Fast embedding of spanning trees in biased Maker-Breaker games

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Abstract

Given a tree $T = (V, E)$ on n vertices, we consider the $(1 : q)$ Maker-Breaker tree embedding game $\mathcal{T}_n$. The board of this game is the edge set of the complete graph on n vertices. Maker wins $\mathcal{T}_n$ if and only if he is able to claim all edges of a copy of T. We prove that there exist real numbers $\alpha, \varepsilon > 0$ such that, for sufficiently large n and for every tree T on n vertices with maximum degree at most n^ε, Maker has a winning strategy for the $(1 : q)$ game $\mathcal{T}_n$, for every $q \leq n^\alpha$. Moreover, we prove that Maker can win this game within $n + o(n)$ moves which is clearly asymptotically optimal.

1   Introduction

In this paper we are interested in biased Maker-Breaker games played on the edges of the complete graph K_n on n vertices, where Maker's goal is to build a copy of a given tree T on n vertices whose maximum degree might grow with n. We pay special attention to the duration of the game, aiming to prove that Maker has a strategy to build a copy of T very fast – within $n + o(n)$ moves.

The problem of embedding large or nearly spanning bounded degree trees in random graphs on n vertices (where by a nearly spanning tree we mean a tree whose number of vertices is at most $(1 - c)n$ for some constant $c > 0$) is a rather well studied subject (see e.g. [7], [10], [1]). In particular, Alon, Sudakov and the third author proved in [1] that for given $\varepsilon > 0$ and integer

d there exists $C = C(d, \varepsilon) > 0$ such that with high probability the random graph $G(n, p)$ with $p = C/n$ admits a copy of a tree T on $(1-\varepsilon)n$ vertices with maximum degree at most d. Hence, one can embed nearly spanning trees with bounded degree into very sparse random graphs. Much less is known about embedding spanning trees into sparse random graphs, though some progress was achieved in recent years (see e.g. [18], [13], [16]). In particular, the third author proved in [18] that any given spanning bounded degree tree T can be embedded with high probability into the random graph $G(n, p)$, with $p = n^{-1+\varepsilon}$ for an arbitrarily small constant $\varepsilon > 0$. In [16] a universality type result is proved. Namely, (in particular) it is proved that with high probability $G(n, p)$, with $p = cdn^{-1/3} \log n$ for an appropriate constant $c > 0$, contains simultaneously a copy of every tree on n vertices with maximum degree d (note that d is allowed to grow with n). In this paper we prove a tree embedding result in the context of Maker-Breaker games.

Let X be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets. In the $(p : q)$ Maker-Breaker game $(X, \mathcal{F})$, two players, called Maker and Breaker, take turns in claiming previously unclaimed elements of X, with Breaker going first. The set X is called the *board* of the game and the members of $\mathcal{F}$ are referred to as the *winning sets*. Maker claims p board elements per turn, whereas Breaker claims q. The parameters p and q are called the *bias* of Maker and of Breaker respectively. Maker wins the game as soon as he occupies all elements of some winning set. If Maker does not fully occupy any winning set by the time every board element is claimed by some player, then Breaker wins the game. We say that the $(p : q)$ game $(X, \mathcal{F})$ is *Maker's win* if Maker has a strategy that ensures his win in this game (in some number of moves) against any strategy of Breaker, otherwise the game is *Breaker's win*. Note that p, q, X and $\mathcal{F}$ determine whether the game is Maker's win or Breaker's win.

Let $T = (V, E)$ be a tree on n vertices. In this paper we study the biased tree embedding game $\mathcal{T}_n$. The board of $\mathcal{T}_n$ is $E(K_n)$, that is, the edge set of the complete graph on n vertices. The winning sets of $\mathcal{T}_n$ are the copies of T in K_n.

There are three natural questions which come to one's mind regarding the game $\mathcal{T}_n$:

1. For which trees T can Maker win the $(1 : q)$ game $\mathcal{T}_n$?

2. What is the largest positive integer q for which Maker can win the $(1 : q)$ game $\mathcal{T}_n$?

3. How fast can Maker win the $(1 : q)$ game $\mathcal{T}_n$ (assuming it is Maker's win)?

It is easy to see that, playing a $(1 : q)$ game on $E(K_n)$, Breaker can ensure that the maximum degree of Maker's graph will not exceed $\frac{n-1}{\lfloor q/2 \rfloor + 1}$ (in fact, a lot more is known on games where one player's goal is to limit the degrees of the graph built by the other player; the interested reader is referred to [4]). Hence, Maker cannot build any tree on n vertices whose maximum degree exceeds this bound.

It is well known (see [6]) that if $q \geq (1 + \varepsilon) \frac{n}{\log n}$, then Breaker can isolate a vertex in Maker's graph. In particular, playing against such Breaker's bias, Maker cannot build any spanning graph. On the other hand, it was proved in [17] that, if $q \leq (1 - \varepsilon) \frac{n}{\log n}$, then Maker can

build a Hamilton path (in fact, even a Hamilton cycle). Hence, there are spanning trees for which $\log n/n$ is the breaking point between Maker's win and Breaker's win. Note that the requirement that T is a spanning tree plays a crucial role in the aforementioned bound on Breaker's bias. Indeed, it was proved by Beck [3] that, for sufficiently large n, if $q \leq n/(100d)$, then, playing a $(1 : q)$ game on $E(K_n)$, Maker can build a (q, d)-tree-universal graph, that is, a graph which contains a copy of *every* tree on q vertices with maximum degree at most d.

Clearly, Maker cannot build a spanning tree of K_n in less than $n - 1$ moves. For certain trees this trivial lower bound is tight. Indeed, it was proved in [11] that, playing a $(1 : 1)$ game on $E(K_n)$, Maker can claim all edges of a Hamilton path of K_n in $n - 1$ moves. Moreover, if Maker just wants to build a connected spanning graph, that is, he does not have to declare in advance which spanning tree he intends to build, then he can do so in $n - 1$ moves even in a $(1 : (1 - \varepsilon)n/\log n)$ game (see [8]). On the other hand, it was conjectured by Beck [3] and subsequently proved by Bednarska [5] that, playing a $(1 : q)$ game on $E(K_n)$, where $q \geq cn$ for an arbitrarily small constant $c > 0$, Maker *cannot* build a complete binary tree on q vertices in optimal time, that is, in $q - 1$ moves. It seems plausible that, assuming Maker can win the $(1 : q)$ game $\mathcal{T}_n$, he can in fact win it within $n + o(n)$ moves.

Our main result gives a partial answer to the three aforementioned questions.

Theorem 1.1 *Let $T = (V, E)$ be a tree on n vertices, with maximum degree $\Delta(T) \leq n^{0.05}$. For sufficiently large n Maker (as the first or second player) has a strategy to win the $(1 : b)$ Maker-Breaker tree embedding game $\mathcal{T}_n$, for every $b \leq n^{0.005}$, in $n + o(n)$ moves.*

A central feature of Theorem 1.1 is that Maker builds the tree in asymptotically optimal time. If, instead of stopping the game as soon as Maker builds the desired tree, we would continue playing until every edge of K_n is claimed by some player, then Maker's graph would be much denser than the graphs discussed in the first paragraph of this paper. Moreover, it was proved in [16] that Maker can in fact build a graph which contains simultaneously all spanning trees of maximum degree at most Δ, where the bound on Δ is better than the one given in Theorem 1.1. However, such a graph is very dense and thus cannot be built quickly.

The rest of this paper is organized as follows: in Subsection 1.1 we introduce some notation and terminology that will be used throughout this paper. In Section 2 we state and prove several auxiliary results which will be used in the proof of Theorem 1.1. In Section 3 we prove Theorem 1.1. Finally, in Section 4 we present some open problems.

