Hitting time results for Maker-Breaker games

Extended Abstract

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Abstract

We analyze classical Maker-Breaker games played on the edge set of a randomly generated graph G. We consider the random graph process and analyze, for each of the properties "being spanning k-vertex-connected" , "admitting a perfect matching", and "being Hamiltonian", the first time when Maker starts having a winning strategy for building a graph possessing the target property (the so called hitting time). We prove that typically it happens precisely at the time the random graph process first reaches minimum degree $2k$, 2 and 4, respectively, which is clearly optimal. The latter two statements settle conjectures of Stojaković and Szabó. We also consider a general-purpose game, the *expander game*, which is a main ingredient of our proofs and might be of an independent interest.

1 Introduction

Let X be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets. In the positional game $(X, \mathcal{F})$, two players take turns in claiming one previously unclaimed element of X and the game ends when all of the elements of X have been claimed by either of the players. The set X is often referred to as the *board* of the game. Positional games have attracted a lot of attention in the past decade and a thorough introduction to this field with a plethora of results can be found in a recent monograph of Beck [3]. In a *Maker-Breaker*-type positional game, the two players are called *Maker* and *Breaker* and the members of $\mathcal{F}$ are referred to as the *winning sets*. Maker wins the game if he occupies all elements of some winning set; otherwise Breaker wins. We will always assume that Breaker starts the game. We say that a game $(X, \mathcal{F})$ is a *Maker's win* if Maker has a strategy (that can be adaptive to Breaker's moves) that ensures his win in this game against any strategy of Breaker, otherwise the game is a *Breaker's win*. Note that X and $\mathcal{F}$ alone determine whether the game is a Maker's win or a Breaker's win. A classical example of this Maker-Breaker setting is the popular board game HEX.

1.1 Notation

Our graph-theoretic notation is standard and follows that of [24]. In particular, we use the following. For a graph G, let $V(G)$ and $E(G)$ denote its sets of vertices and edges respectively, and let $e(G) = |E(G)|$. For a set $A \subseteq V(G)$, let $E_G(A)$ denote the set of edges of G with both endpoints in A, and let $e_G(A) = |E_G(A)|$. For disjoint sets $A, B \subseteq V(G)$, let $E_G(A, B)$ denote the set of edges of G with one endpoint in A and the other in B, and let $e_G(A, B) = |E_G(A, B)|$. For a set $S \subseteq V(G)$, let $N_G(S) = \{u \in V(G) \setminus S : \exists v \in S, \{u, v\} \in E(G)\}$ denote the set of neighbors of S in $V(G) \setminus S$. For a vertex $w \in V(G)$, we abbreviate $N_G(\{w\})$ to $N_G(w)$. For a vertex $w \in V(G) \setminus S$ let $d_G(w, S) = |\{u \in S : \{u, w\} \in E(G)\}|$ denote the number of vertices of S that are adjacent to w in G. We abbreviate $d_G(w, V \setminus \{w\})$ to $d_G(w)$ which denotes the degree of w in G. The minimum vertex degree in G is denoted by $\delta(G)$. For a set $S \subseteq V(G)$ let $G[S]$ denote the subgraph of G with vertex set S and edge set $E_G(S)$. Let $c(G)$ and $o(G)$ respectively denote the number of connected components and the number of connected components of odd cardinality in G. Lastly, we will denote by $\ell(G)$ the length of a longest path in G, where the length of a path is the number of its edges.

1.2 Maker-Breaker games on graphs

Let $G = (V, E)$ be a graph and let $\mathcal{P}$ be a monotone increasing graph property on V (a family of graphs on V, closed under isomorphism and addition of edges). Consider the Maker-Breaker game $(E, \mathcal{F}_{\mathcal{P}})$ played on the edge set E as the board of the game. The game is a win for Maker if and only if the graph spanned by the edges selected by Maker throughout the game satisfies the property

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$\mathcal{P}$. We denote the family of graphs G for which the $(E(G), \mathcal{F}_{\mathcal{P}})$ game is a Maker's win by $\mathcal{M}_{\mathcal{P}}$. Although the above game is described in game-theoretic terms, it should be noted that these games are finite perfect information games with no chance moves, and $\mathcal{M}_{\mathcal{P}}$ is some graph property which clearly satisfies $\mathcal{M}_{\mathcal{P}} \subseteq \mathcal{P}$. Moreover, since $\mathcal{P}$ is monotone increasing, $\mathcal{M}_{\mathcal{P}}$ is clearly monotone increasing as well. By considering monotone increasing graph properties, the game can be terminated as soon as the graph spanned by Maker's edges satisfies the property, regardless of whether all edges have been claimed or not. This leads to several natural questions. First, how sparse can a graph $G \in \mathcal{M}_{\mathcal{P}}$ be? In this context, playing on random graphs (where the density of the graph is controlled by the distribution on the graphs) becomes a very natural question. This setting was formally initiated in [23] by Stojaković and Szabó, and this current work is a further exploration of it. Second, one can also study the minimum number of moves needed for Maker in order to win the game (see e.g. [2, 21, 11, 13, 16]), but "winning fast" is not in the focus of this current work.

1.3 Random graphs

The most widely used random graph model is the Binomial random graph, $\mathcal{G}(n,p)$. In this model we start with n vertices, labeled, say, by $V = \{1, \ldots, n\} = [n]$, and select a graph on these n vertices by going over all $\binom{n}{2}$ pairs of vertices, deciding independently with probability p for a pair to be an edge. The model $\mathcal{G}(n,p)$ is thus a probability space of all labeled graphs on the vertex set $[n]$ where the probability of such a graph, $G = ([n], E)$, to be selected is $p^{|E|}(1-p)^{\binom{n}{2}-|E|}$. This product probability space provides us with a wide variety of probabilistic tools for analyzing the behavior of various random graph properties. (See monographs [6] and [17] for a thorough introduction to the subject of random graphs). In the subsequent sections we will need at some point to employ a slightly generalized model. Let $F \subseteq \binom{V}{2}$ be an arbitrary subset and let $\mathcal{G}(n,p)_{-F} := \mathcal{G}(n,p) \setminus F$.

Although the Binomial random graph model is very natural and relatively easy to use, it was not the first model to be considered. In their seminal paper, Erdős and Rényi considered the uniform probability space over all graphs on a fixed set of vertices with exactly M edges, $\mathcal{G}(n,M)$. Note that for any value of p, if we condition the random graph $\mathcal{G}(n,p)$ to have exactly M edges, then we obtain exactly the Erdős-Rényi random graph model. The similarity of the two models enables us to prove the occurrence of events in the $\mathcal{G}(n,p)$ model and get the corresponding result in the $\mathcal{G}(n,M)$ model.

Proposition 1.1. ([17], Proposition 1.13) *Let $\mathcal{P} = \mathcal{P}(n)$ be a sequence of monotone increasing graph properties, $0 \le a \le 1$ and $0 \le M \le \binom{n}{2}$ be an integer. If for every sequence $p = p(n) \in [0,1]$ such that $p = M/\binom{n}{2} \pm O\left(M\left(\binom{n}{2} - M\right)/\binom{n}{2}^3\right)$ it holds that $\lim_{n\to\infty} \Pr[\mathcal{G}(n,p) \in \mathcal{P}] = a$, then $\lim_{n\to\infty} \Pr[\mathcal{G}(n,M) \in \mathcal{P}] = a$.*

The converse result to Proposition 1.1 holds[1] as well (see e.g. Proposition 1.12 in [17]); this enables us to transfer results from one model to the other. Unfortunately, not all properties we will encounter and explore are monotone increasing, and hence Proposition 1.1 cannot be used in those cases. Nonetheless, we would like to take advantage of the "ease" of calculations in the $\mathcal{G}(n,p)$ model (due to the independence of appearance of its edges), and transfer the results to the $\mathcal{G}(n,M)$ model, for the appropriate values of M. To achieve this we will use this more crude estimate (see e.g. [17]), which will suffice for our purposes.

Proposition 1.2. ([17], inequality (1.6)) *Let $\mathcal{P}$ be a property of graphs on n vertices and let $1 \le M \le \binom{n}{2}$ be an integer. Setting $p = M/\binom{n}{2}$ we have $\Pr[\mathcal{G}(n,M) \in \mathcal{P}] \le 3\sqrt{M} \cdot \Pr[\mathcal{G}(n,p) \in \mathcal{P}]$.*

Next, we consider the following generation process of graphs. Given a set V of n vertices and an ordering on the pairs of vertices $\pi : \binom{V}{2} \to \left[\binom{n}{2}\right]$, we define a *graph process* to be a sequence of graphs $\widetilde{G} = \widetilde{G}(\pi) = \{G_t\}_{t=0}^{\binom{n}{2}}$ on V. Starting with $G_0 = (V, \emptyset)$, for every integer $1 \le t \le \binom{n}{2}$, the graph G_t is defined by $G_t := G_{t-1} \cup \pi(t)$. For a given graph process $\widetilde{G}$ on V, we define the *hitting time* of a monotone increasing graph property $\mathcal{P}$ on V as $\tau(\widetilde{G}; \mathcal{P}) = \min\{t : G_t \in \mathcal{P}\}$.

When selecting π uniformly at random, the process $\widetilde{G}(\pi)$ is usually called the *random graph process*. If $\widetilde{G} = \{G_t\}_{t=0}^{\binom{n}{2}}$ is the random graph process, then, for every $0 \le M \le \binom{n}{2}$, the graph G_M is distributed according to $\mathcal{G}(n,M)$, that is, $G_M \sim \mathcal{G}(n,M)$. This entails that analyzing the hitting time of a monotone increasing property $\mathcal{P}$ is in fact a refinement of the study of values of M and p for which $\mathcal{G}(n,M) \in \mathcal{P}$ and $\mathcal{G}(n,p) \in \mathcal{P}$ respectively (where to get the values of p we need to use the converse of Proposition 1.1 as stated above).

