Fast Strategies In Maker-Breaker Games Played on Random Boards

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Abstract

In this paper we analyze classical Maker-Breaker games played on the edge set of a sparse random board $G \sim \mathcal{G}_{n,p}$. We consider the Hamiltonicity game, the perfect matching game and the k-connectivity game. We prove that for $p(n) \geq \mathrm{polylog}(n)/n$, the board $G \sim \mathcal{G}_{n,p}$ is typically such that Maker can win these games asymptotically as fast as possible, i.e. within $n + o(n)$, $n/2 + o(n)$ and $kn/2 + o(n)$ moves respectively.

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1 Introduction

Let X be any finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets. Usually, X is called the *board*, whereas $\mathcal{F}$ is referred to as the family of *winning sets*. In the $(a : b)$ *Maker-Breaker* game $(X, \mathcal{F})$ (also known as a *weak game*), two players called Maker and Breaker play in rounds. In every round Maker claims a previously unclaimed elements of the board X and Breaker responds by claiming b previously unclaimed elements of the board. Maker wins as soon as he fully claims all elements of some $F \in \mathcal{F}$. If Maker does not fully claim any winning set by the time all board elements are claimed, then Breaker wins the game. The most basic case is $a = b = 1$, the so called *unbiased* game. Notice that being the first player is never a disadvantage in a Maker-Breaker game. Therefore, in order to prove that Maker can win some Maker-Breaker game as the first or the second player it is enough to prove that he can win this game as a second player. Hence, we will always assume that Maker is the second player to move.

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It is natural to play Maker-Breaker games on the edge set of a graph $G = (V, E)$ with $|V| = n$. In this case, $X = E$. In the *connectivity game*, Maker wins if and only if his edges contain a spanning tree. In the *perfect matching* game $\mathcal{M}_n(G)$ the winning sets are all sets of $\lfloor n/2 \rfloor$ independent edges of G. Note that if n is odd, then such a matching covers all vertices of G but one. In the Hamiltonicity game $\mathcal{H}_n(G)$ the winning sets are all edge sets of Hamilton cycles of G. Given a positive integer k, in the k-connectivity game $\mathcal{C}_n^k(G)$ the winning sets are all edge sets of k-connected spanning subgraphs of G.

Maker-Breaker games played on the edge set of the complete graph K_n are well studied. In this case, it is easy to see (and also follows from [16]) that for every $n \geq 4$, Maker can win the unbiased connectivity game in $n - 1$ moves, which is clearly the best possible. It was proved in [12] that Maker can win the unbiased perfect matching game on K_n within $n/2 + 1$ moves (which is clearly the best possible), the unbiased Hamiltonicity game within $n + 2$ moves and the unbiased k-connectivity game within $kn/2 + o(n)$ moves.

In [13], it was shown that Maker can win the unbiased Hamiltonicity game on K_n within $n + 1$ moves which is clearly the best possible and recently it was proved (see [8]) that Maker can win the unbiased k-connectivity game within $kn/2 + 1$ moves which is clearly the best possible.

It follows from all these results that many natural games played on the edge set of the complete graph K_n are drastically in favor of Maker. Hence, it is natural to try to make his life a bit harder and to play on different types of boards or to limit his number of moves. In this paper we are mainly interested in the following two questions.

(i) Given a sparse board $G = (V, E)$, can Maker win the game played on this board?

(ii) How fast can Maker win this game?

In [17] it was suggested to play Maker-Breaker games on the edge set of a random graph $G \sim \mathcal{G}_{n,p}$ and some games were examined such as the perfect matching game, the Hamiltonicity game, the connectivity game and the k-clique game.

Later on, in [5], it was proved that the edge set of $G \sim \mathcal{G}_{n,p}$ with $p = (1 + o(1))\frac{\ln n}{n}$ is typically such that Maker has a strategy to win the unbiased perfect matching game, the Hamiltonicity game and the k-connectivity game. This is best possible since $p = \frac{\ln n}{n}$ is the threshold probability for the property of $\mathcal{G}_{n,p}$ having an isolated vertex. Moreover, the proof in [5] is of a "hitting-time" type. That means, in the random graph process, i.e. when adding one new edge randomly every time, typically at the moment the graph reaches the needed minimum degree for winning the desired game, Maker indeed can win this game. For example, at the first time the graph process achieves minimum degree 2 the board is typically such that Maker wins the perfect matching game.

Another type of games is the following. Let X be any finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets. In the *strong game* $(X, \mathcal{F})$, two players called *Red* and *Blue*, take turns in claiming one previously unclaimed element of X, with Red going first. The winner of this game is the first player to fully claim all the elements of some $F \in \mathcal{F}$. If no one wins by the time all the elements of X are claimed, then the game ends in a *draw*. For example, the classic Tic-Tac-Toe is such a game. It is well known from classic Game Theory, that for every strong game $(X, \mathcal{F})$, either Red has a winning strategy or Blue has a drawing strategy. For certain games, a hypergraph coloring argument can be used to prove that a draw is impossible and thus these games are won by Red. However, these arguments are purely existential. That is, even if it is known that Red has a winning strategy for some strong game $(X, \mathcal{F})$, it might be very hard to describe such a strategy explicitly.

Using fast strategies for Maker in certain games, explicit strategies for Red were given for games such as the perfect matching game, the Hamiltonicity game and the k-connectivity game played on the edge set of K_n (see [8] and [7]). This provides substantial motivation for studying fast winning strategies in Maker-Breaker games.

Regarding the strong game played on $G \sim \mathcal{G}_{n,p}$, not much is known yet. Hence, as a first step for finding explicit strategies for Red in the strong game played on a random board, it is natural to look for fast winning strategies for Maker in the analogous games. Therefore the following question is quite natural.

Question(s) : Given $p = p(n)$, how fast can Maker win the perfect matching, the Hamiltonicity and the k-connectivity games played on the edge set of a random board $G \sim \mathcal{G}_{n,p}$?

In this paper we resolve these questions for a wide range of the values of $p = p(n)$. We prove the following theorems:

Theorem 1.1 *Let $b \geq 1$ be an integer, let $K > 12$, $p = \frac{\ln^K n}{n}$, and let $G \sim \mathcal{G}_{n,p}$. Then a.a.s. G is such that in the $(1 : b)$ weak game $\mathcal{M}_n(G)$, Maker has a strategy to win within $\frac{n}{2} + o(n)$ moves.*

Theorem 1.2 *Let $b \geq 1$ be an integer, let $K > 100$, $p = \frac{\ln^K n}{n}$, and let $G \sim \mathcal{G}_{n,p}$. Then a.a.s. G is such that in the $(1 : b)$ weak game $\mathcal{H}_n(G)$, Maker has a strategy to win within $n + o(n)$ moves.*

Theorem 1.3 *Let $b \geq 1$, $k \geq 2$ be two integers, let $K > 100$, $p = \frac{\ln^K n}{n}$, and let $G \sim \mathcal{G}_{n,p}$. Then a.a.s. G is such that in the $(1 : b)$ weak game $\mathcal{C}_n^k(G)$, Maker has a strategy to win within $\frac{kn}{2} + o(n)$ moves.*

Due to obvious monotonicity the results are valid for any $p = p(n)$ larger than stated in the theorems above.

For the sake of simplicity and clarity of presentation, we do not make a particular effort to optimize the constants obtained in our proofs. We do not believe that the order of magnitude we assume for p in the above theorems is optimal. We also omit floor and ceiling signs whenever these are not crucial. Most of our results are asymptotic in nature and whenever necessary we assume that n is sufficiently large.