1.1   Notation and terminology

For the sake of simplicity and clarity of presentation, we do not make a particular effort to optimize the constants obtained in our proofs. We also omit floor and ceiling signs whenever these are not crucial. Most of our results are asymptotic in nature and whenever necessary we assume that n is sufficiently large. Throughout the paper, log stands for the natural logarithm,

unless stated otherwise. Our graph-theoretic notation is standard and follows that of [19]. In particular, we use the following.

For a graph G, let $V(G)$ and $E(G)$ denote its sets of vertices and edges respectively, and let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For disjoint sets $A, B \subseteq V(G)$, let $E_G(A, B)$ denote the set of edges of G with one endpoint in A and one endpoint in B, and let $e_G(A, B) = |E_G(A, B)|$. For a set $S \subseteq V(G)$, let $N_G(S) = \{u \in V(G) \setminus S : \exists v \in S, (u, v) \in E(G)\}$ denote the set of neighbors of the vertices of S in $V(G) \setminus S$. For a vertex $w \in V(G)$ we abbreviate $N_G(\{w\})$ to $N_G(w)$, and let $d_G(w) = |N_G(w)|$ denote the degree of w in G. The maximum degree of a graph G is denoted by $\Delta(G)$. For vertices $u, v \in V(G)$ let $dist_G(u, v)$ denote the *distance* between u and v in G, that is, the number of edges in a shortest path of G, connecting u and v. Often, when there is no risk of confusion, we omit the subscript G from the notation above. For a set $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G, induced on the vertices of S. Let $P = (v_0, \ldots, v_k)$ be a path in a graph G. The vertices v_0 and v_k are called the *endpoints* of P, whereas the vertices of $V(P) \setminus \{v_0, v_k\}$ are called the *interior vertices* of P. We denote the set of endpoints of a path P by $End(P)$. A path of a tree T is called *a bare path* if all of its interior vertices are of degree 2 in T. Given two graphs G and H on the same set of vertices V, let $G \setminus H$ denote the graph with vertex set V and edge set $E(G) \setminus E(H)$. A graph $G = (V, E)$ is said to be *Hamilton connected* if, for every two vertices $u, w \in V$, there is a Hamilton path in G whose endpoints are u and w. A *triangle factor* of a graph G is a spanning 2-regular subgraph of G, every connected component of which is isomorphic to K_3.

Let G be a graph, let T be a tree, and let $S \subseteq V(T)$ be an arbitrary set. An *S-partial embedding* of T in G is an injective mapping $f : S \to V(G)$, such that $(f(x), f(y)) \in E(G)$ whenever $\{x, y\} \subseteq S$ and $(x, y) \in E(T)$. For every vertex $v \in f(S)$ we denote $v' = f^{-1}(v)$. If $S = V(T)$, we call an S-partial embedding of T in G simply an embedding of T in G. We say that the vertices of S are *embedded*, whereas the vertices of $V(T) \setminus S$ are called *new*. An embedded vertex is called *closed* with respect to T if all its neighbors in T are embedded as well. An embedded vertex that is not closed with respect to T, is called *open* with respect to T. The vertices of $f(S)$ are called *taken*, whereas the vertices of $V(G) \setminus f(S)$ are called *available*. With some abuse of this terminology, for a closed (respectively open) vertex $u \in S$, we will sometimes refer to $f(u)$ as being closed (respectively open) as well.

Assume that some Maker-Breaker game, played on the edge set of some graph G, is in progress. At any given moment during this game, we denote the graph spanned by Maker's edges by M, and the graph spanned by Breaker's edges by B. At any point of the game, the edges of $G \setminus (M \cup B)$ are called *free*. We also denote by $d_M(v)$ and $d_B(v)$ the degree of a given vertex $v \in V$ in M and in B respectively.

2 Auxiliary results

In this section we present some auxiliary results that will be used in the proof of Theorem 1.1.

The following fundamental theorem, due to Beck [2], is a useful sufficient condition for Breaker's

win in the $(p : q)$ game $(X, \mathcal{F})$. It will be used extensively throughout the paper.

Theorem 2.1 *Let X be a finite set and let $\mathcal{F} \subseteq 2^X$. If $\sum_{B \in \mathcal{F}} (1+q)^{-|B|/p} < \frac{1}{1+q}$, then Breaker (as the first or second player) has a winning strategy for the $(p : q)$ game $(X, \mathcal{F})$.*

While Theorem 2.1 is useful in proving that Breaker wins a certain game, it does not show that he wins this game quickly. The following lemma is helpful in this respect.

Lemma 2.2 (Trick of fake moves) *Let X be a finite set and let $\mathcal{F} \subseteq 2^X$. Let $q' < q$ be positive integers. If Maker has a winning strategy for the $(1 : q)$ game $(X, \mathcal{F})$, then he has a strategy to win the $(1 : q')$ game $(X, \mathcal{F})$ within $1 + |X|/(q+1)$ moves.*

The main idea of the proof of Lemma 2.2 is that, in every move of the $(1 : q')$ game $(X, \mathcal{F})$, Maker (in his mind) gives Breaker $q - q'$ additional board elements. The straightforward details can be found in [4].

Let $T = (V, E)$ be an arbitrary tree on n vertices. For every $1 \leq i \leq n - 1$ let $D_i := \{v \in V : d_T(v) = i\}$ denote the set of vertices of V whose degree in T is exactly i. Moreover, let $D_{>i} := \bigcup_{k=i+1}^{n-1} D_k$ denote the set of vertices of V whose degree in T is strictly larger than i.

Lemma 2.3 *Let T be a tree on $n \geq 2$ vertices, then $|D_{>2}| \leq |D_1| - 2$.*

Proof.

$$
\begin{aligned}
2n - 2 \;=\; & \sum_{v \in V} d(v) \\
=\; & \sum_{v \in D_1} d(v) + \sum_{v \in D_2} d(v) + \sum_{v \in D_{>2}} d(v) \\
\geq\; & |D_1| + 2(n - |D_1| - |D_{>2}|) + 3|D_{>2}|.
\end{aligned}
$$

It follows that $2n - 2 \geq 2n - |D_1| + |D_{>2}|$, and thus $|D_{>2}| \leq |D_1| - 2$ as claimed. □

Lemma 2.4 *Let k be a sufficiently large positive integer and let $G = (V, E)$ be a graph on k vertices with maximum degree at most $k^{0.95}$. Let $\ell \geq 1$ be an integer and let $L := \{a_1, \ldots, a_\ell, b_1, \ldots, b_\ell\}$ be a set of 2ℓ vertices of G. For every $1 \leq i \leq \ell$, let k_i be an integer such that $\sum_{i=1}^{\ell} k_i = k - 2\ell$, and $k_i \geq k^{0.2}$. Then, there exists a partition $V \setminus L = V_1 \cup \ldots \cup V_\ell$ such that the following two properties hold for every $1 \leq i \leq \ell$:*

(i) $|V_i| = k_i$.

(ii) The maximum degree of the graph $G_i := G[V_i \cup \{a_i, b_i\}]$ is at most $10k_i k^{-0.05}$.