For every positive integer k let δ_k denote the graph property of having minimum degree at least k, let $\mathcal{EC}_k$ denote the graph property of being k-edge connected, let $\mathcal{VC}_k$ denote the graph property of being k-vertex connected, and let $\mathcal{HAM}$ denote the graph property of

[1] In fact, when moving from $\mathcal{G}(n,M)$ to $\mathcal{G}(n,p)$ the monotonicity requirement is not necessary

admitting a Hamilton cycle. Two cornerstone results in the theory of random graphs are that of Bollobás and Thomason [8] who proved that for every $1 \leq k \leq n-1$, with high probability (or w.h.p. for brevity)[2] $\tau(\widetilde{G};\delta_k) = \tau(\widetilde{G};\mathcal{EC}_k) = \tau(\widetilde{G};\mathcal{VC}_k)$, and that of Komlós and Szemerédi [18] who proved that w.h.p. $\tau(\widetilde{G};\delta_2) = \tau(\widetilde{G};\mathcal{HAM})$ (see also [5]). Note that these two results (and many other which have succeeded) provide a very strong indication that the "bottleneck" for such properties in random graphs is in fact the vertices of minimum degree. The results of this paper are of the very same nature.

Before we proceed we stress that for the sake of simplicity and clarity of presentation, we do not make a particular effort to optimize the constants obtained in our proofs. We also omit floor and ceiling signs whenever these are not crucial. Most of our results are asymptotic in nature and whenever necessary we assume that n is sufficiently large.

1.4 Motivation and previous results Given a graph G with minimum degree at most $2k-1$ Breaker can keep claiming edges incident to a vertex of minimum degree, and with the advantage of playing first will leave Maker with a graph containing a vertex of degree at most $k-1$. This implies that Breaker wins the k-edge-connectivity game $(E(G), \mathcal{F}_{\mathcal{EC}_k})$ for such graphs, and therefore $\tau(\widetilde{G};\mathcal{M}_{\mathcal{EC}_k}) \geq \tau(\widetilde{G};\delta_{2k})$ for every graph process $\widetilde{G}$. In [23] Stojaković and Szabó were the first to consider Maker-Breaker games played on random graphs. By combining theorems of Lehman [19] and of Palmer and Spencer [20], they observed that for every fixed positive integer k, if $\widetilde{G}$ is the random graph process, then w.h.p. $\tau(\widetilde{G};\mathcal{M}_{\mathcal{EC}_k}) = \tau(\widetilde{G};\delta_{2k})$, thus providing a very precise hitting time result for the edge-connectivity game[3]. Similarly to the edge-connectivity case we have that for every graph process $\widetilde{G}$

$$\tau(\widetilde{G};\delta_{2k}) \leq \tau(\widetilde{G};\mathcal{M}_{\mathcal{VC}_k}). \tag{1.1}$$

Let $\mathcal{PM}$ denote the graph property of admitting a matching of size $\lfloor n/2 \rfloor$ in a graph on n vertices. Every graph on G an *even* number of vertices with minimum degree at most 1 is a win for Breaker in the perfect matching game $(E(G), \mathcal{F}_{\mathcal{PM}})$. Hence, for every graph process $\widetilde{G}$ on an even number of vertices

$$\tau(\widetilde{G};\delta_2) \leq \tau(\widetilde{G};\mathcal{M}_{\mathcal{PM}}). \tag{1.2}$$

In [23] Stojaković and Szabó conjectured that if $\widetilde{G}$ is the random graph process, then w.h.p. equality holds in (1.2). Although they did not prove this conjecture, in [23] they proved that if $p > \frac{64\ln n}{n}$, then w.h.p. $\mathcal{G}(n,p) \in \mathcal{M}_{\mathcal{PM}}$. Note that this result is optimal in p up to multiplicative constant factor, for if $p \leq \frac{\ln n + \ln\ln n - \omega(1)}{n}$, where $\omega(1)$ is some function which tends to infinity with n arbitrarily slowly, then w.h.p. $\delta(\mathcal{G}(n,p)) \leq 1$, and hence by (1.2), w.h.p. $\mathcal{G}(n,p) \notin \mathcal{M}_{\mathcal{PM}}$.

Clearly, every graph G with minimum degree at most 3 is a win for Breaker in the Hamiltonicity game $(E(G), \mathcal{F}_{\mathcal{HAM}})$. Hence, we have that for every graph process $\widetilde{G}$

$$\tau(\widetilde{G};\delta_4) \leq \tau(\widetilde{G};\mathcal{M}_{\mathcal{HAM}}). \tag{1.3}$$

In [23] Stojaković and Szabó conjectured that if $\widetilde{G}$ is the random graph process, then w.h.p. equality holds in (1.3).

One of the first results in the field of Maker-Breaker games on graphs is due to Chvátal and Erdős in their seminal paper [9], which states that $K_n \in \mathcal{M}_{\mathcal{HAM}}$ for sufficiently large values of n (in [16] the third author and Stich proved that $n \geq 38$ suffices). The problem of finding sparse graphs which are a win for Maker was addressed by Hefetz et. al. [15] where they showed that, for sufficiently large values of n, there exists a graph $G \in \mathcal{M}_{\mathcal{HAM}}$ on n vertices with $e(G) \leq 21n$. Playing the Hamiltonicity game $(E(G), \mathcal{F}_{\mathcal{HAM}})$ on the random graph $\mathcal{G}(n,p)$ was first considered in the original paper of Stojaković and Szabó [23] where they proved that if $p > \frac{32\ln n}{\sqrt{n}}$, then w.h.p. $\mathcal{G}(n,p) \in \mathcal{M}_{\mathcal{HAM}}$. Later, Stojaković [22] found the correct order of magnitude proving that $p > 5.4\ln n/n$ suffices for $\mathcal{G}(n,p)$ to be w.h.p. Maker's win in the Hamiltonicity game. This requirement on p was subsequently improved to $p \geq \frac{\ln n + (\ln\ln n)^s}{n}$, where s is some large but fixed constant, by Hefetz et. al. [14]. Note that this result is very close to being optimal, for if $p = \frac{\ln n + 3\ln\ln n - \omega(1)}{n}$, where $\omega(1)$ is some function which tends to infinity with n arbitrarily slowly, then w.h.p. $\delta(\mathcal{G}(n,p)) < 4$ and hence by (1.3) w.h.p. $\mathcal{G}(n,p) \notin \mathcal{M}_{\mathcal{HAM}}$. Lastly, in [4] the first and fourth authors with Sudakov studied the Hamiltonicity game played on the edges of random regular graphs (the uniform probability measure over all d-regular graphs on a fixed vertex set) and proved that for large enough constant values of d this game is Maker's win.

1.5 Our results In this paper we address the above mentioned Maker-Breaker games on random graphs, namely when Maker's goal is to build graphs which satisfy the properties of being k-vertex connected, admitting a perfect matching, and being Hamiltonian. Specif-

[2] In this paper, we say that a sequence of events $\mathcal{A}_n$ in a random graph model occurs w.h.p. if the probability of $\mathcal{A}_n$ tends to 1 as the number of vertices n tends to infinity.

[3] In [23] only the case of $k = 1$ is explicitly mentioned, but it can be generalized for any positive integer k in a straightforward manner.

ically, the main objective of this paper is to prove that the trivial minimum degree requirement as stated in (1.1), (1.2), and (1.3) is actually the bottleneck for a typical random graph to be a win for Maker in all of the above mentioned games. The following results will thus be proved.

THEOREM 1.1. *For every fixed integer $k \geq 1$, if $\widetilde{G}$ is the random graph process, then w.h.p.* $\tau(\widetilde{G}; \mathcal{M}_{\mathcal{VC}_k}) = \tau(\widetilde{G}; \delta_{2k})$.

For every positive integer k it holds that $\mathcal{VC}_k \subseteq \mathcal{EC}_k$, hence Theorem 1.1 is in fact an improvement of the aforementioned result of Stojaković and Szabó in [23]. We also note that, by using the theorem of Lehman [19], we can get the result of Palmer and Spencer [20] for even values of k as a corollary of Theorem 1.1.

The following result for the perfect matching game is also proved.

THEOREM 1.2. *If $\widetilde{G}$ is the random graph process on an even number of vertices, then w.h.p.* $\tau(\widetilde{G}; \mathcal{M}_{\mathcal{PM}}) = \tau(\widetilde{G}; \delta_2)$.

Theorem 1.2 settles a conjecture raised in [23]. By the connection between the random graph models as described in Section 1.3 and by known results on the distribution of the minimum degree of $\mathcal{G}(n,p)$, Theorem 1.2 implies that w.h.p. $\mathcal{G}(n,p) \in \mathcal{M}_{\mathcal{PM}}$ for every $p \geq \frac{\ln n + \ln\ln n + \omega(1)}{n}$, where $\omega(1)$ tends arbitrarily slowly to infinity with n, improving on the result of Stojaković and Szabó in [23].

THEOREM 1.3. *If $\widetilde{G}$ is the random graph process, then w.h.p.* $\tau(\widetilde{G}; \mathcal{M}_{\mathcal{HAM}}) = \tau(\widetilde{G}; \delta_4)$.

Theorem 1.3 settles a conjecture raised in [23]. Moreover, similarly to the above, Theorem 1.3 improves on the result of Hefetz et. al. in [14] by implying that w.h.p. $\mathcal{G}(n,p) \in \mathcal{M}_{\mathcal{HAM}}$ for every $p \geq \frac{\ln n + 3\ln\ln n + \omega(1)}{n}$, where $\omega(1)$ tends arbitrarily slowly to infinity with n.