The remaining part of the paper is organized as follows. First, we introduce the necessary notation. In Section 2, we assemble several results that we need. We give some basic results of positional games in Section 2.1, of graph theory in Section 2.2, and about $\mathcal{G}_{n,p}$ in Section 2.3. The strategy of Maker (in each of the three games) includes building a suitable expander on a subgraph, which then contains the desired structure. We therefore include results about expanders in Section 2.4. We prove Theorem 1.1, 1.2 and 1.3 in Sections 3, 4 and 5, respectively. Finally, in Section 6 we pose some open problems connected to our results.

1.1 Notation and terminology

Our graph-theoretic notation is standard and follows that of [18]. In particular, we use the following.

For a graph G, let $V(G)$ and $E(G)$ denote its sets of vertices and edges, respectively. Let $S, T \subseteq V(G)$ be subsets. Let $G[S]$ denote the subgraph of G, induced on the vertices of S, and let $E_G(S) = E(G[S])$. Further, let $E_G(S,T) := \{st \in E(G) : s \in S, t \in T\}$, and let $N_G(S) := \{v \in V : \exists s \in S \text{ s.t. } vs \in E(G)\}$ denote the neighborhood of S. Further, for $v \in V(G)$, let $d_G(v,S) = |E_G(\{v\},S)|$, and $d_G(v) := d_G(v, V(G))$. For an edge $e \in E(G)$ we denote by $G - e$ the graph with vertex set $V(G)$ and edge set $E(G) \setminus \{e\}$. We omit the subscript G whenever there is no risk of confusion.

Assume that some Maker-Breaker game, played on the edge set of some graph G, is in progress. At any given moment during the game, we denote the graph formed by Maker's edges by M, and the graph formed by Breaker's edges by B. At any point during the game, the edges of $F := G \setminus (M \cup B)$ are called *free edges*.

2 Auxiliary results

In this section we present some auxiliary results that will be used throughout the paper.

First, we will need to employ bounds on large deviations of random variables. We will mostly use the following well-known bound on the lower and the upper tails of the Binomial distribution due to Chernoff (see [1], [14]).

Lemma 2.1 *If* $X \sim \mathrm{Bin}(n,p)$*, then*

- $\mathbb{P}[X < (1-a)np] < \exp\left(-\frac{a^2np}{2}\right)$ *for every* $a > 0$.
- $\mathbb{P}[X > (1+a)np] < \exp\left(-\frac{np}{3}\right)$ *for every* $a \geq 1$.

Lemma 2.2 *Let* $X \sim \mathrm{Bin}(n,p)$*,* $\mu = \mathbb{E}(X)$ *and* $k \geq 7\mu$*, then* $\mathbb{P}(X \geq k) \leq e^{-k}$.

2.1 Basic positional games results

The following fundamental theorem, due to Beck [2], is a useful sufficient condition for Breaker's win in the $(a : b)$ game $(X, \mathcal{F})$. It will be used extensively throughout the paper.

Theorem 2.3 *Let* X *be a finite set and let* $\mathcal{F} \subseteq 2^X$*. If* $\sum_{F\in\mathcal{F}}(1+b)^{-|F|/a} < \frac{1}{1+b}$*, then Breaker (as the first or second player) has a winning strategy for the* $(a : b)$ *game* $(X, \mathcal{F})$.

While Theorem 2.3 is useful in proving that Breaker wins a certain game, it does not show that he wins this game quickly. The following lemma is helpful in this respect.

Lemma 2.4 (Trick of fake moves) *Let X be a finite set and let $\mathcal{F} \subseteq 2^X$. Let $b' < b$ be positive integers. If Maker has a winning strategy for the $(1 : b)$ game $(X, \mathcal{F})$, then he has a strategy to win the $(1 : b')$ game $(X, \mathcal{F})$ within $1 + |X|/(b + 1)$ moves.*

The main idea of the proof of Lemma 2.4 is that, in every move of the $(1 : b')$ game $(X, \mathcal{F})$, Maker (in his mind) gives Breaker $b - b'$ additional board elements. The straightforward details can be found in [3].

We will also use a variant of the classical *Box Game* first introduced by Chvátal and Erdős in [6]. The *Box Game with resets* $rBox(m, b)$, first studied in [9], is played by two players, called BoxMaker and BoxBreaker. They play on a hypergraph $\mathcal{H} = \{A_1, \ldots, A_m\}$, where the sets A_i are pairwise disjoint. BoxMaker claims b elements of $\bigcup_{i=1}^m A_i$ per turn, and then BoxBreaker responds by *resetting* one of BoxMaker's *boxes*, that is, by deleting all of BoxMaker's elements from the chosen hyperedge A_i. Note that the chosen box does *not* leave the game. At every point during the game, and for every $1 \leq i \leq m$, we define *the weight* of box A_i to be the number of BoxMaker's elements that are currently in A_i, that is, the number of elements of A_i that were claimed by BoxMaker and have not been deleted yet by BoxBreaker.

Theorem 2.5 (Theorem 2.3 in [9]) *For every integer $k \geq 1$, BoxBreaker has a strategy for the game $rBox(m, b)$ which ensures that, at any point during the first k rounds of the game, every box A_i has weight at most $b(1 + \ln(m + k))$.*

We will use this theorem to provide Maker with a strategy to obtain some minimum degree in his graph. To that end, let $G = (V, E)$ be some graph, and let $V_1, V_2 \subseteq V$ be arbitrary subsets. By $(1 : b) - \mathcal{Deg}(V_1, V_2)$ we denote the $(1 : b)$ positional game where the board is E and Maker tries to get a large degree $d_M(v, V_2)$ for every $v \in V_1$. Occasionally, we shall simply refer back to this as the *degree game*. The following is an immediate conclusion of Theorem 2.5.

Claim 2.6 (The degree game) *Let $G = (V, E)$ be a graph on $|V| = n$ vertices, $V_1, V_2 \subseteq V$, and let b be an integer. Then, in the $(1 : b) - \mathcal{Deg}(V_1, V_2)$ game, Maker can ensure that $d_B(v, V_2) \leq 10b(d_M(v, V_2) + 1) \ln n$ for every vertex $v \in V_1$.*

Proof Maker pretends he is BoxBreaker and that he is playing the $rBox(n, 2b)$ game with the boxes $\{vu \in E : u \in V_2\}$, $v \in V_1$. Notice that these boxes are not necessarily disjoint, since we did not require V_1 and V_2 to be disjoint. However, any edge belongs to at most two of these boxes. So BoxBreaker can pretend that the boxes are disjoint and that BoxMaker claims $2b$ elements in every move (using the Trick of fake moves). Now, according to Theorem 2.5, BoxBreaker can ensure that at any point during the first k rounds of the game, every box has weight at most $2b(1 + \ln(n + k))$. Hence, at the end of the game every box has weight at most $2b(1 + \ln(n + \binom{n}{2})) \leq 10b \ln n$. So, for every vertex $v \in V_1$, Maker (BoxBreaker) has claimed at least one incident edge of v for every $10b \ln n$ incident edges Breaker (BoxMaker) has claimed. Hence $d_B(v, V_2) \leq 10b(d_M(v, V_2) + 1) \ln n$. □

2.2 General graph theory results

We will use the following graph which was introduced in [8]. Let $k \geq 2$ and $n \geq 3(k-1)$ be positive integers such that $(k-1) \mid n$. Let $m := \frac{n}{k-1}$. Let $C_1, \ldots, C_{k-1}$ be $k-1$ pairwise vertex disjoint cycles, each of length m. For every $1 \leq i < j \leq k-1$ let P_{ij} be a perfect matching in the bipartite graph $(V(C_i) \cup V(C_j), \{uv : u \in V(C_i),\ v \in V(C_j)\})$. Let $\mathcal{G}_k$ be the family of all graphs $G_k = (V_k, E_k)$ where $V_k = \bigcup_{i=1}^{k-1} V(C_i)$ and $E_k = \left(\bigcup_{i=1}^{k-1} E(C_i)\right) \cup \left(\bigcup_{1 \leq i < j \leq k-1} P_{ij}\right)$.