Proof. Let $V \setminus L = V_1 \cup \ldots \cup V_\ell$ be a partition, chosen uniformly at random amongst all partitions of $V \setminus L$ into ℓ parts such that $|V_i| = k_i$ for every $1 \leq i \leq \ell$. Fix some $1 \leq i \leq \ell$, and set $m_i := 10k_i k^{-0.05}$. Let $u \in V$ be an arbitrary vertex. The probability that u has more than m_i neighbors in V_i is at most e^{-m_i} (see Theorem 2.10 and Corollary 2.4 in [15]). It follows by a union bound argument, that the probability that there exists a vertex $u \in V$ such that $d_G(u) \geq m_i$ is at most k/e^{m_i}. It thus follows by another union bound argument that the probability that there exists an index $1 \leq i \leq \ell$ such that G_i does not satisfy property (ii) above, is at most $\sum_{i=1}^\ell \frac{k}{e^{m_i}} \leq \frac{k\ell}{e^{k^{0.15}}} = o(1)$. In particular, there exists a partition that satisfies both properties of the lemma. □

2.1 Playing several biased games in parallel

Let m be a positive integer. For every $1 \leq i \leq m$, let $\mathcal{H}_i = (V_i, E_i)$ be a hypergraph, where $V_i \cap V_j = \emptyset$ for every $1 \leq i < j \leq m$. Let $\mathcal{H} = (V, E)$ be the hypergraph with $V = \bigcup_{i=1}^m V_i$ and $E = \prod_{i=1}^m E_i = \{\bigcup_{i=1}^m e_i : e_i \in E_i\}$. Consider a $(1 : q)$ Maker-Breaker game played on $\mathcal{H}$. If $q = 1$, then Maker can play all m games in parallel, that is, whenever Breaker claims a vertex of the board $\mathcal{H}_i$, Maker responds by also claiming a vertex of $\mathcal{H}_i$ according to a fixed winning strategy for the game $\mathcal{H}_i$ (if Breaker claims the last vertex of $\mathcal{H}_i$, then Maker responds by claiming an arbitrary free vertex of $\mathcal{H}$). It follows that Maker wins the $(1 : 1)$ game $\mathcal{H}$ if and only if he wins the $(1 : 1)$ game $\mathcal{H}_i$ for every $1 \leq i \leq m$. If $q > 1$, then this is no longer true because Breaker can play in q different boards in one turn whereas Maker can only respond in one board per turn. Nevertheless, we prove the following result.

Theorem 2.5 *If, for every $1 \leq i \leq m$, Maker has a strategy to win the* $\left(1 : q\left(1 + \log\left(m + \left\lceil \frac{\sum_{i=1}^m |V_i|}{q+1} \right\rceil\right)\right)\right)$ *game $\mathcal{H}_i$ in t_i moves, then he has a strategy to win the* $(1 : q)$ *game $\mathcal{H}$ in $\sum_{i=1}^m t_i$ moves.*

Before proving Theorem 2.5, we introduce an auxiliary game, which is a variation on the classical *Box Game*, first introduced by Chvátal and Erdős [6]. In the Chvátal-Erdős *Box Game Box*(m, ℓ, q) there are m pairwise disjoint boxes $A_1, \ldots, A_m$, each of size ℓ. In every round, the first player, called BoxMaker, claims q elements of $\bigcup_{i=1}^m A_i$ and then the second player, called BoxBreaker, destroys one box. BoxMaker wins the game *Box*(m, ℓ, q) if and only if he is able to claim all elements of some box before it is destroyed. In our variation, the *Box Game with resets r Box*(m, q), the m boxes $A_1, \ldots, A_m$ have infinite size. As in *Box*(m, ℓ, q), BoxMaker claims q elements of $\bigcup_{i=1}^m A_i$ per round. BoxBreaker then responds by *resetting* one of the boxes, that is, by deleting all of BoxMaker's elements from the chosen box A_i. Note that the chosen box does *not* leave the game. We prove the following result.

Theorem 2.6 *For every integer $k \geq 1$, BoxBreaker has a strategy for the game $r Box(m, q)$ which ensures that, at any point during the first k rounds of the game, every box A_i has weight at most $q(1 + \log(m + k))$.*

In the proof of Theorem 2.6 we will use the following restatement of a special case of a theorem of Chvátal and Erdős [6] (note that the proof of this result given in [6] is false; however, the result itself is correct as proved in [9]. The theorem is stated in this altered fashion to account for the fact that, unlike in our case, BoxBreaker is assumed to be the first player in [6]).

Theorem 2.7 *If $\ell > q \sum_{i=1}^{m} 1/i$, then BoxBreaker has a winning strategy in $Box(m, \ell, q)$.*

Proof of Theorem 2.6. Observe that, if one aims to prove that BoxBreaker has a strategy to ensure that every box has weight at most s in $rBox(m, q)$, then one can view resetting a box in $rBox(m, q)$ as deleting this box and adding a new empty box of size $s + 1$ in the standard Box Game (the original boxes are assumed to have size $s + 1$ as well). Indeed, if BoxBreaker wins this standard Box Game, then BoxMaker must have claimed at most s elements in every box before it was destroyed, or equivalently, reset in $rBox(m, q)$. Formally, while playing the game $rBox(m, q)$ with boxes $A_1, \ldots, A_m$ against some strategy of BoxMaker, BoxBreaker also pretends to play a standard Box Game $\mathcal{B}$ with $m + k$ boxes $B_1, \ldots, B_{m+k}$, each of size $q(1 + \log(m + k)) + 1$. At any point during the game and for every $1 \le i \le m$, he maintains a unique $1 \le j \le m + k$ for which the box B_j corresponds to the box A_i. Initially, B_i corresponds to A_i for every $1 \le i \le m$. Whenever BoxMaker claims an element of some box A_i in the game $rBox(m, q)$, BoxBreaker pretends that BoxMaker claims an element of the box B_j which corresponds to A_i at that point of the game $\mathcal{B}$. Whenever BoxBreaker resets a box A_i in $rBox(m, q)$, he destroys the box B_j which corresponds to A_i at that point of the game. He then assigns a new box B_t to correspond to A_i. The index $1 \le t \le m + k$ is chosen to be the smallest integer that does and did not at any point of the game $\mathcal{B}$ up to now correspond to any of the boxes $A_1, \ldots, A_m$. If BoxBreaker does not have a strategy as indicated in the theorem, then there exists a strategy of BoxMaker for which this course of play leads to BoxMaker's win in $\mathcal{B}$. In particular, there exists a winning strategy of BoxMaker in $Box(m + k, q(1 + \log(m + k)) + 1, q)$ contrary to Theorem 2.7. This concludes the proof of the theorem. □

Proof of Theorem 2.5. Let $k = \left\lceil \frac{\sum_{i=1}^{m} |V_i|}{q+1} \right\rceil$. For every $1 \le i \le m$, let $\mathcal{S}_i$ be a strategy for Maker in the $(1 : q(1 + \log(m + k)))$ game $\mathcal{H}_i$ which ensures his win in at most t_i moves. Since the game $\mathcal{H}$ clearly lasts at most k rounds, it follows by Theorem 2.6, that Maker (assuming the role of BoxBreaker in $rBox(m, q)$) has a strategy $\mathcal{S}$ to ensure that, for every $1 \le i \le m$, for every $j \ge 0$, and at any point during the game, if Maker has claimed exactly j vertices of V_i, then Breaker has claimed at most $(j + 1)q(1 + \log(m + k))$ vertices of V_i. For every $i \ge 1$, in his ith move in the game $\mathcal{H}$, Maker will use $\mathcal{S}$ for choosing a board $\mathcal{H}_j$ in which to play in this move. If Maker has already won $\mathcal{H}_j$, then he chooses an arbitrary $1 \le r \le m$ for which he has not yet won $\mathcal{H}_r$ and plays his ith move there. Since Maker chooses a board according to $\mathcal{S}$, it follows by Theorem 2.6 that Breaker has not claimed more than $q(1 + \log(m + k))$ vertices of V_j since Maker has last played on this board. Hence, Maker can follow $\mathcal{S}_j$ whenever he plays in $\mathcal{H}_j$ and thus win this game by assumption. Since this holds for every $1 \le j \le m$, it follows that Maker has a winning strategy for the $(1 : q)$ game $\mathcal{H}$. Moreover, since whenever Maker plays in $\mathcal{H}_i$ he follows $\mathcal{S}_i$ and since he never plays in $\mathcal{H}_i$ if he had already won this game, it follows

that, for every $1 \leq i \leq m$, Maker plays at most t_i moves in $\mathcal{H}_i$. Hence, he has a strategy to win the $(1 : q)$ game $\mathcal{H}$ within at most $\sum_{i=1}^{m} t_i$ moves, as claimed. $\qquad\square$

2.2 A perfect matching game

Maker's strategy for embedding a spanning tree, which we will propose in Section 3, will involve building a perfect matching on some part of the board. Hence, we prove the following result.

