. $\square$

6 Concluding remarks

We presented a general approach, based on permanent estimates, for counting and packing Hamilton cycles in dense graphs and oriented graphs. Using this method we derived some known results in a simpler way and proved some new results as well. In particular, we showed how to apply our technique to find many edge-disjoint Hamilton cycles in dense graphs.

It would be interesting to decide whether our approach can be also used to find many edge-disjoint Hamilton cycles in dense oriented graphs. The main obstacle here is that apparently there is no good analog of Pósa's rotation extension technique for digraphs.

In Proposition 1.4 we obtained a lower bound on $h(G)$ in terms of $\text{reg}_{even}(G)$, for a Dirac graph G. For graphs which are not close to being regular our result is worse than the result of Cuckler and Kahn in [8]. It would be very interesting to try and approach their result using our method.

Another natural question is to obtain a variant of Theorem 1.7 for non-regular oriented graphs similar to the result of Cuckler and Kahn for the non-oriented case. The goal here is to estimate the minimum number of Hamilton cycles in an oriented graph on n vertices with semi-degree $\delta^{\pm}(G) \geq (3/8+o(1))n$. Observe that our technique allows to prove easily that an oriented graph G on n vertices with $\delta^{\pm}(G) \geq (3/8+\varepsilon)n$ contains at least $\left(\frac{\varepsilon n}{3e}\right)^n$ Hamilton cycles. Indeed, applying repeatedly the result of Keevash, Kühn and Osthus [18] we can extract $\frac{\varepsilon n}{2}$ edge-disjoint Hamilton cycles in such graph, whose union is an $\frac{\varepsilon n}{2}$-factor F in G. The rest of the proof is quite similar to our argument in Theorem 1.7. This establishes a weak(er) version of the result of Sárközy, Selkow and Szemerédi [31] for the oriented case.

Finally it would be also nice to extend the result of Keevash, Kühn and Osthus [18] and determine the number of edge disjoint Hamilton cycles that oriented graphs with $\delta^{\pm}(G) \geq 3n/8$ must contain as a function of $\delta^{\pm}(G)$.

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