We now prove the following lemma.

Lemma 2.7 *For all integers $k \geq 2$ and $n \geq 3(k-1)$ such that $(k-1) \mid n$, every $G_k \in \mathcal{G}_k$ is k-regular and k-vertex-connected.*

Proof For $k = 2$, the lemma is trivial. So assume $k \geq 3$. It is obvious that G_k is k-regular. Let $S \subseteq V_k$ be an arbitrary set of size at most $k-1$. We will prove that $G_k \setminus S$ is connected. Assume first that there exists some $1 \leq i \leq k-1$ for which $S \cap V(C_i) = \emptyset$. Then $(G_k \setminus S) \cap C_i = C_i$ is connected and $V(C_i)$ is a dominating set of $G_k \setminus S$. It follows that $G_k \setminus S$ is connected. Assume then that $|S \cap V(C_i)| = 1$ for every $1 \leq i \leq k-1$. Hence, $(G_k \setminus S) \cap C_i$ is a path on $k-1$ vertices for every $1 \leq i \leq k-1$. Since $k-1 \geq 2$ and $|S \cap V(C_i)| = 1$ for every $1 \leq i \leq k-1$ hold by the assumption, it follows that there is at least one edge between $(G_k \setminus S) \cap C_i$ and $(G_k \setminus S) \cap C_j$ for every $1 \leq i < j \leq k-1$. It follows that $G_k \setminus S$ is connected. $\square$

The following lemma shows that if a directed graph satisfies some pseudo-random properties then it contains a long directed path. We will use it in the proof of Theorem 1.2.

Lemma 2.8 (Lemma 4.4 [4]) *Let m be an integer and let $D = (V, E)$ be an oriented graph with the following property: There exists an edge from S to T between any two disjoint sets $S, T \subseteq V$ such that $|S| = |T| = m$. Then, D contains a directed path of length at least $|V| - 2m + 1$.*

The next lemma provides a sufficient (Hall-type) condition for a bipartite graph to contain a perfect matching.

Lemma 2.9 *Let $G = (U_1 \cup U_2, E)$ be a bipartite graph with $|U_1| = |U_2| = n$. Let $r \leq n/2$ be an integer such that:*

$(B1)$ *For $i \in \{1, 2\}$ and every $X \subset U_i$ of size $|X| \leq r$, $|N(X)| \geq |X|$.*

$(B2)$ *For every $X \subset U_1$ and $Y \subset U_2$ with $|X| = |Y| = r$, $|E(X, Y)| > 0$.*

Then G has a perfect matching.

Proof In order to prove that G admits a perfect matching we will prove that G satisfies Hall's condition, that is, $|N(X)| \geq |X|$ for every $X \subseteq U_i$ (see e.g. [18]).

We distinguish three cases:

Case 1: If $|X| \leq r$, then by $(B1)$ we have that $|N(X)| \geq |X|$ and we are done.

Case 2: If $r < |X| \leq n - r$. By $(B2)$ we have that $|N(X)| \geq n - r \geq |X|$.

Case 3: $|X| > n - r$. Let $X \subseteq U_1$ and assume towards a contradiction that $|N(X)| \leq |X| - 1$. Let $Y \subset U_2$ be a subset of size $n - |X| + 1$ for which $N(X) \subseteq U_2 \setminus Y$. Since $|X| > n - r$ we get that $|Y| \leq r$ and by Case 1 we have that $|N(Y)| \geq |Y| = n - |X| + 1$. Moreover, $N(X) \subseteq U_2 \setminus Y$ implies that $N(Y) \subseteq U_1 \setminus X$, which is clearly a contradiction since we have that $n - |X| + 1 \leq |N(Y)| \leq n - |X|$. The case $X \subseteq U_2$ is treated similarly. □

2.3 Properties of $\mathcal{G}_{n,p}$

This subsection specifies properties (A1)-(A3) that a graph $G \sim \mathcal{G}_{n,p}$ fulfils a.a.s. It turns out that these properties are all we need to prove our main theorems. So in fact, we could strengthen them to hold for any graph G that has suitable pseudo-random properties.

Lemma 2.10 *Let $K \geq 2$ and let $G \sim \mathcal{G}_{n,p}$ with $p = \ln^K n/n$. Further, let $\alpha \in \mathbb{R}$ such that $1 \leq \alpha < K$, and let $f = f(n)$ be some function that satisfies $1 \leq f = O((\ln\ln n)^3)$. Then, a.a.s.*

(A1) $\delta(G) = \Theta\left(\ln^K n\right)$ *and* $\Delta(G) = \Theta\left(\ln^K n\right)$.

(A2) *For every subset $U \subseteq V$, $|E(U)| \leq \max\{100\,|U| \ln n,\ 100\,|U|^2 p\}$.*

(A3) *For any two disjoint subsets $U, W \subseteq V$ with $|U| = |W| = nf^{-1}\ln^{-\alpha} n$, $|E(U,W)| = \Omega\left(nf^{-2}\ln^{K-2\alpha} n\right)$ and $|E(U)| = \Omega\left(nf^{-2}\ln^{K-2\alpha} n\right)$.*

Proof To prove (A1), let $v \in V$. Since $d_G(v) \sim \text{Bin}(n-1, p)$ we conclude $\mathbb{E}(d_G(v)) = (n-1)p = (1 - o(1))\ln^K n$. Hence, by Lemma 2.1,

$$\mathbb{P}\Big(d_G(v) \leq (1 - 1/2)\ln^K n\Big) \leq \exp\left(-\frac{\ln^K n}{8}(1 - o(1))\right) = o(1/n).$$

Now, by the union bound argument we conclude that

$$\mathbb{P}\Big(\exists\, v \in V :\ d_G(v) \leq \frac{\ln^K n}{2}\Big) \leq n \cdot o(1/n) = o(1).$$

Similarly, by Lemma 2.1, we obtain

$$\mathbb{P}\Big(\exists\, v \in V :\ d_G(v) \geq 2\ln^K n\Big) = o(1).$$

To prove (A2), let $U \subseteq V$ be a fixed subset of size $t := |U|$. Then

$$7\,\mathbb{E}(|E(U)|) \leq 7t^2 p < 100t^2 p \leq \max\{100t \ln n, 100t^2 p\}.$$

Thus, by Lemma 2.2,

$$\begin{aligned}\mathbb{P}\Big(|E(U)| \geq \max\{100t\ln n, 100t^2p\}\Big) &\leq \exp\Big(-\max\{100t\ln n, 100t^2p\}\Big)\\ &\leq \exp\Big(-100t\ln n\Big).\end{aligned}$$

It follows that

$$
\begin{aligned}
\mathbb{P}\Big(\exists\, U \subseteq V \ :\ |E(U)| > \max\{100|U|\ln n, 100|U|^2 p\}\Big) \\
&\le \sum_{t=1}^{n} \binom{n}{t} \exp(-100t\ln n) \\
&\le \sum_{t=1}^{n} \exp\Big(t\ln n - 100t\ln n\Big) \\
&\le \sum_{t=1}^{n} \exp\Big(-99\ln n\Big) \\
&= \exp(-98\ln n) = o(1).
\end{aligned}
$$