Proposition 2.8 *Let r be a sufficiently large integer and let $q \leq \frac{r}{12 \log_2 r}$. Let G be a spanning subgraph of $K_{r,r}$ with minimum degree at least $r - g(r)$, where g is an arbitrary function satisfying $g(r) = o(r)$. Then, playing a $(1 : q)$ game on $E(G)$, Maker can claim the edges of a perfect matching of G, within $O(r \log r)$ moves.*

Proof. Let A and B denote the two partite sets of G. In order to show that Maker can claim the edges of a perfect matching of G, we will prove that Maker can build a graph which satisfies Hall's condition, that is, a graph M which satisfies $|N_M(X)| \geq |X|$ for every $X \subseteq A$ (see e.g. [19]).

We define an auxiliary game $\mathcal{M}_G$, which we refer to as the *Hall game*. It is a $(q : 1)$ game, played by two players, called HallMaker and HallBreaker. The board of this game is $E(G)$ and the winning sets are the edge sets of all induced subgraphs of G with one partite set of size $1 \leq t \leq r$ and the other of size $r - t + 1$. It is straightforward to verify that if HallBreaker has a winning strategy for the $(q : 1)$ game $\mathcal{M}_G$, then, playing a $(1 : q)$ game on $E(G)$, Maker can claim all edges of some perfect matching of G. In order to prove that HallBreaker can win the $(q : 1)$ game $\mathcal{M}_G$ for $q \leq \frac{r}{12 \log_2 r}$, we apply Theorem 2.1. We have

$$
\begin{aligned}
\sum_{D \in \mathcal{M}_G} 2^{-\frac{|D|}{q}} \ &\leq \ \sum_{t=1}^{r} \binom{r}{t} \binom{r}{r-t+1} 2^{-\frac{t(r-t+1) - g(r) \cdot \min\{t, r-t+1\}}{q}} \\
&\leq \ 2 \sum_{t=1}^{r/2} \binom{r}{t} \binom{r}{t-1} 2^{-\frac{t(r-t+1) - g(r) \cdot t}{q}} \\
&\leq \ 2 \sum_{t=1}^{r/2} \binom{r}{t}^2 2^{-\frac{tr}{3q}} \\
&\leq \ \sum_{t=1}^{r/2} \left[r^2 2^{-4 \log_2 r} \right]^t \\
&\leq \ \sum_{t=1}^{r/2} r^{-2t} \\
&= \ o(1).
\end{aligned}
$$

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It follows that Maker can indeed build the required perfect matching. Moreover, it follows by Lemma 2.2 that Maker can do so within at most $1 + \frac{|E(G)|}{r/(12\log_2 r)} = O(r\log r)$ moves. This concludes the proof of the proposition. $\square$

2.3 A Hamiltonicity game

Maker's strategy for embedding a spanning tree, which we will propose in Section 3, will involve building a Hamilton connected subgraph of some part of the board. Hence, we prove the following result.

Proposition 2.9 *Let k be a sufficiently large integer and let $q \leq \frac{k}{\log^2 k}$. Let $G = (V, E)$ be a graph on k vertices, with minimum degree at least $k - g(k)$, where g is an arbitrary function satisfying $g(k) = o(k/\log k)$. Then, playing a $(1 : q)$ game on E, Maker can build a Hamilton connected graph within $O(k\log^2 k)$ moves.*

Proof. In the proof of this proposition we will make use of the following sufficient condition for a graph to be Hamilton connected (see [12]).

Theorem 2.10 *Let $D(k) = \log\log k$ and let $G = (V, E)$ be a graph on k vertices satisfying the following two properties:*

- *For every $S \subseteq V$, if $|S| \leq \frac{k}{\log k}$, then $|N_G(S)| \geq D|S|$;*

- *There is an edge in G between any two disjoint subsets $A, B \subseteq V$ with $|A|, |B| \geq \frac{k}{\log k}$.*

Then G is Hamilton connected, for sufficiently large k.

Let $\mathcal{H}_1$ be the hypergraph whose vertices are the edges of G and whose set of hyperedges is $\{E_G(A, B) : A, B \subseteq V, A \cap B = \emptyset, 1 \leq |A| \leq \frac{k}{\log k}, |B| = k - (D+1)|A|\}$. Note that by our assumption on the minimum degree in G, it follows that

$$e_G(A, B) \geq |A|(|B| - g(k)) \geq (1 - o(1))|A|k.$$

for every A, B as above.

Let $\mathcal{H}_2$ be the hypergraph whose vertices are the edges of G and whose set of hyperedges is $\{E_G(A, B) : A, B \subseteq V, A \cap B = \emptyset, |A| = |B| = \frac{k}{\log k}\}$. Note that by our assumption on the minimum degree in G, it follows that

$$
\begin{aligned}
e_G(A, B) &\geq \frac{k}{\log k}\left(\frac{k}{\log k} - g(k)\right) \\
&\geq \left(\frac{k}{2\log k}\right)^2
\end{aligned}
$$

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for every A, B as above.

By Theorem 2.10, in order to prove that playing a $(1 : q)$ game Maker can build a Hamilton connected subgraph of G, it suffices to prove that Breaker can win the $(q : 1)$ game $\mathcal{H}_1 \cup \mathcal{H}_2$. This however follows from Theorem 2.1. Indeed, for every $1 \leq a \leq \frac{k}{\log k}$, we have

$$
\begin{aligned}
&\binom{k}{a}\binom{k}{k - (D+1)a} 2^{-(1-o(1))ak/q} \\
&\leq k^a k^{(D+1)a} 2^{-(1-o(1))ak/q} \\
&\leq \exp\left\{a(D+2)\log k - \log 2 \cdot (1 - o(1))ak \cdot \frac{\log^2 k}{k}\right\} \\
&= o(1/k).
\end{aligned}
$$

Hence,

$$
\sum_{B \in \mathcal{H}_1} 2^{-|B|/q} \leq \sum_{a=1}^{\frac{k}{\log k}} \binom{k}{a}\binom{k}{k - (D+1)a} 2^{-(1-o(1))ak/q} = o(1).
$$

Similarly,

$$
\begin{aligned}
&\sum_{B \in \mathcal{H}_2} 2^{-|B|/q} \\
&\leq \binom{k}{\frac{k}{\log k}}^2 \exp\left\{-\log 2 \left(\frac{k}{2\log k}\right)^2 \frac{\log^2 k}{k}\right\} \\
&\leq \exp\left\{\frac{(2 + o(1))k \log\log k}{\log k} - \log 2 \left(\frac{k}{2\log k}\right)^2 \frac{\log^2 k}{k}\right\} \\
&= o(1).
\end{aligned}
$$

Hence,

$$
\sum_{B \in \mathcal{H}_1 \cup \mathcal{H}_2} 2^{-|B|/q} = \sum_{B \in \mathcal{H}_1} 2^{-|B|/q} + \sum_{B \in \mathcal{H}_2} 2^{-|B|/q} = o(1).
$$

It follows that Maker can build the required Hamilton connected graph. Moreover, it follows by Lemma 2.2 that Maker can do so within at most $1 + \frac{|E|}{k/(\log^2 k)} = O(k \log^2 k)$ moves. This concludes the proof of the proposition. $\qquad\square$

3   Embedding a spanning tree quickly

In this section, we prove Theorem 1.1. For the sake of presentation we denote $\varepsilon = 0.05$ and $\alpha = 0.005$ throughout the proof. We begin by describing Maker's strategy, then prove that it is indeed a winning strategy and that Maker can follow all of its stages.