We note that, by using a slight modification of our proofs, Theorems 1.2 and 1.3 can in fact be extended. For every positive integer $k \geq 1$, let $\mathcal{PM}^k$ and $\mathcal{HAM}^k$ denote the graph properties of admitting k paiwise edge-disjoint perfect matchings, and k pairwise edge-disjoint Hamilton cycles.

THEOREM 1.4. *For every fixed integer $k \geq 1$, if $\widetilde{G}$ is the random graph process, then w.h.p.* $\tau(\widetilde{G}; \mathcal{M}_{\mathcal{PM}^k}) = \tau(\widetilde{G}; \delta_{2k})$.

THEOREM 1.5. *For every fixed integer $k \geq 1$, if $\widetilde{G}$ is the random graph process, then w.h.p.* $\tau(\widetilde{G}; \mathcal{M}_{\mathcal{HAM}^k}) = \tau(\widetilde{G}; \delta_{4k})$.

Theorem 1.5 can be viewed as a Combinatorial game analog of the classical result of Bollobás and Frieze [7] who proved that w.h.p. $\tau(\widetilde{G}; \mathcal{HAM}^k) = \tau(\widetilde{G}; \delta_{2k})$ (see also [12] for an extension to non-constant minimum degree in the $\mathcal{G}(n,p)$ model).

1.6 Organization The rest of the paper is organized as follows. Section 2 is devoted to the introduction of expander graphs and the analysis of a general game in which Maker's goal is to build such a graph. This will give us a framework from which we can build on to prove the concrete results on the more natural games mentioned above. We then move on to provide the full proofs of Theorems 1.1 and 1.2 and 1.3 in Section 3. These proofs will rely heavily on the general expander game and the properties of random graphs and random graph processes which we discussed in the preceding two sections. Section 4 presents a sketch of a proof of Theorems 1.4 and 1.5 as these proofs follow quite closely the footsteps of the proofs presented in Section 3. In the Appendix we provide some additional background and technical details which will be of use throughout the course of our proofs.

2 An expander game on pseudo-random graphs and its application to random graphs

The main object of this section is to describe a general Maker-Breaker game which will reside in the core of all of our proofs. Let us first define the type of expanders we wish to study.

DEFINITION 2.1. *For every $c > 0$ and every positive integer R we say that a graph $G = (V,E)$ is an (R,c)-*expander *if every subset of vertices $U \subseteq V$ of cardinality $|U| \leq R$ satisfies $|N_G(U)| \geq c \cdot |U|$. We denote the graph property of being an (R,c)-expander by $\mathcal{X}_{R,c}$.*

REMARK 2.1. *From the above definition it clearly follows that for every $c > 0$ and every positive integer R (both c and R can be functions of the number of vertices of the graph in question), the graph property $\mathcal{X}_{R,c}$ is monotone increasing.*

Next, we consider some structural properties of (R,c)-expanders. The following two propositions show that the removal or addition of subsets that satisfy certain properties result in graphs that are still expanders. These properties will allow us to slightly modify certain expanders without losing their expansion properties.

PROPOSITION 2.1. *If $G = (V,E)$ is an (R,c)-expander and $U \subseteq V$ is a subset of vertices such that no two vertices of U have a common neighbor in G, then $G[V \setminus U]$ is an $(R,c-1)$-expander.*

Proof. Let $S \subseteq V \setminus U$ be a set of cardinality $|S| \leq R$. It follows by our assumption on U that $|N_G(v) \cap U| \leq 1$ holds for every vertex $v \in S$. Hence $|N_{G[V\setminus U]}(S)| \geq |N_G(S)| - |S| \geq (c-1)|S|$.

PROPOSITION 2.2. *Let $G = (V, E)$ be a graph, let $c > 0$, and let R be a positive integer. Let $U \subseteq V$ be a subset of vertices such that $|N_G(U')| \geq (c-1)|U'|$ for every $U' \subseteq U$, and, moreover, there is no path of length at most 4 in G whose (possibly identical) endpoints lie in U. If $G[V \setminus U]$ is an (R, c)-expander, then G is an $(R, c-1)$-expander.*

Proof. Let $V' = V \setminus U$ and let $H = G[V']$. Let $S \subseteq V$ be of cardinality $s \leq R$, and set $S_1 = S \cap U$ and let $S_2 = S \setminus S_1$ with respective cardinalities s_1 and $s_2 = s - s_1$. Our assumption on U implies, in particular, that it is independent. It follows that $N_G(S_1) \subseteq V \setminus U$. Furthermore, $N_G(S_1)$ can contain at most one vertex from each set $\{\{t\} \cup N_H(t)\}_{t\in V'}$, and hence $|N_G(S_1) \cap (S_2 \cup N_H(S_2))| \leq |S_2|$. It follows that $N_G(S) \supseteq N_G(S_1) \cup (N_H(S_2) \setminus (N_G(S_1) \cap (S_2 \cup N_H(S_2))))$, and that $|N_G(S)| \geq (c-1)s_1 + (c \cdot s_2 - s_2) = (c-1)s_1 + (c-1)(s-s_1) = (c-1)s$ as claimed.

Next, we describe some sufficient conditions for a graph $G = (V, E)$ to be an expander (with appropriate parameters).

M1 $e_G(U) \leq \frac{\delta(G)|U|}{2(c+1)}$ for every subset of vertices $U \subseteq V$ of cardinality $1 \leq |U| < (c+1)r$;

M2 $e_G(U, W) > 0$ for every pair of disjoint subsets of vertices $U, W \subseteq V$ of cardinality $|U| = |W| = r$.

LEMMA 2.1. *For every $c > 0$, if $G = (V, E)$ is a graph which satisfies properties **M1** and **M2** for some positive integer $r \leq \frac{|V|}{c+2}$, then G is a $(\frac{|V|-r}{c+1}, c)$-expander.*

Proof. Set $R = \frac{|V|-r}{c+1}$; note that $R \geq r$ holds by the assumption of the lemma. Assume for the sake of contradiction that there exists a set $S \subseteq V$ of cardinality $|S| \leq R$ for which $|N_G(S)| < c|S|$. Let $T = S \cup N_G(S)$, then $|T| < (c+1)|S|$. If $1 \leq |S| \leq r$, then $|T| < (c+1)r$. Moreover, since all edges that have at least one endpoint in S are spanned by the vertices of T, it follows that $e_G(T) \geq \frac{\delta(G)|S|}{2} > \frac{\delta(G)|T|}{2(c+1)}$, which contradicts property **M1**. If $r < |S| \leq R$, then, since $e_G(S, V \setminus T) = 0$ and $|V \setminus T| > |V| - (c+1)|S| \geq |V| - (c+1)R = r$, we obtain a contradiction to property **M2**. This concludes the proof of the lemma.

The reason we study (R, c)-expanders is the fact that they entail some pseudo-random properties from which (under some conditions on R and c) all of the natural properties that are considered in this paper, namely, admitting a perfect matching, being k-vertex-connected and being Hamiltonian, follow. We will provide a sufficient conditions for an (R, c)-expander to be k-vertex connected and to admit a perfect matching. Hence by playing for an (R, c)-expander, Maker will be able to win the two games whose goals are the aforementioned two properties (each posing different conditions on R and c). The sufficient condition for a graph to be Hamiltonian, that we will use in the course of the proof, is more delicate than the conditions for k-vertex connectivity and for admitting a perfect matching, and requires some additional ideas, but the crux of the proof will still rely on expanders.

Next, we provide sufficient conditions for $G \in \mathcal{M}_{\mathcal{X}_{R,c}}$, or namely, for a graph G to be Maker's win when Maker's goal is to build an (R, c)-expander. Although this game may seem at first to be an unnatural and artificial game to study, it turns out that this game will lie in the heart of our proofs of all of the results presented in this paper. Given parameters $c > 0$, $0 < \varepsilon < 1$, $K > 0$ and a positive integer $r \leq \frac{|V|}{c+1}$, we define the following two properties of a graph $H = (V, E)$ on n' vertices. These properties, which are closely related to properties **M1** and **M2**, will be needed in the proof of the main result of this section.

Q1 $e_H(U) \leq \frac{\varepsilon\delta(H)|U|}{10(c+1)}$ for every subset of vertices $U \subseteq V$ of cardinality $1 \leq |U| < (c+1)r$;

Q2 $e_H(U, W) \geq Kr \ln\left(\frac{n'}{r}\right)$ for every pair of disjoint subsets of vertices $U, W \subseteq V$ of cardinality $|U| = |W| = r$.

REMARK 2.2. *Whenever we will cite property **Q2** we will give an explicit expression for K which will not necessarily be a constant.*

THEOREM 2.1. *There exists an integer $n_0 > 0$ such that for every graph $G' = (V, E)$ on $n' \geq n_0$ vertices with minimum degree $\delta(G') > 0$ and for every choice of parameters $\frac{1}{2\delta(G')} < \varepsilon < \frac{1}{2}$, $c > 0$, and integer $0 < r \leq \min\{\frac{n'}{c+2}, \frac{n'}{e^{30}}\}$ for which G' satisfies properties **Q1** and **Q2** with $K = \frac{n'}{r(1-2\varepsilon)}$, Maker can win the $(\frac{n'-r}{c+1}, c)$-expander game on G', that is, $G' \in \mathcal{M}_{\mathcal{X}_{R,c}}$ with $R = \frac{n'-r}{c+1}$.*

The main idea of the proof is to show that we can split the graph G' into two parts where one has a large enough minimal degree and the other satisfies **Q2** for some other value of K. From the minimal degree of the first graph and the **Q1** property of G'

we derive the property **M1**. To guarantee the property **M2** we resort to a classical result in Maker-Breaker theory, namely the Erdős-Selfridge criteria which we introduce in Section A of the Appendix as Theorem A.2. Our proof of Theorem 2.1 will be presented as a series of three lemmata (Lemmata 2.2, 2.3, and 2.4) whose composition implies the theorem directly.