For (A3), let $U, W \subseteq V$ be two disjoint subsets such that $|U| = |W| = nf^{-1}\ln^{-\alpha} n$. Note that $|E(U,W)|$ is binomially distributed with expectation $\mu_n := nf^{-2}\ln^{K-2\alpha} n$. So by Lemma 2.1 we have that $\mathbb{P}\left(|E(U,W)| \le \frac{\mu_n}{2}\right) \le \exp\left(-\frac{\mu_n}{8}\right)$. Applying union bound we get that

$$
\begin{aligned}
\mathbb{P}\left(\exists \text{ disjoint } U, W \subseteq V \ :\ |U| = |W| = \frac{n}{f\ln^{\alpha} n} \text{ and } |E(U,W)| \le \frac{\mu_n}{2}\right) \\
&\le \binom{n}{\frac{n}{f\ln^{\alpha} n}}^2 \exp\left(-\frac{\mu_n}{8}\right) \\
&\le \exp\left(\frac{2n}{f\ln^{\alpha} n}(\alpha\ln\ln n + 1 + \ln f) - \frac{n\ln^{K-2\alpha} n}{8f^2}\right) \\
&= o(1),
\end{aligned}
$$

since $K > \alpha$. Finally, since $|E(U)| \sim \mathrm{Bin}\left(\binom{|U|}{2}, p\right)$, it follows analogously that a.a.s. $|E(U)| = \Omega\left(nf^{-2}\ln^{K-2\alpha} n\right)$ for all U with $|U| = nf^{-1}\ln^{-\alpha} n$. □

2.4 Expanders

Definition 2.11 *Let $G = (V,E)$ be a graph with $|V| = n$. Let $R := R(n)$ and $c := c(n)$ be two positive integers. We say that the graph G is an (R,c)-*expander *if it satisfies the following two properties:*

$(E1)$ *For every subset $X \subseteq V$ with $|X| \le R$, $|N(X) \setminus X| \ge c|X|$.*

$(E2)$ *$|E(X,Y)| > 0$ for every two disjoint subsets $X, Y \subseteq V$ of size $|X| = |Y| = R$.*

Recall that a graph $G = (V,E)$ is called *Hamilton-connected* if for every $x, y \in V$, the graph G contains a Hamilton path with x and y as its endpoints. The following sufficient condition for a graph to be Hamilton-connected was introduced in [10].

Theorem 2.12 *Let n be sufficiently large, and let $G = (V,E)$ be an $\Big(n/\ln n, \ln\ln n\Big)$-expander on n vertices. Then G is Hamilton-connected.*

That is, by ensuring expander properties (locally), we can enforce a Hamilton cycle (global property).

The following theorem lies in the heart of all of our proofs. It says that in a subgraph of G of sublinear order where certain properties hold Maker is able to build a suitable expander fast, that is in $o(n)$ moves.

Theorem 2.13 *Let b be an integer, $K > 12$, let n be a sufficiently large integer and let $p = \ln^K n/n$. Let $H = (V_H, E_H)$ be a graph on $|V_H| = \Theta\left(n/\ln^4 n\right)$ vertices and let M and F be two edge disjoint subgraphs of H, where E_M already belongs to Maker and F consists of free edges. Assume that the following properties hold:*

(1) $E_M \cup E_F = E_H$.

(2) There exist a constant $c_1 > 0$ and a partition $V_H = A_1 \cup (V_H \setminus A_1)$ such that

$$\begin{aligned} &d_M(v) \geq c_1 \ln^{K-6} n \text{ for every } v \in A_1,\\ &d_F(v) \geq c_1 \ln^{K-4} n \text{ for every } v \notin A_1\\ &\text{and } |A_1| = O(n \cdot \ln^{6-K} n). \end{aligned}$$

(3) For any two disjoint subsets $U, W \subseteq V_H$ of size $|U| = |W| = \frac{n}{(\ln n)^5 (\ln\ln n)^3}$,
$|E_H(U,W)| = \Omega\left(\frac{n \ln^{K-10} n}{(\ln\ln n)^6}\right)$.

(4) For every subset $U \subseteq V_H$, $|E_H(U)| \leq \max\{100\,|U| \ln n,\ 100\,|U|^2 p\}$.

Then, for every $c \leq \ln\ln|V_H|$ and $R = |V_H|/\ln|V_H|$, in the $(1:b)$ Maker-Breaker game played on E_H, Maker has a strategy to build an (R,c)-expander within $o(n)$ moves.

Before we prove this theorem we need an auxiliary result. Consider a graph H with (edge-disjoint) subgraphs M and F such that the assumptions of Theorem 2.13 hold. Given a subgraph $H_1 = (V_H, E_1)$ of H, we denote M_1 and F_1 to be the restrictions of M and F respectively to the subgraph H_1. The following lemma says that in H we can find a sparse subgraph with suitable properties that will guarantee Maker's win in the expander game.

Lemma 2.14 *Under the assumptions of Theorem 2.13 there exists a subgraph $H_1 = (V_H, E_1)$ of H with the following properties:*

(i) For every $v \notin A_1$, $d_{F_1}(v) = \Omega(\ln^3 n)$.

(ii) For any two disjoint subsets $U, W \subseteq V_H$ such that $|U| = |W| = \frac{n}{(\ln n)^5 (\ln\ln n)^3}$,
$|E_1(U,W)| = \Omega\left(\frac{n}{(\ln n)^3 (\ln\ln n)^6}\right)$.

(iii) For every $U \subseteq V_H$, $|E_1(U)| \leq \max\{1000|U|\ln n, 1000|U|^2 \ln^7 n/n\}$.

(iv) $|E_1| = o(n)$.

Note that all size parameters in this lemma do not depend on K anymore. We want to stress that this is crucial for obtaining $|E_1| = o(n)$.

Proof Let $\rho = \ln^{7-K}(n)$. Pick every edge of H to be an edge of H_1 with probability ρ independently of all other choices. Let $s(n) := \frac{n}{(\ln n)^5 (\ln\ln n)^3}$. The properties (i)-(iv) will all be proven by identifying the correct binomial distribution and by applying Chernoff- and union-bound-type arguments.

To prove property (i), notice that for every $v \notin A_1$ the degree of v in F_1 is binomially distributed, that is, $d_{F_1}(v) \sim \mathrm{Bin}(d_F(v), \rho)$ with mean $\mathbb{E}(d_{F_1}(v)) \geq c_1 \ln^3 n$.

Therefore, by Lemma 2.1 we have $\mathbb{P}\Big(d_{F_1}(v) \leq \frac{1}{2} c_1 \ln^3 n\Big) \leq \exp\Big(-\frac{1}{8} c_1 \ln^3 n\Big)$. Hence, by the union bound we conclude $\mathbb{P}\Big(\exists\, v \in V_H \setminus A_1 : d_{F_1}(v) \leq \frac{1}{2} c_1 \ln^3 n\Big) = o(1)$.

For property (ii), let $c_2 > 0$ be such that $|E_H(U,W)| \geq c_2 n \ln^{K-10} n/(\ln\ln n)^6$ for every two disjoint subsets $U, W \subseteq V_H$ with $|U| = |W| = s(n)$. This c_2 clearly exists by assumption (3). Let U, W be such subsets. Since $|E_1(U,W)| \sim \mathrm{Bin}(E_H(U,W), \rho)$ with mean $\mathbb{E}(|E_1(U,W)|) \geq \frac{c_2 n}{(\ln n)^3 (\ln\ln n)^6}$, by Lemma 2.1,

$$\mathbb{P}\left(|E_1(U,W)| \leq \frac{c_2 n}{2(\ln n)^3 (\ln\ln n)^6}\right) \leq \exp\left(-\frac{c_2 n}{8(\ln n)^3 (\ln\ln n)^6}\right).$$