Maker's strategy: Maker distinguishes between two cases, according to the number of neighbors of the leaves of T. Throughout this section, let L denote the set of leaves of T.

Case I: $|N_T(L)| \geq n^{2/3}$.

Maker's strategy for this case is divided into two stages.

Stage 1: Let $L' \subseteq L$ be a set of exactly $n^{2/3}$ leaves, every two of which have no common neighbor, that is, $|N_T(L')| = |L'|$. Maker's goal in this stage is to embed a subtree T'' of T such that $T' := T \setminus L' \subseteq T''$ and $|V(T'')| \leq n - \frac{1}{2}n^{2/3}$, in at most $n + o(n)$ moves.

At any point during this stage, a vertex $v \in V(K_n)$ is called *dangerous* if $d_B(v) \geq \sqrt{n}$ and v is either an available or an open vertex with respect to T. Throughout this stage, Maker maintains a set $D \subseteq V(K_n)$ of dangerous vertices, a set $S \subseteq V(T)$ of embedded vertices, and an S-partial embedding f of T in $K_n \setminus B$. Initially, $D = \emptyset$, $S = \{w'\}$, where w' is an arbitrary vertex of T', and $f(w') = w$, where w is an arbitrary vertex of K_n. If at some point Maker is unable to follow the proposed strategy (including the time limits it sets), then he forfeits the game. Moreover, if after claiming $2n$ edges, Maker has not yet won, then he forfeits the game (as noted above, we will in fact prove that Maker can win within $n + o(n)$ moves; however, the technical upper bound of $2n$ will suffice for the time being).

For as long as $V(T') \setminus S \neq \emptyset$, Maker plays as follows:

(1) If $D \neq \emptyset$, then Maker plays as follows. Let $v \in D$ be an arbitrary dangerous vertex. We distinguish between two subcases:

    (1.1) v is taken. Let $v'_1, \ldots, v'_r$ be the new neighbors of $v' := f^{-1}(v)$ with respect to T. Maker sequentially claims r free edges $\{(v, v_i) : 1 \leq i \leq r\}$, where, for every $1 \leq i \leq r$, v_i is an arbitrary available vertex. Subsequently, he updates D, f and S by adding v'_i to S and setting $f(v'_i) = v_i$ for every $1 \leq i \leq r$, and by deleting v from D and adding (if necessary) new dangerous vertices to D.

    (1.2) $v \in D$ is available. First, Maker adds v to his tree. If there exists an open vertex u such that (u, v) is a free edge, then Maker claims this edge, and then closes v and updates D, f and S as in (1.1). If there is no such edge, that is, v is connected by Breaker's edges to every open (with respect to T) vertex of Maker's tree, then Maker connects v to an open vertex u via a path of length three, within at most $11n^\alpha$ moves. Finally, Maker closes v and updates D, f and S as in (1.1).

(2) If $D = \emptyset$, then Maker claims some free edge (u, v) such that v is an open vertex with respect to T' and u is an available vertex.

Stage 2: Maker embeds the vertices of $T \setminus T''$ (that is, all vertices of T that were not embedded in Stage 1) into the set of available vertices, within $o(n)$ moves.

Case II: $|N_T(L)| < n^{2/3}$. It follows that T has strictly less than $n^{2/3+\varepsilon}$ leaves.

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Maker's strategy for this case is divided into two stages.

Stage 1: Let F denote the forest which is obtained from T by removing the interior vertices of all inclusion maximal bare paths whose length is at least $n^{0.2}$. Maker embeds F in K_n without paying any attention to Breaker's moves.

Stage 2: Maker embeds the edges of $T \setminus F$, thus completing the embedding of T. In order to do so, he splits the rest of the board into parts of appropriate sizes and, playing on all parts in parallel, he embeds the missing pieces of T in the appropriate parts.

Note that if Maker can indeed follow all parts of the proposed strategy, then he clearly wins the $(1 : q)$ game $\mathcal{T}_n$ (though possibly not fast enough).

3.1 Following Maker's strategy for Case I

In this subsection we prove that Maker can indeed follow every part of his strategy for Case I. We prove this separately for Stage 1 and for Stage 2. First, we prove the following two lemmas.

Claim 3.1 *At any point during Stage 1, the number of vertices that are or previously were dangerous is $O(n^{1/2+\alpha}) = o(n^{0.6})$.*

Proof. By Maker's strategy, the game lasts at most $2n$ moves. Since, moreover, every dangerous vertex has degree at least $\sqrt{n}$ in Breaker's graph, it follows that there can be at most $2 \cdot \frac{2n^{1+\alpha}}{\sqrt{n}} = O(n^{1/2+\alpha})$ such vertices. $\qquad\square$

Claim 3.2 *The following two properties hold at any point during Stage 1:*

(i) $n - |S| \geq \frac{1}{2}n^{2/3}$.

(ii) $d_B(v) = o(n^{0.6})$ *holds for every* $v \in V(K_n) \setminus f(S)$ *and for every* $v \in f(S)$ *which is open with respect to T.*

Proof.

(i) By Maker's strategy, for every dangerous vertex $v \in D$, Maker embeds less than $2n^\varepsilon$ other vertices (n^ε vertices for closing v and 3 more if he is forced to first add v to his tree via a path of length 3 as described in part (1.2) of Maker's strategy). Other than these vertices, Maker embeds only vertices of T'. It follows by Claim 3.1 that $|S| \leq |V(T')| + o(n^{0.6+\varepsilon}) \leq (n - n^{2/3}) + o(n^{0.6+\varepsilon}) \leq n - \frac{1}{2}n^{2/3}$ holds at any point during Stage 1.

(ii) This holds trivially for every non-dangerous vertex. By Maker's strategy, closing any dangerous vertex, requires at most $n^\varepsilon + 11n^\alpha$ moves (if Maker is unable to close the vertex fast enough, then he forfeits the game and the assertion of the lemma still holds).

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It follows by Claim 3.1 there are at most $O(n^{1/2+\alpha})$ dangerous vertices at any point during Stage 1. Moreover, as long as $D \neq \emptyset$, all of Maker's moves are dedicated to closing dangerous vertices and, after each move of Maker, Breaker claims just n^α free edges of K_n. Hence, when Maker tries to close a dangerous vertex v, it holds that $d_B(v) \leq \sqrt{n} + O(n^{\max\{\varepsilon,\alpha\}} n^\alpha n^{1/2+\alpha}) = o(n^{0.6})$. Since, unless he has already won, Maker continues closing dangerous vertices for as long as $D \neq \emptyset$, it follows that $d_B(w) = o(n^{0.6})$ holds for every $w \in V(K_n) \setminus f(S)$ and for every $w \in f(S)$ which is open with respect to T, as claimed.