LEMMA 2.2. *There exists an integer $n_0 > 0$ such that for every graph $G' = (V, E)$ on $n' \geq n_0$ vertices with minimum degree $\delta(G') > 0$ and for every choice of parameters $\frac{1}{2\delta(G')} < \varepsilon < \frac{1}{2}$ and integer $0 < r \leq n'/e^4$ for which G' satisfies property **Q2** with $K = \frac{n'}{r(1-2\varepsilon)}$, the edge set E can be split into two disjoint subsets $E = E_1 \cup E_2$ such that the graph $G_1 = (V, E_1)$ has minimum degree $\delta(G_1) \geq \varepsilon\delta(G')$ and the graph $G_2 = (V, E_2)$ satisfies property **Q2** with $K = 3$.*

Proof. Pick every edge of G' to be an edge in G_1 with probability 2ε independently of all other choices. The degree in G_1 of every vertex $v \in V$ is binomially distributed, that is, $d_{G_1}(v) \sim \text{Bin}(d_{G'}(v), 2\varepsilon)$ and thus its median is at least $\lfloor 2\varepsilon\delta(G') \rfloor$. By our choice of ε we have that $\lfloor 2\varepsilon\delta(G') \rfloor > \varepsilon\delta(G')$ and therefore $\Pr[d_{G_1}(v) \geq \varepsilon\delta(G')] > 1/2$. Since the degrees of every two vertices are positively correlated, we have that

$$\Pr[\delta(G_1) \geq \varepsilon\delta(G')] > 2^{-n'}.$$

Let U, W be a pair of disjoint subsets of vertices of cardinality $|U| = |W| = r$. By our assumption on G' we have that $e_{G'}(U, W) \geq \frac{n' \ln\left(\frac{n'}{r}\right)}{1-2\varepsilon}$. As $e_{G_2}(U, W) \sim \text{Bin}(e_{G'}(U, W), 1 - 2\varepsilon)$ we have $\mathbf{E}[e_{G_2}(U, W)] \geq n' \ln\left(\frac{n'}{r}\right)$. Applying Theorem A.1 we have

$$\begin{aligned} &\Pr\left[e_{G_2}(U, W) < 3r \ln\left(\frac{n'}{r}\right)\right] \\ \leq\ &\exp\left(-\frac{\left(1 - \frac{3r}{n'}\right)^2 n' \ln\left(\frac{n'}{r}\right)}{2}\right) \\ \leq\ &\exp\left(-\frac{n' \ln\left(\frac{n'}{r}\right)}{3}\right). \end{aligned}$$

By applying the union bound over all pairs of disjoint subsets of vertices of cardinality r each, we conclude that the probability that G_2 violates property **Q2** with $K = 3$ is at most

$$\begin{aligned} &\binom{n'}{r}\binom{n'-r}{r} \cdot \exp\left(-\frac{n' \ln\left(\frac{n'}{r}\right)}{3}\right) \\ \leq\ &\left(\frac{en'}{r}\right)^{2r} \cdot \exp\left(-\frac{n' \ln\left(\frac{n'}{r}\right)}{3}\right) \\ =\ &\exp\left(2r\left(1 + \ln\left(\frac{n'}{r}\right)\right) - \frac{n' \ln\left(\frac{n'}{r}\right)}{3}\right) \\ \leq\ &\exp\left(-\frac{n' \ln\left(\frac{n'}{r}\right)}{4}\right) \\ <\ &2^{-n'}, \end{aligned}$$

and therefore there exists a partition of G' as claimed.

The following lemma provides a sufficient condition on a graph $G = (V, E)$ for it to be a Maker's win in the game $(E, \mathcal{F}_{\mathbf{M2}})$, that is, the game on G in which Maker's goal is to build a subgraph which satisfies the (monotone increasing) property **M2**. In order to prove this result, we invoke a rather standard technique of studying a dual game in which the roles of Maker and Breaker are exchanged. Note that in the dual game, Breaker (which was the original Maker) is the second player.

LEMMA 2.3. *There exists an integer $n_0 > 0$ such that for every graph $G_2 = (V, E_2)$ on $n' \geq n_0$ vertices and for every integer $0 < r \leq n'/e^{30}$ for which G_2 satisfies property **Q2** with $K = 3$, playing on E_2 Maker can build a subgraph of G_2 which satisfies property **M2**.*

Proof. Let G_2 be any graph with vertex set V. In order for Maker to build a graph which satisfies property **M2**, he can adopt the role of Breaker in the game $(E_2, \mathcal{L})$, where $\mathcal{L}$ is the family of edge-sets of all induced bipartite subgraphs of G_2 with both parts of size r. Recall that, by property **Q2** with $K = 3$, every such winning set $L \in \mathcal{L}$ spans at least $3r \ln\left(\frac{n'}{r}\right)$ edges. It follows that

$$\begin{aligned} &\sum_{L \in \mathcal{L}} 2^{-|L|} \\ \leq\ &\sum_{U \subseteq V\ ;\ |U|=r}\ \sum_{W \subseteq V \setminus U\ ;\ |W|=r} 2^{-e_{G_2}(U,W)} \\ \leq\ &\binom{n'}{r}\binom{n'-r}{r} \cdot \exp\left(-3r \ln\left(\frac{n'}{r}\right) \ln 2\right) \\ \leq\ &\left(\frac{en'}{r}\right)^{2r} \cdot \exp\left(-3r \ln\left(\frac{n'}{r}\right) \ln 2\right) \\ \leq\ &\exp\left(r \cdot \left(2 \ln\left(\frac{n'}{r}\right) + 2 - \ln 2 \cdot 3 \ln\left(\frac{n'}{r}\right)\right)\right) \\ <\ &\frac{1}{2}. \end{aligned}$$

The assertion of the lemma follows readily by Theorem A.2.

LEMMA 2.4. *There exists an integer $n_0 > 0$ such that for every graph $G' = (V, E)$ on $n' \geq n_0$ vertices and for every choice of parameters $0 < \varepsilon < 1$, $c > 0$ and integer $0 < r \leq \frac{n'}{c+2}$ for which G' satisfies property* ***Q1*** *and whose edge set can be partitioned into two disjoint sets $E = E_1 \cup E_2$ where $G_1 = (V, E_1)$ is of minimum degree $\delta(G_1) \geq \varepsilon \cdot \delta(G')$, and $G_2 = (V, E_2)$ satisfies* ***Q2*** *with $K = 3$, Maker can win the $(\frac{n'-r}{c+1}, c)$-expander game, that is, $G' \in \mathcal{M}_{\mathcal{X}_{R,c}}$ with $R = \frac{n'-r}{c+1}$.*

Proof. Before the game starts, Maker splits the board into two parts, $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ as indicated in the lemma. Maker then plays two separate games in parallel, one on E_1 and the other on E_2. In every turn in which Breaker claims some edge of E_i, for $i = 1, 2$, Maker responds by claiming an edge of E_i as well (except for maybe once if Breaker has claimed the last edge of E_i). Let H denote the graph built by Maker by the end of the game and set $H_1 = (V, E(H) \cap E_1)$ and $H_2 = (V, E(H) \cap E_2)$.

The game on E_1 is played according to Lemma A.2. Hence, at the end of the game, Maker's graph H_1 will have minimum degree at least $\delta(H_1) \geq \frac{\delta(G_1)}{5}$. Since G' satisfies property **Q1** and $\delta(G_1) \geq \varepsilon\delta(G')$ it follows that, for every $U \subseteq V$ of cardinality $1 \leq |U| < (c+1)r$, the number of Maker's edges with both endpoints in U is $e_M(U) \leq e_{G'}(U) \leq \frac{\varepsilon\delta(G')|U|}{10(c+1)} \leq \frac{\delta(G_1)|U|}{10(c+1)} \leq \frac{\delta(H_1)|U|}{2(c+1)} \leq \frac{\delta(H)|U|}{2(c+1)}$. Hence, H satisfies property **M1**.

The game on E_2 is played according to Lemma 2.3, and therefore at the end of the game, Maker will build a graph H_2 which satisfies property **M2**. By the monotonicity of **M2**, this property also holds for H. Noting that H, n', r and c satisfy the conditions of Lemma 2.1, we deduce that $H \in \mathcal{M}_{\mathcal{X}_{R,c}}$, that is, Maker's graph is an (R, c)-expander as claimed.

Let $G = (V, E)$ be a graph on n vertices and for a positive integer t denote $\mathcal{D}_t(G) = \{v \in V \ : \ d_G(v) < t\}$. Note that if G is a subgraph of H then $\mathcal{D}_t(G) \supseteq \mathcal{D}_t(H)$. Based on Theorem 2.1 we show that removing vertices of small degree from the random graph typically leaves a graph on which the expander game can be won by Maker. In fact, we even show that Maker can win the game when this graph is thinned substantially (i.e. the vast majority of the edges are removed). Note that this stronger property (i.e. of the existence of a "good" thinning of the graph) will play a crucial role in the proof of Theorem 1.3. To this end we will need some structural properties of the set $\mathcal{D}_t$ in the random graph model $\mathcal{G}(n, M)$, which we have seen to be equivalent to stopping the random graph process at G_M. The following propositions are central in all of our arguments, as we will need to take special care of vertices of small degree.