Therefore, by union bound we conclude that

$$\begin{aligned}
&\mathbb{P}\left(\exists\, U, W \subseteq V_H : |U| = |W| = s(n) \text{ and } |E_1(U,W)| \leq \frac{c_2 n}{2(\ln n)^3 (\ln\ln n)^6}\right)\\
&\quad\leq \binom{n}{\frac{n}{\ln^5 n}}^2 \exp\left(-\frac{c_2 n}{8(\ln n)^3 (\ln\ln n)^6}\right)\\
&\quad\leq \exp\left(\frac{2n}{\ln^4 n} - \frac{c_2 n}{8(\ln n)^3 (\ln\ln n)^6}\right) = o(1).
\end{aligned}$$

To prove property (iii), note that $|E_1(U)| \sim \mathrm{Bin}(E_H(U), \rho)$ with expectation $\mathbb{E}(|E_1(U)|) \leq \max\{100\,|U| \ln n,\ 100\,|U|^2 \ln^7 n/n\}$. Again, by Lemma 2.2 and union bound we get that:

$$\begin{aligned}
&\mathbb{P}\Big(\exists\, U \subseteq V_H : |E_1(U)| \geq \max\Big\{1000\,|U| \ln n,\ 1000\,|U|^2 \ln^7 n/n\Big\}\Big)\\
&\quad\leq \sum_{t=1}^{|V_H|} \binom{|V_H|}{t} \exp\Big(-\max\Big\{1000t \ln n,\ 1000t^2 \ln^7 n/n\Big\}\Big)\\
&\quad\leq \sum_{t=1}^{n} \exp\Big(t \ln n - 1000t \ln n\Big)\\
&\quad\leq n \exp(-999 \ln n) = o(1).
\end{aligned}$$

For property (iv), notice that $|E_1| \sim \mathrm{Bin}(|E_H|, \rho)$. By condition (4), and since $|V_H| = \Theta(n/\ln^4 n)$, $|E_H| = O(n \ln^{K-8} n)$. So the expected size of E_1 is $\mu = O(n\rho \ln^{K-8} n) = o(n)$. Hence, again, by Lemma 2.1 we conclude that $|E_1| = o(n)$ with probability tending to 1.

We have shown that in the randomly chosen subgraph the properties $(i)-(iv)$ hold a.a.s. In particular, there exists an instance where all hold. $\square$

Now, we are ready to prove Theorem 2.13.

Proof [of Theorem 2.13]

Let $H_1=(V_H,E_1)$ be a subgraph of H as given by Lemma 2.14. For achieving his goal, Maker will play two games in parallel on E_1. In the odd moves Maker plays the $(1:2b)$ degree game on F_1 and in the even moves he plays as $\mathcal{F}$-Breaker the $(2b:1)$ game $(E_1,\mathcal{F})$, where the winning sets are

$$\mathcal{F}=\left\{E_1(U,W): U,W\subseteq V_H,\ U\cap W=\emptyset \text{ and } |U|=|W|=\frac{n}{(\ln n)^5\,(\ln\ln n)^3}\right\}.$$

Combining Claim 2.6 and Lemma 2.14, Maker can ensure with his odd moves that

$$\text{for every } v\in V_H\setminus A_1\ : d_{M\cap H_1}(v)=\Omega(\ln^2 n). \tag{2.1}$$

Also, by Lemma 2.14 (ii),

$$\begin{aligned}\sum_{F\in\mathcal{F}}2^{-|F|/2b}&\le\binom{n}{\frac{n}{\ln^5 n}}^2 2^{-\Omega\left(n/((\ln n)^3\,(\ln\ln n)^6)\right)}\\&\le\exp\left(\frac{2n}{\ln^4 n}-\Omega\left(\frac{n}{(\ln n)^3\,(\ln\ln n)^6}\right)\right)=o(1).\end{aligned}$$

So by Theorem 2.3 Maker (as $\mathcal{F}$-Breaker) wins the game $(E_1,\mathcal{F})$. That is, for any two disjoint subsets $U,W\subseteq V_H$ of size $|U|=|W|=\frac{|V_H|}{\ln|V_H|}=\Theta\left(\frac{n}{\ln^5 n}\right)=\omega(\frac{n}{(\ln n)^5\,(\ln\ln n)^3})$, Maker can claim an edge between U and W. Note that this gives condition $(E2)$ of the expander definition, with $R=|V_H|/\ln|V_H|$. Furthermore, by Lemma 2.14 (iv), the game lasts $|E_1|=o(n)$ moves.

To prove that by the end of this game Maker's graph is indeed a $\Big(|V_H|/\ln|V_H|,\ \ln\ln|V_H|\Big)$-expander, it remains to check condition (E1).

Assume for a contradiction that there exists a set $X\subseteq V_H$ such that

$$|X|\le|V_H|/\ln|V_H| \quad\text{and}\quad |X\cup N_M(X)|\le 2|X|\ln\ln|V_H|. \tag{2.2}$$

We distinguish three cases.

Case 1: $|X\cap A_1|\ge|X|/2$. Then $|X|=O(n\cdot\ln^{6-K}n)$ by assumption (2). Hence, and by assumption (4) and (2.2),

$$\begin{aligned}|E_H(X,N_M(X))|&\le|E_H(X\cup N_M(X))|\\&\le\max\Big\{100\,|X\cup N_M(X)|\,\ln n,\ 100\,|X\cup N_M(X)|^2p\Big\}\\&=O\Big(\max\Big\{|X|(\ln\ln|V_H|)\ln n,|X|(\ln\ln|V_H|)^2\,\ln^6 n\Big\}\Big).\end{aligned}$$

But this implies $|E_H(X, N_M(X))| = o(|X| \ln^{K-6} n)$ since $K > 12$. However, since every vertex $v \in A_1$ has Maker degree at least $c_1 \cdot \ln^{K-6} n$ we also conclude that $|E_M(X, N_M(X))| = \Omega(|X| \ln^{K-6} n)$, a contradiction.

Case 2: $|X \setminus A_1| \geq |X|/2$ and $|X| < \frac{n}{(\ln n)^5 (\ln \ln n)^3}$. By (2.1), for every $v \in X \setminus A_1$, $d_{M \cap H_1}(v) = \Omega(\ln^2 n)$. Hence, $|E_{M \cap H_1}(X, N_M(X))| = \Omega(|X| \ln^2 n)$.
On the other hand, by Lemma 2.14,

$$\begin{aligned} |E_{H_1}(X, N_M(X))| &\leq \max\Big\{1000\,|X \cup N_M(X)|\, \ln n,\ 1000\,|X \cup N_M(X)|^2\, \ln^7 n/n\Big\} \\ &= O\Big(\max\Big\{|X|(\ln\ln |V_H|) \ln n, |X|^2 (\ln\ln |V_H|)^2\, \ln^7 n/n\Big\}\Big) \\ &= o(|X| \ln^2 n), \end{aligned}$$

where the first equality follows from (2.2). But this, again, is a contradiction.

Case 3: $\frac{n}{(\ln n)^5 (\ln \ln n)^3} \leq |X| \leq \frac{|V_H|}{\ln |V_H|}$. Since Maker wins (as $\mathcal{F}$-Breaker) the game $(E_1, \mathcal{F})$ we conclude that

$$|N_M(X)| \geq |V_H| - \frac{n}{(\ln n)^5 (\ln \ln n)^3} = \Omega\left(\frac{n}{\ln^4 n}\right) = \omega(|X| \ln\ln n),$$

which contradicts (2.2). This completes the proof. $\square$

3 The Perfect Matching Game

In this section we prove Theorem 1.1 and a variant for random bipartite graphs.

Proof [of Theorem 1.1] First we describe a strategy for Maker and then we prove it is a winning strategy. At any point during the game, if Maker cannot follow the proposed strategy (including the time limits) then he forfeits the game. Before the game starts, Maker picks a subset $U_0 \subseteq V$ of size $|U_0| = \frac{n}{\ln^4 n}$ such that for every $v \in V$, $d(v, U_0) = \Omega(\ln^{K-4} n)$. Such a subset exists because a randomly chosen subset of size $\frac{n}{\ln^4 n}$ has this property by a Chernoff-type argument a.a.s. Now, we divide Maker's strategy into two main stages.