$\square$

A proof that Maker can follow Stage 1 of his strategy for Case I:

For as long as $D = \emptyset$ (part (2) of Maker's strategy), Maker claims a free edge (v, u), where $v' = f^{-1}(v)$ is open with respect to T' and $u \in V(K_n)$ is available. This is always possible since, unless Maker has already won, there must exist an open vertex v' with respect to T'. Moreover, since $D = \emptyset$, it follows that $d_B(v) < \sqrt{n}$. However, it follows by part (i) of Claim 3.2 that $|V(K_n) \setminus f(S)| \geq \frac{1}{2} n^{2/3}$. Hence, there exists an available vertex u for which (v, u) is free.

It remains to consider part (1) of Maker's strategy, that is, the case $D \neq \emptyset$.

Part (1.1) of Maker's strategy – v is already taken. It follows that $v' = f^{-1}(v)$ is open with respect to T. Maker's goal is to close v in T, that is, to embed all vertices of $N_T(v') \setminus S$ into available vertices of $V(K_n)$. We claim that the required edges exist. Indeed, it follows by part (ii) of Claim 3.2 that $d_B(v) = o(n^{0.6})$ holds for as long as v is open. Since $|V(K_n) \setminus f(S)| \geq \frac{1}{2} n^{2/3} > n^{0.6}$ holds by part (i) of Claim 3.2, it follows that the required available vertices and corresponding free edges exist.

Part (1.2) of Maker's strategy – v is an available vertex. In this case, Maker must first add v to his current tree. As noted in his strategy, if there exists an open vertex u such that the edge (u, v) is free, then Maker claims this edge. Otherwise, Maker connects v to an open (with respect to T) vertex of his current tree via a path of length three. First, we claim that there exists a taken vertex u, such that $u' := f^{-1}(u)$ is open with respect to T, and new vertices $v', x', y' \in V(T)$ such that $(u', x'), (x', y'), (y', v') \in E(T)$. Indeed, assume for the sake of contradiction that this is not the case. It follows from our assumption that, for every new vertex $a' \in V(T)$, there exists an open vertex $b' \in V(T)$ such that $dist_T(a', b') \leq 2$. Hence, every new vertex is an element of $L \cup N_T(L)$. It follows by the definition of L' and by part (i) of Claim 3.2 that at least $\frac{1}{2} n^{2/3}$ of the vertices of $N_T(L)$ are either open or new (in which case it must have an open neighbor). Hence, the number of open vertices in Maker's tree with respect to T is at least $\frac{n^{2/3}/2}{\Delta(T)} \geq \frac{1}{2} n^{2/3-\varepsilon} > n^{0.6}$. On the other hand, since Maker is trying to follow part (1.2) of his strategy, it follows that there is no free edge $(w, v) \in E(K_n)$ such that w is taken and $w' := f^{-1}(w)$ is open with respect to T. Hence, the number of open vertices with respect to T at this point is at most $d_B(v) = o(n^{0.6})$, where this equality holds by part (ii) of Claim 3.2. This is clearly a contradiction.

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Next, we prove that Maker can embed v', x', and y' into appropriate available vertices, that is, that he can claim free edges $(u, x), (x, y)$, and (y, v), where x and y are available vertices. Moreover, we prove that this entire phase takes at most $11n^\alpha$ moves. Finally, we prove that Maker can then close v. It follows that Maker can follow this part of his strategy without forfeiting the game. Let $A := V(K_n) \setminus (f(S) \cup \{v\} \cup N_B(u) \cup N_B(v))$. Let $I \subseteq A$ be an independent set in Breaker's graph of size $|I| \geq |A|/(\Delta(B)+1) \geq 100n^{2\alpha}$. Such an independent set exists since $|A| \geq \frac{1}{2}n^{2/3} - 1 - o(n^{0.6})$ by parts (i) and (ii) of Claim 3.2, and since Breaker's graph, induced on the vertices of A, has maximum degree at most $o(n^{0.6})$ by property (ii) of Claim 3.2. In the first $5n^\alpha$ moves of this phase, Maker claims $5n^\alpha$ arbitrary edges of $\{(v, w) : w \in I\}$. This is possible as $|\{(v, w) : w \in I\}| = |I| > 5n^\alpha + 5n^{2\alpha}$, and Breaker can claim at most $5n^{2\alpha}$ of these edges. Let $N_v \subseteq I \cap N_M(v)$ be an arbitrary set of size $5n^\alpha$. In his next $5n^\alpha$ moves, Maker claims $5n^\alpha$ edges of $\{(u, w) : w \in I \setminus N_v\}$. This is possible as $|\{(u, w) : w \in I \setminus N_v\}| = |I \setminus N_v| > 5n^\alpha + 10n^{2\alpha}$, and during this entire phase, Breaker has claimed at most $10n^{2\alpha}$ of these edges. Note that it is possible that Maker has already claimed some edges of $\{(v, w) : w \in I\} \cup \{(u, w) : w \in I \setminus N_v\}$ during some previous stage of the game. In this case he will need less than $10n^\alpha$ moves to achieve his goal; clearly this does not harm him. Now, Maker can claim a free edge (x, y), where $y \in N_v$ and $x \in (I \setminus N_v) \cap N_M(u)$. The required edge exists since there are at least $|N_v||(I \setminus N_v) \cap N_M(u)| \geq 25n^{2\alpha}$ such edges in K_n, and during this entire phase, Breaker has claimed at most $10n^{2\alpha}$ of them. Maker embeds x' into x, y' into y, and v' into v. This completes the required path of length three. Note that embedding all these vertices takes at most $11n^\alpha$ moves. Finally, Maker closes v by embedding all of its neighbors (as in part (1.1) of his strategy).

A proof that Maker can follow Stage 2 of his strategy for Case I:

Let $L'' \subseteq L'$ denote the set of leaves which have not been embedded in Stage 1. Let $H = (X \cup Y, F)$ be the bipartite graph with $X = V(K_n) \setminus f(S)$, $Y = f(N_T(L''))$, and $F = \{(u, v) \in E(K_n \setminus B) : u \in X, v \in Y\}$. Note that, by the choice of L', no two leaves of L'' have a common neighbor in T. Hence, in order to complete the embedding of T in K_n, Maker has to claim the edges of a perfect matching of H, in a $(1 : b)$ game against Breaker, where $b \leq n^\alpha$. This is possible by Proposition 2.8, since $|Y| = |f(N_T(L''))| = |L''| = |X|$, $|L''| \geq \frac{1}{2}n^{\frac{2}{3}}$ by part (i) of Claim 3.2, $\Delta(B) \leq n^{0.6} \leq |X|^{0.95}$ in the beginning of Stage 2 by part (ii) of Claim 3.2, and $n^\alpha \leq \frac{|X|}{12 \log_2 |X|}$.

Note that Stage 1 and Stage 2 together last at most $(n + o(n^{0.6+\alpha})) + O(n^{2/3} \log(n^{2/3})) = n + o(n)$ moves.

This concludes the proof that Maker can win the $(1 : b)$ game $\mathcal{T}_n$ if T is as in Case I.

3.2 Following Maker's strategy for Case II

In this subsection we prove that Maker can indeed follow every part of his strategy for Case II. We prove this separately for Stage 1 and for Stage 2. First, we prove the following lemma, which will be used in our proof that Maker can follow Stage 2 of his strategy for Case II.

Lemma 3.3 *Let $\beta > 0$ and $0 < \gamma < 1/8$ be real numbers such that $\beta + 2\gamma < 1$. Let k be sufficiently large and let $q \leq k^\gamma$. Let $G = (V, E)$ be a graph on k vertices, with minimum degree at least $k - k^\beta$, and let $a, b \in V$ be two vertices. Then, playing a $(1 : q)$ game on E, Maker can build a Hamilton path, whose endpoints are a and b, within $k + o(k)$ moves.*

Proof. We present a strategy for Maker, and then prove that it is a winning strategy.