PROPOSITION 2.3. *For every integer $t \leq \ln^{0.9} n$, if $\widetilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$ is the random graph process and $M \geq \tau(\widetilde{G}; \delta_1)$ then w.h.p. $|\mathcal{D}_t(G_M)| \leq n^{0.3}$.*

The proof of this proposition will appear in the full version of this paper.

PROPOSITION 2.4. *For every fixed integers $k \geq 1$ and $t \leq \ln^{0.9} n$, if $\widetilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$ is the random graph process and $M = \tau(\widetilde{G}; \delta_k)$ then w.h.p $G = G_M$ does not contain a path of at most 4 distinct edges where both (possibly identical) endpoints lie in $\mathcal{D}_t(G_M)$.*

In order to prove the above results and other structural properties of the random graph process we resort to the use of $\mathcal{G}(n, p)$, where the analysis is much simpler, and use Proposition 1.2 to transfer the results to the original random graph model. The full proof of Proposition 2.4 will appear in the full version of this paper, but we provide here a brief sketch of it. First one can use Proposition 2.3 to show that w.h.p. the set $U = \mathcal{D}_t(G_{m_k})$ is small. At time m_k, one can also prove that there is no undesired short path and that w.h.p. at time M_k all degrees are $O(\log n)$ (using, say, Proposition 1.2). Note that it follows that when running the random graph process from time m_k to M_k, while assuming that the final point satisfies the above condition, we have that during the run all degrees are still $O(\log n)$. Now, consider the probability that the ith edge, e_i, of the process, for any $m_k < i \leq M_k$ closes an undesired path (a path of length at most 4 in G_i between two vertices of U). For this to happen both endpoints of e_i should be at distance at most 3 in G_i from the set U. Since U is small, and all degrees are $O(\log n)$ in G_i, the number of such pairs is $O((|U| \log^3 n)^2) = o(n^{0.8})$, and therefore the probability that e_i is such is at most $o(n^{0.8}/n^2) < n^{-1.2}$. Then one can apply a union bound argument over all steps i between m_k and M_k to complete the proof.

The following lemma, whose proof will appear in the full version of this paper, guarantees that after the removal of vertices of small degree, Maker can win the expander game when R and c are within a specified range. The main idea is to show that once the vertices of small degree are removed from the graph, all of the conditions of Theorem 2.1 are met with high probability. This lemma is the main structural property of the random graph process which we use in the course of the proof of Theorem 1.3.

LEMMA 2.5. *For every $\alpha > 0$ and fixed positive integer $k \geq 2$ if $\widetilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$ is the random graph process* *and $M = \tau(\widetilde{G};\delta_k)$ then w.h.p. $G' = (V', E') = G_M \setminus \mathcal{D}_{\ln^{0.9} n}(G_M)$ on n' vertices contains a spanning subgraph $\widehat{G} \subseteq G_M$ with at most $2n' \ln^{0.97} n'$ edges such that $\widehat{G} \in \mathcal{M}_{\mathcal{X}_{R,c}}$ for every $c \leq \ln^{0.02} n'$ and $R \leq (1-\alpha)\frac{n'}{c+1}$.*

Remark 2.3. *As was noted in Remark 2.1, by the monotonicity of $\mathcal{X}_{R,c}$, the above Lemma can be used to deduce that the graph $G' \in \mathcal{M}_{\mathcal{X}_{R,c}}$.*

3 Proofs of Theorems 1.1, 1.2 and 1.3

This section is devoted to the proofs of Theorems 1.1, 1.2 and 1.3. These three theorems make use of some simple sufficient condition on (R,c)-expanders to have the required property at hand and some simple applications of the properties of random graphs as presented in the previous sections.

Lemma 3.1. *For every positive integer k, if $G = (V, E)$ is an (R,c)-expander such that $c \geq k$, and $Rc \geq \frac{1}{2}(|V| + k)$, then $G \in \mathcal{VC}_k$.*

Proof. Assume for the sake of contradiction that there exists some set $S \subseteq V$ of size $k-1$ whose removal disconnects G. Denote the connected components of $G \setminus S$ by $S_1, \ldots, S_t$, where $t \geq 2$ and $1 \leq |S_1| \leq \ldots \leq |S_t|$. If $|S_1| \leq R$, then $k-1 = |S| \geq |N_G(S_1)| \geq c|S_1| \geq c \geq k$, which is clearly a contradiction. Assume then that $|S_1| > R$. For $i \in \{1,2\}$, let $A_i \subseteq S_i$ be an arbitrary subset of size R. It follows that $|V| \geq |S_1 \cup S_2 \cup N_G(S_1) \cup N_G(S_2)| \geq |N_G(A_1) \cup N_G(A_2)| = |N_G(A_1)| + |N_G(A_2)| - |N_G(A_1) \cap N_G(A_2)| \geq 2Rc - |S| \geq |V| + 1$, which is clearly a contradiction. It follows that G is k-vertex-connected as claimed.

In order to prove Theorem 1.1 it thus suffices to show that w.h.p. at the moment the random graph process first reaches minimum degree $2k$, Maker has a winning strategy for the (R,c)-expander game for suitably chosen values of R and c. In doing so we will heavily rely on Theorem 2.1.

Proof. [Proof of Theorem 1.1] Fix some positive integer $k \geq 1$ and let $\widetilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$ denote the random graph process. Set $M = \tau(\widetilde{G};\delta_{2k})$, let $G = G_M$, Small $= \mathcal{D}_{\ln^{0.9} n}(G)$, $G' = G[V \setminus \text{Small}]$ and denote by n' the number of vertices in G'. Setting $c = k+2$, and $R = \frac{n'}{k+4}$, the conditions of Lemma 2.5 are met, and thus $G' \in \mathcal{M}_{\mathcal{X}_{\frac{n'}{k+4},k+2}}$.

Maker's strategy is quite natural. He splits the board into $F_1 = E(G')$ and $F_2 = E_G(\text{Small}, V \setminus \text{Small})$, and plays the corresponding two games in parallel, that is, in each move Maker will claim an edge of the board Breaker chose his last edge from (except for possibly his last move in one of the two games). Playing on the edges of F_1, Maker aims to build an $(\frac{n'}{k+4}, k+2)$-expander. As noted above, Maker has a winning strategy for this game. Playing on the edges of F_2, Maker follows a simple pairing strategy which guarantees that, by the end of the game, the graph H which Maker constructs will satisfy $d_H(v) \geq \lfloor d_G(v)/2 \rfloor$ for every $v \in$ Small. To achieve this goal, whenever Breaker claims an edge which is incident with some vertex $v \in$ Small, Maker responds by claiming a different edge incident with v if such an edge exists, and otherwise he claims an arbitrary free edge of $F_1 \cup F_2$. Since the minimum degree in G is $2k$, it follows by Maker's strategy for the game on F_2 and by Proposition 2.4, that in Maker's graph H, the vertices of Small form an independent set with k edges emitting out of each vertex. Since the graph $H' = H[V \setminus \text{Small}]$ is an $(\frac{n'}{k+4}, k+2)$-expander, and since $(k+2) \cdot \frac{n'}{k+4} \geq \frac{1}{2}(n+k)$ holds for every $k \geq 1$ by Proposition 2.3, Lemma 3.1 implies that $H' \in \mathcal{VC}_k$. Adding to H' the vertices of Small with their incident edges clearly keeps the k-vertex connectivity property, as connecting a new vertex to at least k vertices of a k-vertex connected graph produces a k-vertex connected graph. This concludes the proof of the theorem.

In order to show that expansion entails admitting a perfect matching, we make use of the well-known Berge-Tutte formula for the size of a maximum matching in a graph (see e.g. [24, Corollary 3.3.7]).

Theorem 3.1. (Berge-Tutte) *The maximum number of vertices which are saturated by a matching in a graph $G = (V, E)$ is $\min_{S \subseteq V} \{|V| + |S| - o(G - S)\}$.*

The following lemma is applicable regardless of the parity of the number of vertices in the graph.

Lemma 3.2. *If $G = (V, E)$ is an (R,c)-expander such that $c \geq 2$ and $(c+1)R \leq |V| \leq \min\left\{\frac{2R^2c(c-1)}{c+R(c-1)}, \frac{2Rc(c-1)-6c^2}{c-1}\right\}$, then $G \in \mathcal{PM}$.*

Proof. From the conditions on R and c it follows that $Rc > |V|/2$ and, combined with G being an (R,c)-expander, this trivially implies that the graph G must be connected. Setting $S = \emptyset$, we have that $o(G - S) = 1$ for odd $|V|$, and that $o(G - S) = 0$ for even $|V|$. By Theorem 3.1 we can thus assume that $S \neq \emptyset$. We will in fact prove that $|S| \geq c(G - S)$ holds for every non-empty $S \subseteq V$. It clearly suffices to prove this for every $\emptyset \neq S \subseteq V$ of cardinality $|S| \leq |V|/2$. Let S be such a set, let $t = c(G - S)$, and let $S_1, \ldots, S_t$ denote the connected components of $G - S$, where