Stage I: At this stage, Maker builds a matching M_0 of size $n/2 - n/\ln^4 n$ which does not touch U_0. Moreover, Maker wants to ensure that by the end of this stage, for every $v \in V$,

$$d_F(v, U_0) = \Omega\left(\ln^{K-4} n\right), \quad \text{or} \quad d_M(v, U_0) = \Omega\left(\ln^{K-6} n\right). \tag{3.1}$$

Initially, set $M_0 = \emptyset$. For $i \leq n$, as long as $|M_0| < n/2 - n/\ln^4 n$, Maker plays his i-th move as follows:

(1) If there exists an integer j such that $i = j\lfloor \ln n \rfloor$, then Maker plays the degree game $(1 : b \ln n) - \mathcal{Deg}(V, U_0)$.

(2) Otherwise, Maker claims an arbitrary free edge $e_i \in E$ s.t. $e_i \cap e = \emptyset$ for every $e \in M_0$ and $e_i \cap U_0 = \emptyset$. Then, Maker updates M_0 to $M_0 \cup \{e_i\}$.

When Stage I is over, i.e. $|M_0| = n/2 - n/\ln^4 n$, Maker proceeds to Stage II.

Stage II: Let $V_H = V \setminus V(M_0)$ with $|V_H| = 2n/\ln^4 n$, and let $H := (G-B)[V_H]$. We will show that H together with the subgraphs M consisting of Maker's edges and F consisting of the free edges satisfies the conditions of Theorem 2.13. That is, Maker can play on H according to the strategy suggested by the theorem and build a suitable expander in $o(n)$ moves.

Indeed, stage I and stage II constitute a winning strategy, i.e. if Maker can follow the proposed strategy, he will get a perfect matching of G. By Theorem 2.13, Maker's subgraph of H will be an (R,c)-expander with $R = |V_H|/\ln|V_H|$ and $c = \ln\ln|V_H|$, for large n. By Theorem 2.12, this subgraph will be Hamilton-connected and that is why it will contain a perfect matching M_1. Together with M_0 this forms a perfect matching of G. Furthermore, Maker will win in $n/2 + o(n)$ moves, since Stage I lasts at most $n/2 + o(n)$ rounds, whereas in Stage II Maker needs only $o(n)$ moves. Thus, we only need to guarantee that Maker can follow the strategy.

By Lemma 2.10, the properties (A1), (A2) and (A3) hold a.a.s. for G. We condition on these, and henceforth assume that G satisfies (A1), (A2) and (A3), where $f \in \{1, (\ln\ln n)^3\}$. We consider each stage separately.

Stage I: First, consider part (2), that is when Maker tries to build the matching M_0 greedily. Assume that Maker has to play his i-th move in Stage I and $i \neq j\lfloor \ln n \rfloor$ for any $j \in \mathbb{N}$. Furthermore, assume that still $|M_0| < n/2 - n/\ln^4 n$. Let $T := V \setminus (V(M_0) \cup U_0)$. Then $|T| > n/\ln^4 n$. Thus, by (A3) ($f = 1$), $|E(T)| = \omega(n)$. Since $i \leq n$, Maker and Breaker have claimed $O(n)$ edges so far. In particular, Maker can find a free edge in T to be added to M_0. Thus, he can follow part (2) of Stage I.

Secondly, consider part (1). It is clear that Maker can play the degree game. Thus, we only need to prove that the desired degree condition (3.1) will hold. We already know that there exists a constant $c_1 > 0$ with $d(v, U_0) \geq c_1 \ln^{K-4} n$ for every $v \in V$. If at the end of Stage I Breaker has $d_B(v, U_0) \leq 0.5c_1 \ln^{K-4} n$ for some $v \in V$, then (3.1) holds trivially for this v. Thus, we can assume that $d_B(v, U_0) \geq 0.5c_1 \ln^{K-4} n$. In this case, Claim 2.6 gives $d_M(v, U_0) \geq 0.04c_1 \ln^{K-6} n/b$.

Stage II: We only need to check whether the conditions of Theorem 2.13 hold for $H = (V_H, E(H))$. Firstly, $|V_H| = 2n/\ln^4 n$. Also, condition (1) holds trivially by the definition of H.

For condition (2), note that because of the degree condition (3.1) we can find a constant c_2 such that $V_H = A_1 \cup (V \setminus A_1)$, where $d_M(v) \geq c_2 \ln^{K-6} n$ for every $v \in A_1$ and $d_F(v) \geq c_2 \ln^{K-4} n$ for every $v \notin A_1$. Since Stage I took at most n rounds, $|A_1| = O(n \cdot \ln^{6-K} n)$.

Towards condition (3), note that by (A3) ($f = (\ln\ln n)^3$) for every disjoint $U, W \subseteq V$ of size $\frac{n}{(\ln n)^5 (\ln\ln n)^3}$, $|E(U,W)| = \Omega\Big(\frac{n \ln^{K-10} n}{(\ln\ln n)^6}\Big)$. Since Stage I took at most n rounds, Breaker has claimed $O(n)$ edges. Hence, in the reduced graph (where Breaker's edges are deleted), property (3) is satisfied.

Condition (4) follows by (A2) and since $H \subseteq G$. $\square$

In the light of Theorem 1.3, i.e. the k-connectivity game, we would like to get a similar result for a random bipartite graph. That is, for even n we denote by $\mathcal{B}_{n,p}$ a bipartite graph with two vertex classes of size $n/2$, where every possible edge is inserted with probability p. We show that Maker can win the perfect matching game on $\mathcal{B}_{n,p}$ fast. The main difference to the proof of Theorem 1.1 is that Maker will not build an expander, but will rather fulfill the conditions of Lemma 2.9.

Theorem 3.1 *Let $b \geq 1$ be an integer, let $K > 12$, $p = \frac{\ln^K(n)}{n}$, and let $G \sim \mathcal{B}_{n,p}$. Then a.a.s. Maker wins the $(1 : b)$ perfect matching game played on G within $\frac{n}{2} + o(n)$ moves.*

Proof The proof is analogous to the proof of Theorem 1.1, so we just sketch it here.

For $G = (U_1 \cup U_2, E(G)) \sim \mathcal{B}_{n,p}$, first choose a subset $U_0 \subset U_1 \cup U_2$ such that $|U_0 \cap U_1| = |U_0 \cap U_2| = \frac{n}{2\ln^4 n}$, and $d(v, U_0) = \Omega(\ln^{K-4} n)$ for every $v \in U_1 \cup U_2$.

Then Maker divides the game into two stages.

Stage I: Maker again builds greedily a matching M_0 of size $n/2 - n/\ln^4 n$ which does not touch U_0. Furthermore, Maker ensures that by the end of this stage for some $c_1 > 0$, $d_F(v, U_0) \geq c_1 \ln^{K-4} n$ or $d_M(v, U_0) \geq c_1 \ln^{K-6} n$ for every $v \in U_1 \cup U_2$.

Stage II: Let $V_H = V \setminus V(M_0)$ with $|V_H \cap U_i| = \frac{n}{\ln^4 n}$ and let $H = (G - B)[V_H]$. Maker plays similarly to the strategy given in the proof of Theorem 2.13. This time, he will not build an expander like before. But he will ensure that after $o(n)$ rounds his subgraph of H will satisfy conditions $(B1)$ and $(B2)$ of Lemma 2.9 with $r = |V_H|/\ln(|V_H|)$.