Maker's strategy: Maker's strategy is divided into two stages.

Stage 1: Let δ and δ' be real numbers such that $\delta' > \max\{\frac{1}{2} + 2\gamma, \beta\}$ and $\delta' + 2\gamma < \delta < 1$. Maker builds two vertex disjoint paths $P_a = (a, v_1, v_2, \ldots, v_i)$ and $P_b = (b, u_1, u_2, \ldots, u_j)$, such that $k^\delta \leq k - (|V(P_a)| + |V(P_b)|) \leq 2k^\delta$. While building these two paths, Maker ensures that $d_B(u) \leq 2k^{\delta'}$ holds for every vertex $u \in (V \setminus (V(P_a) \cup V(P_b))) \cup ((End(P_a) \cup End(P_b)) \setminus \{a, b\})$.

At any point during this stage, a vertex $v \in (V \setminus (V(P_a) \cup V(P_b))) \cup ((End(P_a) \cup End(P_b)) \setminus \{a, b\})$ is called *dangerous* if $d_B(v) \geq k^{\delta'}$. Throughout this stage, Maker maintains a set $D \subseteq V$ of dangerous vertices and two paths of G whose edges he has claimed, P_a and P_b, where a is an endpoint of P_a and b is an endpoint of P_b. Initially, $D = \emptyset$, $P_a = (a)$, and $P_b = (b)$. Maker updates D after each move (by either player). If at some point, Maker is unable to follow the proposed strategy (including the time limits it sets), then he forfeits the game. Moreover, if after claiming $2k$ edges, Maker has not yet won, then he forfeits the game.

For as long as $k - (|V(P_a)| + |V(P_b)|) \geq 2k^\delta$, Maker plays as follows:

(1) If $D = \emptyset$, then Maker extends the shorter of his two paths (breaking ties arbitrarily). Assume without loss of generality that currently P_a is shorter than P_b. If $P_a = (a)$, then let $x = a$, otherwise, let x denote the unique element of $End(P_a) \setminus \{a\}$. Maker extends P_a by claiming a free edge (x, w) for some $w \in V \setminus (V(P_a) \cup V(P_b))$.

(2) If $D \neq \emptyset$, then Maker plays as follows. Let $v \in D$ be an arbitrary dangerous vertex. We distinguish between two sub cases:

    (2.1) $v \in (End(P_a) \cup End(P_b)) \setminus \{a, b\}$. Maker claims a free edge (v, w) for some $w \in V \setminus (V(P_a) \cup V(P_b))$.

    (2.2) $v \in V \setminus (V(P_a) \cup V(P_b))$. First, Maker adds v to P_a. Let x denote the unique element of $End(P_a) \setminus \{a\}$. Maker connects v to x via a path of length three, within at most $11k^\gamma$ moves. Once $v \in V(P_a)$, Maker extends P_a by one more edge, as in (1) above.

Stage 2: Let $G' = G[(V \setminus (V(P_a) \cup V(P_b))) \cup ((End(P_a) \cup End(P_b)) \setminus \{a, b\})]$. Maker builds a Hamilton connected subgraph of G'.

Note that, if Maker can indeed follow all parts of the proposed strategy, then, in particular, he builds a Hamilton path in G whose endpoints are a and b (though possibly not fast enough).

Next, we prove that Maker can indeed follow every part of his suggested strategy. We prove this separately for Stage 1 and for Stage 2. First, we prove the following two propositions.

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Claim 3.4 *At any point during Stage 1, the number of vertices that are or previously were dangerous is $o(\sqrt{k})$.*

Proof. By Maker's strategy, the game lasts at most $2k$ moves. Since, moreover, every dangerous vertex has degree at least $k^{\delta'}$ in Breaker's graph, it follows that there can be at most $2 \cdot \frac{2k^{1+\gamma}}{k^{\delta'}} = o(\sqrt{k})$ such vertices. $\qquad\square$

Claim 3.5 *The following two properties hold at any point during Stage 1:*

(i) $k - |V(P_a)| - |V(P_b)| \geq k^{\delta}$.

(ii) $d_B(v) \leq 2k^{\delta'}$ holds for every $v \in (V \setminus (V(P_a) \cup V(P_b))) \cup ((End(P_a) \cup End(P_b)) \setminus \{a, b\})$.

Proof.

(i) According to his strategy, Maker tries to stop extending his paths at the very moment $k - |V(P_a)| - |V(P_b)| \leq 2k^{\delta}$ happens for the first time. Maker can stop at this exact moment unless he is in the middle of part (2.2) of his strategy. In this case he adds at most 3 additional vertices. It follows that $k - |V(P_a)| - |V(P_b)| \geq 2k^{\delta} - 3 \geq k^{\delta}$ as claimed.

(ii) This holds trivially for any non-dangerous vertex. By Maker's strategy, adding any dangerous vertex to the interior of $P_a \cup P_b$, requires at most $11k^{\gamma}$ moves (if Maker is unable to do this fast enough, then he forfeits the game and the assertion of the lemma still holds). By Claim 3.4 there are $o(\sqrt{k})$ dangerous vertices at any point of the game. Moreover, as long as $D \neq \emptyset$, all of Maker's moves are dedicated to adding dangerous vertices to the interior of $P_a \cup P_b$. Hence, when Maker tries to add a dangerous vertex v to the interior of $P_a \cup P_b$, it holds that $d_B(v) \leq k^{\delta'} + o(k^{\gamma}k^{\gamma}\sqrt{k}) \leq 2k^{\delta'}$. Since, unless he has already won, Maker continues adding dangerous vertices to the interior of $P_a \cup P_b$, for as long as $D \neq \emptyset$, it follows that $d_B(w) \leq 2k^{\delta'}$ holds for every $w \in (V \setminus (V(P_a) \cup V(P_b))) \cup ((End(P_a) \cup End(P_b)) \setminus \{a, b\})$ as claimed.

$\qquad\square$

Stage 1: If $D = \emptyset$, then Maker can extend either path. Indeed, assume that Maker wishes to extend P_a. Let x denote the unique element of $End(P_a) \setminus \{a\}$ (or $x = a$ if $P_a = (a)$). It follows by part (i) of Claim 3.5 that $k - |V(P_a)| - |V(P_b)| \geq k^{\delta}$. Since $d_B(x) \leq k^{\delta'}$, the minimum degree of G is at least $k - k^{\beta}$, and $\delta > \delta' > \beta$, it follows that there exists a vertex $w \in V \setminus (V(P_a) \cup V(P_b))$ for which the edge (x, w) is free.

If $D \neq \emptyset$, then Maker has to add some vertex $v \in D$ to the interior of $P_a \cup P_b$. The fact that he can indeed achieve this goal, and moreover do so within at most $11k^{\gamma}$ moves, follows by essentially the same argument used to show that Maker can follow part (1) of Stage 1 of his strategy for Case I. We omit the straightforward details (note that here is where we use the fact that $\delta > \delta' + 2\gamma$).

Stage2: It follows by the fact that Maker can follow his strategy for Stage 1 without forfeiting the game and by part (i) of Claim 3.5 that $|V(G')| = \Theta(k^\delta)$. It follows by part (ii) of Claim 3.5 that the minimum degree of G' is at least $|V(G')| - 1 - 2k^{\delta'} = |V(G')| - \Theta(|V(G')|^{\delta'/\delta})$. Since, moreover, $q \leq k^\gamma \leq \frac{|V(G')|}{\log^2 |V(G')|}$, it follows by Proposition 2.9 that Maker can indeed build a Hamilton connected subgraph of G'. Furthermore, he can do so within $O(|V(G')| \log^2 |V(G')|) = o(k)$ moves.