$1 \le |S_1| \le \ldots \le |S_t|$. Assume first that there exists a set $A \subseteq \{1,\ldots,t\}$ such that $|S|/c < \left|\bigcup_{i\in A} S_i\right| \le R$. By definition we have $N_G(\bigcup_{i\in A} S_i) \subseteq S$. It follows that $|S| \ge |N_G(\bigcup_{i\in A} S_i)| \ge c\left|\bigcup_{i\in A} S_i\right| > |S|$, which is clearly a contradiction. Hence, no such $A \subseteq \{1,\ldots,t\}$ exists. It follows that there must exist some $0 \le j^* \le t$ such that $\sum_{i=1}^{j^*} |S_i| \le \lfloor |S|/c \rfloor$ and $|S_i| > R - |S|/c$ for every $j^* < i \le t$. If $j^* \ge t-1$, then, since $|S_i| \ge 1$ for every $1 \le i \le t$, it follows that $t \le \sum_{i=1}^{t-1} |S_i| + 1 \le \lfloor |S|/c \rfloor + 1 \le |S|$. Hence, we can assume that $j^* \le t-2$. We claim that, under this assumption, $|S| \ge \frac{Rc}{2}$. Indeed, assume for the sake of contradiction that $1 \le |S| < \frac{Rc}{2}$ or equivalently, that $c(R - |S|/c) > |S|$. If $R - |S|/c \le |S_{j^*+1}| \le R$, then, as $S \supseteq N_G(S_{j^*+1})$ we have that $|S| \ge |N_G(S_{j^*+1})| \ge c(R - |S|/c) > |S|$, a contradiction. Thereofore, $|S_i| > R$ for $j^* < i \le t$. Since $j^* \le t-2$, for $i \in \{t-1, t\}$, we can choose $A_i \subseteq S_i$ to be an arbitrary subset of size R. It follows that $|V| \ge |S_{t-1} \cup S_t \cup N_G(S_{t-1}) \cup N_G(S_t)| \ge |N_G(A_{t-1}) \cup N_G(A_t)| = |N_G(A_{t-1})| + |N_G(A_t)| - |N_G(A_{t-1}) \cap N_G(A_t)| \ge 2Rc - |S| > |V|$, which is, again, clearly a contradiction. We deduce that $|V|/4 < Rc/2 \le |S| \le |V|/2 < Rc$. Note that under our assumption on R and c we have that $R - |S|/c > 4$, and therefore $|S_i| \ge 5$ for all $j^* < i \le t$. Moreover, as all S_i contain at least one vertex $j^* \le |S|/c$. Putting this together we have that $|S| > \frac{1}{3}\sum_{i=1}^{t} |S_i| \ge \frac{j^* + (t-j^*)(R-|S|/c)}{3} \ge \frac{5t-4j^*}{3}$, and therefore $t < \frac{|S|}{5}(3 + \frac{4}{c}) \le |S|$ which completes the proof of the lemma.

In order to prove Theorem 1.2 we proceed very similarly to the proof of Theorem 1.1.

Proof. [Proof of Theorem 1.2] Let $\widetilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$ denote the random graph process. Set $M = \tau(\widetilde{G}; \delta_2)$, let $G = G_M$, $\textsc{Small} = \mathcal{D}_{\ln^{0.9} n}(G)$, $G' = G[V \setminus \textsc{Small}]$ and denote by n' the number of vertices in G'. Setting $c = 8$, and $R = \frac{n'}{10}$, the conditions of Lemma 2.5 are met, and thus $G' \in \mathcal{M}\mathcal{X}_{\frac{n'}{10},8}$.

Maker's strategy follows the same lines as that of the k-connectivity case presented in the proof of Theorem 1.1. He splits the board into $F_1 = E(G')$ and $F_2 = E_G(\textsc{Small}, V \setminus \textsc{Small})$, and plays the corresponding two games in parallel, that is, in each move Maker will claim an edge of the board Breaker chose his last edge from (except for possibly his last move in one of the two games). Playing on the edges of F_1, Maker aims to build an $(n'/10, 8)$-expander. As noted above, Maker has a winning strategy for this game. We denote the restriction of the graph built by Maker by the end of the game to the edges of F_1 by H_1. Playing on the edges of F_2, Maker follows a simple pairing strategy which guarantees that, by the end of the game, the graph H_2 which Maker constructs will satisfy $d_{H_2}(v) \ge \lfloor d_G(v)/2 \rfloor$ for every $v \in \textsc{Small}$. To achieve this goal, whenever Breaker claims an edge which is incident with some vertex $v \in \textsc{Small}$, Maker responds by claiming a different edge incident with v if such an edge exists, and otherwise he claims an arbitrary free edge of $F_1 \cup F_2$. Recalling Proposition 2.4 we can assume that $\textsc{Small}$ is an independent set in G and that no two vertices in $\textsc{Small}$ share a common neighbor. As the minimum degree in G is 2, Maker's graph, $H = H_1 \cup H_2$, will contain at least one edge emitting out of every vertex in $\textsc{Small}$, each incident with a different vertex of $V \setminus \textsc{Small}$. Therefore, there exists a matching in $\mathcal{M}$ which covers all vertices of $\textsc{Small}$. Let T denote the set of vertices of $V \setminus \textsc{Small}$ which are covered by $\mathcal{M}$. Again, by Proposition 2.4 we can assume that no two vertices in T share a common neighbor (as this would create a path of length 4 between two vertices in $\textsc{Small}$). Since, the graph H_2 is an $(n'/10, 8)$-expander, it follows by Proposition 2.1 that the graph $H' = H_2 \setminus T$ is an $(n'/10, 7)$-expander. The values $R = n'/10$ and $c = 7$ satisfy the condition of Lemma 3.2, implying that $H' \in \mathcal{PM}$. Let $\mathcal{M}'$ be some perfect matching of H', then $\mathcal{M} \cup \mathcal{M}'$ is a perfect matching of H. This concludes the proof of the theorem.

Our proof of Theorem 1.3 is fairly similar to the two the proofs of Theorems 1.1 and 1.2. However, after having built an appropriate expander, Maker will need to claim additional edges in order to transform his expander into a Hamiltonian graph. In order to describe the relevant connection between Hamiltonicity and (R,c)-expanders, we require the notion of *boosters*.

Definition 3.1. *For every graph G we say that a non-edge $\{u,v\} \notin E(G)$ is a* booster *with respect to G if $G + \{u,v\}$ is Hamiltonian or $\ell(G + \{u,v\}) > \ell(G)$. We denote by $\mathcal{B}_G$ the set of boosters with respect to G.*

The following is a well-known property of $(R,2)$-expanders (see e.g. [12]).

Lemma 3.3. *If G is a connected non-Hamiltonian $(R,2)$-expander then $|\mathcal{B}_G| \ge R^2/2$.*

Our goal is to show that during a game on an appropriate graph G, assuming Maker can build a subgraph of G which is an (R,c)-expander, he can also claim sufficiently many such boosters, so that his (R,c)-expander becomes Hamiltonian. In order to do so, we further analyze the structure of the random graph process.

Lemma 3.4. *If $\widetilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$ is the random graph process and $M = \tau(\widetilde{G}; \delta_4)$, then w.h.p. G_M does not* *contain a connected non-Hamiltonian $(n/5,2)$-expander Γ with at most $n\ln^{0.98} n$ edges such that $|E(G_M)\cap\mathcal{B}_\Gamma| \leq n\ln^{0.98} n$.*

Proof. First we note that any $(n/5,2)$-expander must be connected, as each connected component must be of size at least $n/5+2n/5>n/2$. Let $m_4 \leq M' \leq M_4$ be an integer, let $p = M'/\binom{n}{2} > \frac{\ln n}{n}$, and let $G=(V,E)\sim \mathcal{G}(n,p)$. Our goal is to prove that the probability that G contains a connected non-Hamiltonian $(n/5,2)$-expander subgraph Γ with at most $n\ln^{0.98} n$ edges such that $|E\cap\mathcal{B}_\Gamma| \leq n\ln^{0.98} n$ is "much smaller" than the probability that $e(G)=M'$. Summing over all integral values of M' in the interval $[m_4, M_4]$, and applying Proposition 1.2 to each of these values, will enable us to complete the proof.

Let $\mathcal{S}$ denote the set of all labeled non-Hamiltonian $(n/5,2)$-expanders on the vertex set V which have at most $n\ln^{0.98} n$ edges. Fix a graph $\Gamma=(V,F)\in\mathcal{S}$, then clearly $\Pr[\Gamma\subseteq G]=p^{|F|}$. Now, let $G'=(V,E\setminus F)\sim\mathcal{G}(n,p)_{-F}$. By definition, every booster with respect to Γ is a non-edge in Γ, hence $\mathcal{B}_\Gamma$ is a subset of the potential pairs of the graph G'. Lemma 3.3 implies that $|\mathcal{B}_\Gamma|\geq n^2/50$, and since $|E(G')\cap\mathcal{B}_\Gamma|\sim \mathrm{Bin}(|\mathcal{B}_\Gamma|,p)$, it follows that $\mathbf{E}[|E(G')\cap\mathcal{B}_\Gamma|]\geq\frac{n^2p}{50}>\frac{n\ln n}{50}$. Applying Theorem A.1 we have

$$\begin{aligned}
&\Pr\left[|E(G')\cap\mathcal{B}_\Gamma|\leq n\ln^{0.98} n\right]\\
\leq\ &\exp\left(-\frac{\left(1-\frac{50}{\ln^{0.02} n}\right)^2 n^2p}{100}\right)\\
\leq\ &\exp\left(-\frac{n^2p}{101}\right).
\end{aligned}$$

Next, we note that by the independence of appearance of edges in $\mathcal{G}(n,p)$, the event $\Gamma\subseteq G$ and the event that some booster e with respect to Γ was chosen among the edges of G', are independent events. We can thus use a union bound argument by going over all $\Gamma\in\mathcal{S}$ to upper bound the probability that G contains a connected non-Hamiltonian $(n/5,2)$-expander Γ with at most $n\ln^{0.98} n$ edges, such that $|E\cap\mathcal{B}_\Gamma|\leq n\ln^{0.98} n$ as follows

$$\begin{aligned}
&\sum_{m=1}^{n\ln^{0.98} n}\binom{\binom{n}{2}}{m}p^m\cdot\exp\left(-\frac{n^2p}{101}\right)\\
\leq\ &\sum_{m=1}^{n\ln^{0.98} n}\left(\frac{en^2p}{2m}\right)^m\cdot\exp\left(-\frac{n^2p}{101}\right)\\
\leq\ &\sum_{m=1}^{n\ln^{0.98} n}\exp\left(m\cdot\left(1+\ln\left(\frac{n^2p}{2m}\right)\right)-\frac{n^2p}{101}\right)\\
\leq\ &\exp\left(-\frac{n^2p}{102}\right).
\end{aligned}$$

Using Proposition 1.2, the above calculation implies that the same event, with $G\sim\mathcal{G}(n,M')$, is upper bounded by $3\sqrt{M'}\cdot\exp\left(-\frac{n^2p}{102}\right)\leq\exp\left(\frac{-n\ln n}{103}\right)$. Taking the union bound over all integral values of $m_4\leq M'\leq M_4$, we conclude that the probability there exists such an integer M' for which $G_{M'}$ violates the claim is at most $(M_4-m_4+1)\cdot\exp\left(\frac{-n\ln n}{103}\right)\leq n\ln\ln\ln n\cdot\exp\left(\frac{-n\ln n}{103}\right)=o(1)$.