Similarly to Lemma 2.14, we find a sparser subgraph $H_1 \subseteq H$ with the analogue properties for bipartite graphs. As in the proof of Theorem 2.13, Maker plays in every even move the $(2b : 1)$ game $(E_1, \mathcal{F})$ as $\mathcal{F}$-Breaker where E_1 is the edge set of H_1, and where

$$\mathcal{F} = \left\{ E_1(U, W) : U \subseteq U_1,\ W \subseteq U_2 \text{ and } |U| = |W| = \frac{n}{(\ln n)^5 (\ln \ln n)^3} \right\}.$$

Winning this game, he will ensure (B2) with $r = |V_H|/\ln |V_H|$.

To obtain (B1), Maker plays in each odd move the $(1 : 2b)$ degree game. □

4 The Hamiltonicity Game

In this section we prove Theorem 1.2.

Proof First we describe a strategy for Maker and then we prove it is a winning strategy. At any point during the game, if Maker cannot follow the proposed strategy (including the time limits) then he forfeits the game. As in the perfect matching game, Maker picks a subset $U_0 \subseteq V$ of size $|U_0| = \frac{n}{10\ln^4 n}$ such that for every $v \in V$, $d(v, U_0) = \Omega(\ln^{K-4} n)$.

We divide Maker's strategy into the following four main stages.

Stage I: At this stage, Maker builds a matching M_0 of size $n/2 - n/(9\ln^4 n)$ which does not touch U_0. Moreover, Maker wants to ensure that by the end of this stage $d_F(v, U_0) = \Omega\left(\ln^{K-4} n\right)$ or $d_M(v, U_0) = \Omega\left(\ln^{K-6} n\right)$ for every $v \in V$. As soon as this stage is over, Maker proceeds to Stage II.

Stage II: For a path P let $End(P)$ denote the set of its endpoints. Throughout this stage, Maker maintains a set M_1 of vertex disjoint paths, a subset $M_2 \subseteq M_1$ and a set $End := \{v \in$

$V : \exists P \in M_1 \setminus M_2$ such that $v \in End(P)\}$. Initially, $M_1 := M_0$ and $M_2 = \emptyset$. Let r be the number of rounds Stage I lasted. For every $i > r$, Maker will play his i-th move of this stage as follows:

(1) If there exists an integer j such that $i = j\lfloor \ln n \rfloor$, then Maker plays the degree game $(1 : b \ln n) - \mathcal{Deg}(V, U_0)$.

(2) Otherwise, Maker claims a free edge xy between two vertices from End which are endpoints of two disjoint paths. Also, Maker updates M_1 by replacing the two old paths merged through xy by a new one. Note that this new path is not deleted from M_1. Maker also updates End accordingly.

(3) If there is any path P of length at least $10 \ln^{K/3} n$ then Maker updates $M_2 := M_2 \cup \{P\}$.

This stage ends when $|M_1| = \lfloor n/\ln^{K/3} n \rfloor$. Thus, Stage II lasts not more than $n/2 + o(n)$ rounds. When this stage ends, Maker proceeds to Stage III.

Stage III: In this stage Maker ensures that his graph on $V \setminus U_0$ will contain a path P of length at least $n - n/\ln^4 n$. Moreover, Maker does so within $o(n)$ moves. Let s be the number of rounds Stage I and II lasted. For every $i > s$, Maker plays his i-th move of this stage as follows:

(1) If there exists an integer j such that $i = j\lfloor \ln n \rfloor$, then Maker plays the degree game $(1 : b \ln n) - \mathcal{Deg}(V, U_0)$.

(2) Otherwise, consider the paths in M_1 of length at least $3 \ln^{K/4} n$. Maker tries to connect these paths, not necessarily through their endpoints, but through points close to their ends. The full details of this partial game will be given in the proof below.

Stage IV: Let x, y be the endpoints of P, the long path created in Stage III. Let $V_H = (V \setminus V(P)) \cup \{x, y\}$. At this stage Maker builds a Hamilton path on $(G - B)[V_H]$ with x, y as its endpoints. Moreover, Maker does so within $o(n)$ moves.

It is evident that if Maker can follow the proposed strategy then he wins the Hamiltonicity game within $n + o(n)$ moves. It thus remains to prove that indeed Maker can follow the proposed strategy without forfeiting the game.

By Lemma 2.10, the properties (A1), (A2) and (A3) hold a.a.s. for G. We condition on these, and henceforth assume that G satisfies (A1), (A2) and (A3), where $f \in \{1, (\ln \ln n)^3\}$. We consider each stage separately.

Stage I: The proof that Maker can follow the proposed strategy for this stage is analogous to the proof that Maker can follow Stage I of the proposed strategy in the proof of Theorem 1.1.

Stage II: Assume $i \neq j\lfloor \ln n \rfloor$. If $|M_1| > n/\ln^{K/3} n$, then $|M_1 \setminus M_2| = \Omega\left(n/\ln^{K/3} n\right)$, since there can be at most $n/(10 \ln^{K/3} n)$ disjoint paths of length at least $10 \ln^{K/3} n$. Hence, $|End| = \Omega\left(n/\ln^{K/3} n\right)$ and by (A3) ($f = 1$), we have that the number of edges of G spanned by End is $\Omega\left(n\ln^{K/3} n\right) = \omega(n)$. Therefore, we conclude that indeed Maker can claim a free edge in $G[End]$.

Stage III: Let $U' = \{v \in V : v \text{ belongs to a path of length } \leq 3\ln^{K/4} n \text{ in } M_1\}$ and update $M_1 := M_1 \setminus \{P : P \text{ is of length } \leq 3\ln^{K/4} n\}$. Notice that

$$|U'| \leq 3\ln^{K/4} n \cdot \frac{n}{\ln^{K/3} n} = o\left(\frac{n}{\ln^4 n}\right).$$

So the sum of the lengths of all paths in M_1 is at least

$$|V(M_0) \setminus U'| - |M_1| \geq n - n/4\ln^4 n. \tag{4.1}$$

For every path $P \in M_1$, define $L(P)$ and $R(P)$ to be the first and last $\ln^{K/4} n$ vertices of P (according to some fixed orientation of the path). Notice that since $|V(P)| > 3\ln^{K/4} n$ it follows that $L(P) \cap R(P) = \emptyset$ for every $P \in M_1$. Now, let $m = n/\ln^{K/2} n$ and let $\mathcal{H} = (X, \mathcal{F})$ be the hypergraph whose vertices are all edges of $G - B$ with both endpoints in $\bigcup_{P \in M_1}(L(P) \cup R(P))$ and whose hyperedges are:

$$\mathcal{F} = \left\{E_{G-B}(S,T) : \exists \text{ distinct } P_1, ..., P_{2m} \in M_1 \text{ s.t } S = \bigcup_{i=1}^{m} L(P_i), T = \bigcup_{i=m+1}^{2m} R(P_i)\right\}.$$

Note that for $E_{G-B}(S,T) \in \mathcal{F}$, $|S| = |T| = m\ln^{K/4} n = n/\ln^{K/4} n$ holds. Thus, by (A3) $(f = 1)$, we have for an element of $\mathcal{F}$ that

$$|E_{G-B}(S,T)| \geq |E_G(S,T)| - (1 + o(1))bn = \Omega\left(n\ln^{K/2} n\right).$$

Moreover, by (A2), we get that $|X| = O\left(\left(\ln^{K/4} n|M_1|\right)^2 p\right) = O\left(n\ln^{5K/6} n\right)$.