$\square$

A proof that Maker can follow Stage 1 of his strategy for Case II:

Clearly, there are at most $|D_1| + |D_{>2}|$ inclusion maximal paths with at least one endpoint in $D_{>2}$. Since, except for vertices of $D_{>2}$, Maker embeds only vertices of bare paths whose lengths are at most $n^{0.2}$, it follows that, during Stage 1 Maker tries to embed at most $|D_{>2}| + (|D_1| + |D_{>2}|)n^{0.2} \leq 3|D_1|n^{0.2} \leq 3n^{2/3+\varepsilon+0.2} = o(n^{0.92})$ vertices, where the first inequality above follows by Lemma 2.3. Since, moreover, $b \leq n^\alpha$, it follows that at any point during Stage 1, the maximum degree in Breaker's graph is $o(n^{0.95})$, whereas the number of vertices that have not yet been embedded is $(1-o(1))n$. Hence, there are always free edges that extend the embedding of F.

A proof that Maker can follow Stage 2 of his strategy for Case II:

By Maker's strategy, it remains to embed ℓ bare paths, for some $1 \leq \ell \leq 2n^{2/3+\varepsilon}$. For every $1 \leq i \leq \ell$, let n_i denote the length of the ith path and let a_i and b_i denote its endpoints. Note that $n_i \geq n^{0.2}$ and that a_i and b_i were already embedded in Stage 1. Let $\tilde{F}$ denote the set of vertices of K_n into which the vertices of F were embedded in Stage 1. First, Maker partitions the remaining board into ℓ disjoint parts $V(K_n) \setminus \tilde{F} = V_1 \cup \ldots \cup V_\ell$ such that

(i) $|V_i| = n_i - 1$.

(ii) The minimum degree of $G_i := (K_n \setminus (M \cup B))[V_i \cup \{a_i, b_i\}]$ is at least $n_i - 10n_i n^{-0.05}$.

Such a partition exists by Lemma 2.4. Maker plays a $(1 : b)$ game on $\bigcup_{i=1}^{\ell} G_i$, which he wins if and only if, for every $1 \leq i \leq \ell$, he is able to build a Hamilton path of G_i between a_i and b_i. Clearly, if Maker wins this game, then he completes the embedding of T in K_n. By Theorem 2.5, in order to prove that Maker wins the aforementioned game, it suffices to prove that for every $1 \leq i \leq \ell$, Maker can win a $\left(1 : b\left(1 + \log\left(\ell + \lceil \frac{n^2}{b+1} \rceil\right)\right)\right)$ game played on $E(G_i)$, which he wins if and only if he is able to build a Hamilton path of G_i whose endpoints are a_i and b_i. However, since $|V(G_i)| \geq n^{0.2}$ and $b\left(1 + \log\left(\ell + \lceil \frac{n^2}{b+1} \rceil\right)\right) \leq 2n^\alpha \log n \leq |V(G_i)|^{1/40}$, this follows by Property (ii) above and by Lemma 3.3. Moreover, since, by Lemma 3.3, for every $1 \leq i \leq \ell$, Maker wins the game on $E(G_i)$ in $n_i + o(n_i)$ moves, it follows by Theorem 2.5 that Maker wins the game on $\bigcup_{i=1}^{\ell} G_i$ in $\sum_{i=1}^{\ell}(n_i + o(n_i))$ moves.

Note that Stage 1 and Stage 2 together last at most $o(n) + (\sum_{i=1}^{\ell}(n_i + o(n_i))) = n + o(n)$ moves.

This concludes the proof that Maker can win the $(1 : b)$ game $\mathcal{T}_n$ if T is as in Case II.

4   Concluding remarks and open problems

Breaker's bias and the maximum degree of T. In this paper it is proved that, given any tree T on n vertices whose maximum degree is not too large, Maker can build a copy of T within $n + o(n)$ moves, when playing a biased game on $E(K_n)$. While the obtained upper bound on the duration of the game is clearly very close to being optimal for sufficiently large n, the upper bounds on the maximum degree of T and on Breaker's bias are probably quite far from being best possible. It would be interesting to improve either bound, even at the expense of the other. In particular we offer the following two questions:

Problem 4.1 *Let n be sufficiently large and let $T = (V, E)$ be a tree on n vertices. What is the largest integer $d = d(n)$ for which Maker has a winning strategy for the $(1 : 1)$ game $\mathcal{T}_n$, assuming that $\Delta(T) \leq d$?*

Problem 4.2 *Let n be sufficiently large and let $T = (V, E)$ be a tree on n vertices with constant maximum degree. What is the largest integer $b = b(n)$ for which Maker has a winning strategy for the $(1 : b)$ game $\mathcal{T}_n$?*

Other spanning graphs. It would be interesting to analyze analogous games for general spanning graphs (not necessarily trees) of bounded degree. It was proved in [14] that, for sufficiently large n, Maker can build a Hamilton cycle of K_n in a $(1 : 1)$ game, within $n + 1$ moves. However, even if we restrict our attention to $(1 : 1)$ games on $E(K_n)$ in which Maker's goal is to build a copy of some predetermined 2-regular spanning graph, we cannot expect him to win very quickly. Indeed, for every positive integer n such that $3 \mid n$, let $\mathcal{T}F_n$ denote the game whose board is $E(K_n)$ and whose winning sets are all triangle factors of K_n. The following result was observed by Tibor Szabó and the third author.

Theorem 4.3 *Maker cannot win the $(1 : 1)$ game $\mathcal{T}F_n$ in less than $7n/6$ moves.*

Proof. We describe Breaker's strategy and then show that indeed, following this strategy delays Maker's win for at least $7n/6$ moves. Without loss of generality we assume that Maker is the first player (otherwise, Breaker claims an arbitrary edge in his first move).

Breaker's strategy: For every $i \geq 1$, let $e_i = (x_i, y_i)$ denote the edge claimed by Maker in his ith move. In his ith move, Breaker plays as follows. If there exists a vertex $z \in V(K_n)$ such that (x_i, z) was previously claimed by Maker and (y_i, z) is free, then Breaker claims (y_i, z) (if there are several such edges (y_i, z), then Breaker claims one of them arbitrarily). If there exists a vertex $z \in V(K_n)$ such that (y_i, z) was previously claimed by Maker and (x_i, z) is free, then Breaker claims (x_i, z) (if there are several such edges (x_i, z), then Breaker claims one of them arbitrarily). Otherwise, Breaker claims an arbitrary edge.

We claim that if Breaker follows the proposed strategy, then there is at least one vertex of degree at least 3 in any triangle in Maker's graph. Indeed, let $(x, y), (y, z)$ and (x, z) be three edges in Maker's graph. Assume that the first of these three edges to be claimed by Maker was (x, y) and the second was (y, z). Let i denote the number of the move in which Maker has claimed (y, z). Since, on his ith move, Breaker did not claim (x, z), it follows by the description of his strategy that he must have claimed an edge (y, u) or (z, u) for some vertex $u \neq x$. Hence, y or z will have degree at least 3 in Maker's graph after he will claim (x, z).

Consider Maker's graph M immediately after it admits a triangle factor for the first time. Clearly the minimum degree of M is at least 2. Let F be an arbitrary triangle factor of M. Since, as proved above, there is at least one vertex of degree at least 3 in every triangle of F, it follows that $2e(M) = \sum_{v \in V} d_M(v) \geq 3\frac{n}{3} + 2\frac{2n}{3} = \frac{7n}{3}$. Thus building M took Maker at least $7n/6$ moves. $\qquad\square$

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