We are now ready to present the full proof of Theorem 1.3.

Proof. [Proof of Theorem 1.3] Let $\widetilde{G}=\{G_i\}_{i=0}^{\binom{n}{2}}$ denote the random graph process. Set $M=\tau(\widetilde{G};\delta_4)$, let $G=G_M$, $\textsc{Small}=\mathcal{D}_{\ln^{0.9} n}(G)$, $G'=G[V\setminus\textsc{Small}]$ and denote by n' the number of vertices in G'. By Proposition 2.3 we can assume that $|\textsc{Small}|\leq n^{0.3}$. Setting $c=3$, and $R=\frac{9n'}{40}$, the conditions of Lemma 2.5 are met, and thus there exists a subgraph $\widehat{G}\subseteq G'$ such that $\widehat{G}\in\mathcal{M}_{\mathcal{X}_{\frac{9n'}{40},3}}$ and $e(\widehat{G})\leq 2n'\ln^{0.97} n'$.

Again, Maker's strategy resembles that presented in the previous two cases, but now it consists of two phases. Let e_i denote the edge selected by Maker in his ith move and let $H_i=(V,\{e_1,\ldots,e_i\})$ denote the graph Maker has built during his first i moves. Let H' denote Maker's graph at the end of the first phase and let H denote Maker's graph at the end of the second phase, i.e. Maker's final graph. Before the game starts, Maker splits the board $E(G)$ into three parts $F_1=E(\widehat{G})$, $F_2=E_G(\textsc{Small},V\setminus\textsc{Small})$ and $F_3=E(G'\setminus\widehat{G})$. During the first phase, Maker plays two games in parallel, one on F_1 and the other on F_2. For every $j\geq 1$, on his jth move of the first phase, Maker claims an edge of $F_1\cup F_2$, according to his strategy for each of the two games. If on his jth move Breaker claims an edge of F_i, for some $i\in\{1,2\}$, then Maker claims an edge of F_i as well (unless he has already achieved his goal in the game on F_i). If Breaker claims an edge of F_3, then Maker claims an edge of $F_1\cup F_2$. Playing on the edges of F_1, Maker aims to build a $(9n'/40,3)$-expander, H'_1. As noted above, Maker has a winning strategy for this game. Moreover, since $|F_1|\leq 2n'\ln^{0.97} n'$, Maker can build such an expander within at most $t_{1,1}:=n'\ln^{0.97} n'$ moves. Playing on the edges of F_2, Maker follows a simple pairing strategy which guarantees that, by the end of the game, the graph, H'_2, which Maker constructs, will satisfy $d_{H'_2}(v)\geq 2$ for every $v\in\textsc{Small}$. To achieve this goal, whenever Breaker claims an edge which is incident with some vertex $v\in\textsc{Small}$, Maker responds by claiming a different edge incident with v, unless his current graph already contains two edges which are incident with v

in which case he claims another free edge of $F_1 \cup F_2$ which brings him closer to his goal in the corresponding game. Hence, the number of moves required for Maker to reach his goal in the game on F_2 is at most $t_{1,2} := 2|\textsc{Small}| \leq 2n^{0.3}$. It follows by Proposition 2.4 that Small is an independent set and that no two edges emitting from Small are incident with the same vertex of $V \setminus \textsc{Small}$. Hence, Maker's graph, H_2', satisfies $N_{H_2'}(U') \geq 2|U'|$ for every $U' \subseteq \textsc{Small}$. Applying Proposition 2.2 and noting that $9n'/40 \geq n/5$, it follows that $H' = H_1' \cup H_2'$ is an $(n/5, 2)$-expander. Clearly, Maker's final graph H is an $(n/5, 2)$-expander as well. A crucial point to keep in mind is that the number of moves required for Maker to construct his $(n/5, 2)$-expander H', is $t_1 = t_{1,1} + t_{1,2} = o(n \ln^{0.98} n)$.

After having completed the construction of H', Maker proceeds to the second phase of his strategy. Let $t_2 \leq n$ denote the number of moves Maker plays during the second phase. For every $t_1 < j \leq t_1 + t_2$, on his jth move, Maker claims an edge of G which is a booster with respect to H_{j-1}. This is possible since, throughout the game Breaker claims at most $t_1 + t_2 \leq t_1 + n$ edges of G, but by Lemma 3.4, w.h.p. either H_{j-1} is Hamiltonian or it has at least $n \ln^{0.98} n > t_1 + n$ boosters among the edges of G. It follows by the definition of a booster that either H_j is Hamiltonian or $\ell(H_j) > \ell(H_{j-1})$. Repeating the same argument $t_2 \leq n$ times, we conclude that H is Hamiltonian as claimed.

4 Proof sketch of Theorems 1.4 and 1.5

We now sketch how the proof of Theorem 1.3 can be adapted so as to entail Theorem 1.5. Similarly, the proof of Theorem 1.4 can be obtained using appropriate modifications to the proof of Theorem 1.2, but as this case is simpler, we omit the details.

It suffices to prove that when removing all vertices of degree at most $\ln^{0.9} n$ from the random graph $\mathcal{G}(n, M)$, where $M = \tau(\widetilde{G}; \delta_{4k})$, playing on this subgraph G' on n' vertices, w.h.p. Maker can quickly (that is, within $o(n' \ln n')$ moves) build an $(9n'/40k, 3k)$-expander H' for which the property **M2** with $r = n'/ln^{0.4}n'$ holds. Moreover, at the same time, Maker can ensure that the minimum degree of his graph will be at least $2k$. After the removal of $0 \leq i \leq k-1$ edge-disjoint Hamilton cycles from the original graph we have removed a $2i$-regular graph from from H' and are left with a graph $\widehat{H}_i$ (which is spanned by the vertices which are not in Small) for which $|N_{\widehat{H}_i}(U)| \geq 3k|U| - 2i|U| \geq (k+2)|U|$ for every $U \subseteq V(H')$ of cardinality $|U| \leq 9n'/40k$. To complete the proof it is left to note that for the choice of the parameter r guarantees that between sets of linear size there is a super-linear number of edges. It is not hard to see that adding back the vertices of Small who are all incident to at least $2k - 2i \geq 2$ edges results in a connected $(n/5, 2)$-expander. This graph has many boosters which Breaker could not have taken them all, and Maker can thus continue playing for another Hamilton cycle using the boosters left in the graph. As there is a super-linear number of boosters and Breaker can claim at most n of them per Hamilton cycle, Maker can keep playing this way until he completely saturates his vertices of minimum degree.

5 Concluding remarks and open problems

In this paper we have proved the hitting time results for the perfect matching, k-vertex-connectivity, and Hamiltonicity Maker-Breaker games along with some generalizations. These results further exemplify the so-called minimum degree phenomena which occurs in random graphs, where the bottleneck for a typical random graph to satisfy some (global) property is the existence of vertices whose degree is too small for the property to hold. The main ingredient of our proofs is to note that once the vertices of small degree are "taken care of" the remainder of the random graph is *rich* enough (and, actually, much richer is needed) so that we can deduce the required properties. This setting, in a sense, emphasizes the true bottleneck as the one just described. In the context of random regular graphs there are no "special" vertices to take care of, and the problem becomes much more intricate. Let $\mathcal{G}_{n,d}$ denote the uniform probability space of all random regular graphs on a fixed set of n vertices. An interesting question to tackle would be

Problem. What is the minimal d for which w.h.p. $\mathcal{G}_{n,d} \in \mathcal{M}_{\mathcal{HAM}}$?

Classical results for this mode imply that w.h.p. $\mathcal{G}_{n,d} \in \mathcal{HAM}$ for all fixed $d \geq 3$, but it follows from a result of Hefetz et. al [15] that d must at least 5 for $\mathcal{G}_{n,d} \in \mathcal{M}_{\mathcal{HAM}}$. The first and fourth author with Sudakov [4] proved that this minimal d is fixed, but to get the minimal value requires further ideas.

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References

[1] N. Alon and J. H. Spencer. *The Probabilistic Method.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, third edition, 2008.

[2] J. Beck. On positional games. *Journal of Combinatorial Theory, Series A*, 30(2):117–133, 1981.

[3] J. Beck. *Combinatorial Games: Tic-Tac-Toe theory.* Cambridge University Press, New York, 2008.

[4] S. Ben-Shimon, M. Krivelevich, and B. Sudakov. Local resilience and Hamiltonicity Maker-Breaker games in random regular graphs. *Combinatorics, Probability, and Computing*, to appear.

[5] B. Bollobás. The evolution of sparse graphs. In B. Bollobás, editor, *Porceedings of Cambridge Combinatorial conference in honor of Paul Erdős*, Graph Theory and Combinatorics, pages 35–57. Academic Press, 1984.