Now,

$$\begin{aligned}
\sum_{F \in \mathcal{F}} 2^{-|F|/\ln^{0.9K} n} &= \sum_{F \in \mathcal{F}} 2^{-\Omega(n\ln^{-0.4K} n)} \\
&\leq \binom{|M_1|}{m}^2 2^{-\Omega(n\ln^{-0.4K} n)} \\
&\leq \left(e\ln^{K/6} n\right)^{2n/\ln^{K/2} n} 2^{-\Omega(n\ln^{-0.4K} n)} \\
&\leq \exp\left(\frac{2n}{\ln^{K/2} n}(1 + K\ln\ln n/6) - \Omega\Big(\frac{n}{\ln^{0.4K} n}\Big)\right) = o(1).
\end{aligned}$$

Thus, by Theorem 2.3 Maker as $\mathcal{F}$-Breaker can win the $(\ln^{0.9K} n, 1)$ game $(X, \mathcal{F})$. Lemma 2.4 therefore tells us that Maker can claim at least one element in every $F \in \mathcal{F}$ within $1 + |X|/(\ln^{0.9K} n + 1) = o(n)$ moves.

To complete Stage III, let us define the auxiliary directed graph $D = (V_D, E_D)$ whose vertices are $\{P : P \in M_1\}$ and whose directed edges are $\{(P,Q) : E_M(R(P), L(Q)) \neq \emptyset\}$. Notice that for every pair of disjoint subsets $S, T \subseteq V_D$ such that $|S| = |T| = m$, there exists an edge in

D from S to T, since Maker wins the game $(X, \mathcal{F})$. Now, we claim that Maker has a path of the desired length in his graph. By Lemma 2.8, D contains a directed path $\mathcal{P} = P_0 \dots P_t$ of length $t \geq |V_D| - 2m + 1$. Further, note that any path in M_1 has length at most $20 \ln^{K/3} n$. Combining this with (4.1), removing paths $P \in M_1$ which do not appear in $\mathcal{P}$ and deleting unnecessary parts of $L(P)$ and $R(P)$ from paths $P \in \mathcal{P}$ we conclude that Maker has thus created a path of length at least $n - n/4 \ln^4 n - 2|M_1| \ln^{K/4} n - 2m \cdot 20 \ln^{K/3} n \geq n - n/\ln^4 n$.

Stage IV: Let P be the long path Maker has created in Stage III, and let x, y be its endpoints. Denote $V_H = (V \setminus V(P)) \cup \{x, y\}$. Analogously to the perfect matching game, we can use Theorem 2.13 and Theorem 2.12 on $H := (G - B)[V_H]$. That is, Maker can build an expander on a sparse subgraph, and thus obtains a Hamilton path in H with x, y as its endpoints in $o(n)$ moves. This completes the proof. □

5 The k-Connectivity Game

In this section we prove Theorem 1.3. It is a simple application of the Hamiltonicity game, the Perfect-matching game on random bipartite graphs, and the degree game.

Proof Let $G \sim \mathcal{G}_{n,p}$, and randomly partition the vertex set into k disjoint sets $V_1, ..., V_{k-1}, W$ where each V_i has size $\left\lfloor \frac{n}{k-1} \right\rfloor$ (W might be empty). For every $1 \leq i \leq k-1$, let $G_i = G[V_i]$, and for every $1 \leq i < j \leq k-1$ let G_{ij} be the bipartite subgraph of G with parts V_i and V_j. From the definition it is clear that $G_i \sim \mathcal{G}_{\lfloor \frac{n}{k-1} \rfloor, p}$ for every $1 \leq i \leq k-1$ and $G_{ij} \sim \mathcal{B}_{2\lfloor \frac{n}{k-1} \rfloor, p}$ for every $1 \leq i < j \leq k-1$.

Now, Maker's strategy is to play the Hamiltonicity game on every G_i, the perfect matching game on every G_{ij}, and for every $w \in W$, Maker wants to claim k distinct edges ww' with $w' \in V \setminus W$ (recall that G is typically such that $d(v) = \Theta(\ln^K n)$ for every vertex $v \in V(G)$). Thus, in total Maker plays on $t \leq k - 1 + \binom{k-1}{2} + k - 2 = \binom{k}{2} + k \leq k^2$ boards. Enumerate all boards arbitrarily, and let Maker play on board $i \mod t$ in his i-th move. Between any two moves on a particular board, Breaker has claimed at most bk^2 new edges on this board. Using the trick of fake moves we can assume that Maker plays the $(1 : bk^2)$ Hamiltonicity game on every G_i, the $(1 : bk^2)$ perfect matching game on every G_{ij}, and the degree-game $(1 : bk^2) - \mathcal{D}eg(\{w\}, V \setminus W)$ for every $w \in W$. By Theorem 1.2 and Theorem 3.1, every Hamiltonicity game and every perfect matching game lasts $\left\lfloor \frac{n}{k-1} \right\rfloor + o(n)$ moves, whereas the games $Deg(\{w\}, V \setminus W)$ last in total at most $k|W| = O(1)$ moves. If Maker succeeds on some board (that is, either he formed a Hamilton cycle on some G_j, or a perfect matching on some $G_{j_1 j_2}$, or $d_M(w, V \setminus W)) \geq k$ for $w \in W$), then he quits playing on that particular board. That is, he ignores this board and plays on another one where he has not won yet.

By Lemma 2.7 Maker is thus able to build a k-connected graph on $G[V_1 \cup \ldots \cup V_{k-1}]$. Also, since for every $w \in W$, $d_M(w, V \setminus W) \geq k$, Maker's final graph will be k-connected. In total, Maker plays at most

$$(k-1)\left(\left\lfloor \frac{n}{k-1} \right\rfloor + o(n)\right) + \binom{k-1}{2}\left(\left\lfloor \frac{n}{k-1} \right\rfloor + o(n)\right) + O(1) \leq \frac{kn}{2} + o(n)$$

moves, as claimed. □

6 Open problems

We conclude with the list of several open problems directly relevant to the results of this paper.

Sparser graphs. For the three games considered in this paper, we would like to find fast winning strategies for Maker when the games are played on $G \sim \mathcal{G}_{n,p}$, where $p = (1+\varepsilon)\frac{\ln n}{n}$ for a constant $\varepsilon > 0$. Our proofs heavily depend on the ability of Maker to build an expander fast (cf. Theorem 2.13), which does not seem possible for such small p. We were not able to prove an analogue to Lemma 2.14 for smaller p's mainly because of Property (iv) in this lemma. Therefore, we find it very interesting to either find fast strategies for Maker substantially different from ours, or alternatively provide Breaker with a strategy for delaying Maker's win by a linear number of moves.

Faster winning strategies for Maker. In this paper we have proved that Maker can win the perfect matching game, the Hamiltonicity game and the k-connectivity game played on $G \sim \mathcal{G}_{n,p}$ within $n/2 + o(n)$, $n + o(n)$ and $kn/2 + o(n)$ moves, respectively. Although this is asymptotically tight it could be that the error term does not depend on n. It would be interesting to find the error term explicitly, or at least to provide tighter estimates on it.

Fast winning strategies for other games. It would be very interesting to prove similar results, i.e. fast winning strategies for Maker, for other games played on $G \sim \mathcal{G}_{n,p}$. We suggest the fixed-spanning-tree game. To be precise, let $\Delta \in \mathbb{N}$ be fixed, and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of trees on n vertices with bounded maximum degree $\Delta(T_n) \leq \Delta$. Maker's goal is to build a copy of T_n within $n + o(n)$ moves. Notice that this problem might be much harder than what we proved since even the problem of embedding spanning trees into $G \sim \mathcal{G}_{n,p}$ is still not completely settled (for more details see, e.g [15], [11]).

Winning strategies for Red. The problems considered in this paper were initially motivated by finding winning strategies for Red in the strong games via fast winning strategies for Maker (see [8], [7]). It would be very interesting to prove that indeed typically Red can win the analogous strong games played on $G \sim \mathcal{G}_{n,p}$.

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