[6] B. Bollobás. *Random Graphs.* Cambridge University Press, 2001.

[7] B. Bollobás and A. Frieze. On matchings and hamiltonian cycles in random graphs. In *Random Graphs (Poznań 1983)*, volume 28 of *Annals of Discrete Mathematics*, pages 23–46. North-Holland, Amsterdam, 1985.

[8] B. Bollobás and A. G. Thomason. Random graphs of small order. In *Random Graphs (Poznań 1983)*, volume 28 of *Annals of Discrete Mathematics*, pages 47–97. North-Holland, Amsterdam, 1985.

[9] V. Chvátal and P. Erdős. Biased positional games. In *Algorithmic aspects of combinatorics (Vancouver 1976)*, volume 2 of *Annals of Discrete Mathematics*, pages 221–229. 1978.

[10] P. Erdős and J. Selfridge. On a combinatorial game. *Journal of Combinatorial Theory, Series A*, 14:298–301, 1973.

[11] O. N. Feldheim and M. Krivelevich. Winning fast in sparse graph construction games. *Combinatorics, Probability and Computing*, 17(6):781–791, 2008.

[12] A. Frieze and M. Krivelevich. On two Hamilton cycle problems in random graphs. *Israel Journal of Mathematics*, 166:221–234, 2008.

[13] D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó. Fast winning strategies in Maker-Breaker games. *Journal of Combinatorial Theory, Series B*, 99(1):39–47, 2009.

[14] D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó. A sharp threshold for the Hamilton cycle Maker-Breaker game. *Random Structures and Algorithms*, 34(1):112–122, 2009.

[15] D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó. Global Maker-Breaker games on sparse graphs. *European Journal of Combinatorics*, to appear.

[16] D. Hefetz and S. Stich. On two problems regarding the Hamilton cycle game. *The Electronic Journal of Combinatorics*, 16(1):R28, 2009.

[17] S. Janson, T. Łuczak, and A. Ruciński. *Random Graphs.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, 2000.

[18] J. Komlós and E. Szemerédi. Limit distributions for the existence of Hamilton circuits in a random graph. *Discrete Mathematics*, 43(1):55–63, 1983.

[19] A. Lehman. A solution of the Shannon switching game. *Journal of the Society for Industrial and Applied Mathematics*, 12(4):687–725, 1964.

[20] E. M. Palmer and J. J. Spencer. Hitting time for k edge-disjoint spanning trees in a random graph. *Periodica Mathematica Hungarica*, 31(3):235–240, 1995.

[21] A. Pekeč. A winning strategy for the Ramsey graph game. *Combinatorics, Probability and Computing*, 5(3):267–276, 1996.

[22] M. Stojaković. *Games on Graphs.* PhD thesis, ETH Zürich, 2005.

[23] M. Stojaković and T. Szabó. Positional games on random graphs. *Random Structures and Algorithms*, 26(1-2):204–223, 2005.

[24] D. B. West. *Introduction to Graph Theory.* Prentice Hall, 2001.

A Preliminaries

In this section we cite some tools which we will make use of in the succeeding sections. First, we will need to employ bounds on large deviations of random variables. We will mostly use the following well-known bound on the lower and the upper tails of the Binomial distribution due to Chernoff (see e.g. [1, Appendix A]).

Theorem A.1. (Chernoff bounds) *If $X \sim B(n,p)$ then*

1. $\Pr[X < (1-\varepsilon)np] < \exp(-\frac{\varepsilon^2 np}{2})$ *for every* $\varepsilon > 0$;
2. $\Pr[X > (1+\varepsilon)np] < \exp(-\frac{np}{3})$ *for every* $\varepsilon \geq 1$.

It will sometimes be more convenient to use the following bound on the upper tail of the Binomial distribution.

Lemma A.1. *If $X \sim Bin(n,p)$ and $k \geq np$, then* $\Pr[X \geq k] \leq (enp/k)^k$.

Note that the bound given in Lemma A.1 is especially useful when k is "much larger" than np.

A.1 Basic positional games results

The following theorem is a classical result of Erdős and Selfridge [10] which provides a useful sufficient condition for Breaker's win in the $(X, \mathcal{F})$ game.

Theorem A.2. (Erdős and Selfridge [10]) *For any hypergraph $(X, \mathcal{F})$, if*

$$\sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{1}{2},$$

then Breaker, playing as the first or second player, has a winning strategy for the $(X, \mathcal{F})$ game.

The following simple lemma is useful when a player is trying to ensure expansion of small sets. A similar lemma appeared in [14].

Lemma A.2. *For every integer $k > 0$, if H is a graph on n vertices with minimum degree $\delta(H) \geq 5k$, then $H \in \mathcal{M}_{\delta_k}$. Moreover, Maker can win the minimum degree k game on the edge set of H in at most kn moves.*

Proof. We define a new graph H^*, where $H^* = H$ if all the degrees in H are even, and otherwise H^* is the graph obtained from H by adding a new vertex v^* and connecting it to every vertex of odd degree in H. Since all degrees of H^* are even, it admits an Eulerian orientation $\overrightarrow{H}^*$. For every $v \in V(H)$, let $E(v) = \{\{v,u\} \in E(H) : \overrightarrow{(v,u)} \in E(\overrightarrow{H}^*)\}$. Clearly, $|E(v)| \geq d_{H^*}(v)/2 \geq \lfloor d_H(v)/2 \rfloor \geq \lfloor 5k/2 \rfloor$ and the sets $\{E(v)\}_{v \in V(H)}$ are pairwise disjoint. In every round, if Breaker claims an edge of $E(v)$, then Maker responds by claiming an edge of $E(v) \setminus \{\{v, v^*\}\}$, unless he already has k edges incident with v in which case Maker proceeds by claiming an edge of $E(u)$, where u is some vertex such that Maker did not yet claim k of its incident edges (if no such vertex exists, then the game was already won by Maker). Note that since $\lfloor |E(v)|/2 \rfloor \geq k$, Maker can always play according to this strategy, and is never forced to pick an edge incident with v^*. Hence, Maker claims only edges of the original graph H. Disregarding the orientation, after at most kn moves, the graph spanned by Maker's edges has minimum degree at least k as claimed.

B Properties of random graphs and random graph processes

We start with a very simple claim regarding the number of edges in the Binomial random graph model $\mathcal{G}(n,p)$ whose proof is standard and is therefore omitted.

Proposition B.1. *If $p \geq \frac{\ln n}{n}$, then w.h.p. $e(\mathcal{G}(n,p)) \leq n^2 p$.*

The following estimates the probability of a vertex to be in $\mathcal{D}_t(\mathcal{G}(n,p))$.

Proposition B.2. *For every integer $t \leq \ln^{0.9} n$ and every vertex v, if $\frac{\ln n}{2} < p < \frac{2 \ln n}{n}$,*

$$\Pr\left[v \in \mathcal{D}_t(\mathcal{G}(n,p))\right] \leq n^{-1+o(1)}.$$

Proof. Let $G = (V,E) \sim \mathcal{G}(n,p)$, then for every vertex $v \in V$ we have that $d_G(v) \sim \mathrm{Bin}(n-1,p)$, and therefore we have that for every integer $t \leq \ln^{0.9} n$

$$\begin{aligned} & \Pr\left[v \in \mathcal{D}_t(G)\right] \\ \leq\ & \Pr\left[\mathrm{Bin}(n-1,p) < t\right] \\ \leq\ & t\binom{n-1}{t} p^t (1-p)^{n-1-t} \\ \leq\ & t \cdot \left(\frac{enp}{t}\right)^t e^{-p(n-1-t)} \\ <\ & \ln^{0.9} n \cdot n^{o(1)} \cdot n^{-1+o(1)} \\ <\ & n^{-1+o(1)}. \end{aligned}$$

For every fixed integer $k \geq 1$ we define two functions as follows:

$$\begin{aligned} m_k &= \binom{n}{2} \frac{\ln n + (k-1)\ln\ln n - \ln\ln\ln n}{n}; \\ M_k &= \binom{n}{2} \frac{\ln n + (k-1)\ln\ln n + \ln\ln\ln n}{n}. \end{aligned}$$

The following lemma (see e.g. [6]) gives a precise behavior of the minimum degree of the random graph process.

Lemma B.1. *For every fixed integer $k \geq 1$, if $\widetilde{G}$ is the random graph process then w.h.p. $m_k < \tau(\widetilde{G}; \delta_k) < M_k$.*

Lastly, we state some structural properties of the the random graph model $\mathcal{G}(n,M)$, which we have seen to be equivalent to stopping the random graph process at G_M.

Proposition B.3. *For every fixed integer $k \geq 2$, if $\widetilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$ is the random graph process and $M = \tau(\widetilde{G}; \delta_k)$ then w.h.p. G_M satisfies that $e_G(U) < |U| \ln^{0.8} n$ for every subset of vertices $U \subseteq V$ of cardinality $1 \leq |U| \leq \frac{n}{\ln^{0.3} n}$.*

Proposition B.4. *For every fixed integers $k \geq 1$, and $r = \lfloor \frac{n}{2\ln^{0.4} n} \rfloor$ if $\widetilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$ is the random graph process and $M = \tau(\widetilde{G}; \delta_k)$ then w.h.p. $e_{G_M}(U,W) \geq n \ln^{0.1} n$ for every disjoint pair of subsets of vertices $U, W \subseteq V$ of cardinality $|U| = |W| = r$.*

In order to prove these results one can resort to the use of $\mathcal{G}(n,p)$, where the analysis is much simpler, and use Proposition 1.2 to transfer the results to the original random graph model. The full proofs of these two propositions will appears in the full version of this paper.