Minors in expanding graphs

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1 Brief summary of results

In this paper we address several extremal problems related to graph minors. In all of our results we assume essentially that a given graph G is expanding, where expansion is either postulated directly, or G can be shown to contain a large expanding subgraph, or G is locally expanding due to the fact that G does not contain a copy of a fixed bipartite graph H. We need the following definitions to state our results. A graph $\Gamma = (U, F)$ with vertex set $U = \{u_1, \ldots, u_k\}$ is a *minor* of a graph $G = (V, E)$ if the vertex set V of G contains a sequence of disjoint subsets $A_1, \ldots, A_k$ such that the induced subgraphs $G[A_i]$ are connected, and there is an edge of G between A_i and A_j whenever the

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corresponding vertices u_i, u_j of Γ are connected by an edge. A graph $G = (V, E)$ is (t, α)-*expanding* if every subset $X \subset V$ of size $|X| \leq \alpha |V|/t$ has at least $t|X|$ external neighbors in G. A graph $G = (V, E)$ is called (p, β)-*jumbled* if

$$\left| e(X) - p\frac{|X|^2}{2} \right| \leq \beta |X|$$

for every subset $X \subseteq V$, where $e(X)$ stands for the number of edges spanned by X in G. Informally, this definition indicates that the edge distribution of G is similar to that of the random graph $G_{|V|,p}$, where the degree of similarity is controlled by parameter β.

Here are the main results of this paper.

Theorem 1 *Let $0 < \alpha < 1$ be a constant. Let G be a (t, α)-expanding graph of order n, and let $t \geq 10$. Then G contains a minor with average degree at least*

$$c\frac{\sqrt{nt \log t}}{\sqrt{\log n}},$$

where $c = c(\alpha) > 0$ is a constant.

This is an extension of results of Alon, Seymour and Thomas [5], Plotkin, Rao and Smith [34], and of Kleinberg and Rubinfeld [17], who cover basically the case of expansion by a constant factor $t = \Theta(1)$.

Theorem 2 *Let G be a (p, β)-jumbled graph of order n such that $\beta = o(np)$. Then G contains a minor with average degree $cn\sqrt{p}$, for an absolute constant $c > 0$.*

This statement is an extension of results of A. Thomason [39, 40], who studied the case of constant p. It can be also used to derive some of the results of Drier and Linial [12].

Theorem 3 *Let $2 \leq s \leq s'$ be integers. Let G be a $K_{s,s'}$-free graph with average degree r. Then G contains a minor with average degree $cr^{1+\frac{1}{2(s-1)}}$, where $c = c(s, s') > 0$ is a constant.*

This confirms a conjecture of Kühn and Osthus from [23].

Theorem 4 *Let $k \geq 2$ and let G be a C_{2k}-free graph with average degree r. Then G contains a minor with average degree $cr^{\frac{k+1}{2}}$, where $c = c(k) > 0$ is a constant.*

This theorem generalizes results of Thomassen [43], Diestel and Rempel [11], and Kühn and Osthus [22], who proved similar statements under the (much more restrictive) assumption that G has girth at least $2k + 1$.

All of the above results are, up to a constant factor, asymptotically tight (Theorems 1, 2), or are allegedly tight (Theorems 3, 4), where in the latter case the tightness hinges upon widely accepted conjectures from Extremal Graph Theory about the asymptotic behavior of the Turán numbers of $K_{s,s'}$ and of C_{2k}.

2 Background

This paper is devoted to two of the most fundamental, yet normally quite distant, concepts in modern Graph Theory – minors and expanding graphs. Their prominent role in mathematics is reflected by the fact that both have been featured in a popular column "What is...?" of the AMS Notices [29], [37]. The purpose of this section is to provide a basic information for both of these concepts, and also for several related notions in Graph Theory, relevant for this paper. Before going into technicalities, we would like to state notational agreements to be used in this paper. All graphs considered here are finite, without loops and without multiple edges, unless stated explicitly otherwise. Most of our notation is rather standard and can be found in any textbook in Graph Theory. Here we define several less common pieces of notation, used throughout the paper.

Let $G = (V, E)$ be a graph. For a subset $X \subseteq V$ we denote by $e_G(X)$ or simply by $e(X)$ the number of edges of G spanned by X, and by $N(X)$ the external neighborhood of X:

$$N(X) := \{u : u \notin X, u \text{ has a neighbor in } X\} .$$

In case $X = \{v\}$ we simply write $N(\{v\}) = N(v)$; obviously, the cardinality of $N(v)$ is the degree of v in G. For two disjoint sets $X, Y \subset V$, we denote the number of edges of G connecting X and Y by $e(X, Y)$.

As quite customary in Extremal Graph Theory, our approach to the problems researched will be asymptotic in nature. We thus assume that an underlying parameter (normally the order n of a graph) tends to infinity and is therefore assumed to be sufficiently large whenever necessary. We also do not make any serious attempt to optimize absolute constants in our statements and proofs. All logarithms are in the natural basis. We omit systematically rounding signs for the sake of clarity of presentation.

The following (standard) asymptotic notation will be utilized extensively: for two functions $f(n)$, $g(n)$ of a natural valued parameter n, we write $f(n) = o(g(n))$, whenever $\lim_{n\to\infty} f(n)/g(n) = 0$; $f(n) = O(g(n))$ if there exists a constant $C > 0$ such that $f(n) \leq Cg(n)$ for all n. Also, $f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$, and $f(n) = \Theta(g(n))$ if both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ are satisfied.

2.1 Minors

Definition 1 *A graph Γ is a* minor *of a graph G is for every vertex $u \in \Gamma$ there is a connected subgraph G_u of G such that all subgraphs G_u are vertex disjoint, and G contains an edge between G_u and $G_{u'}$ whenever (u, u') is an edge of Γ.*

An equivalent definition is through edge deletions and contractions: we can obtain a minor Γ of a graph G by first deleting all edges except those in subgraphs G_u, $u \in \Gamma$, and those connecting G_u, $G_{u'}$ for $(u, u') \in E(\Gamma)$, and then contracting all edges inside each of the connected subgraphs G_u. (Given an edge $e = (v', v'')$ of a graph G, *contracting* e results in replacing v', v'' by a single new vertex v, and connecting $w \in V(G) - \{v', v''\}$ to the new vertex v if and only if w is connected to v' or to v'' or to both in G).

Though the notion of graph minors appears at the first sight to be purely graph theoretic, it turns out to be absolutely essential in bridging between Graph Theory on one side, and Topology and Geometry on the other – one of the most non-trivial and fundamental connections in Mathematics. Indeed, the famous theorem of Kuratowski [24] (in its reformulation due to Wagner [46]) postulates that a graph G can be embedded in the plane (is planar) if and only if neither the complete graph K_5 on five vertices nor a complete bipartite graph $K_{3,3}$ with three vertices at each side are minors of G. This was the beginning of Topological Graph Theory, whose culmination is without a doubt the celebrated Robertson-Seymour theory of graph minors. In a series of twenty papers, spanning over two decades (with [36] being the concluding paper of the series), Robertson and Seymour proved the so called Wagner conjecture: in every infinite collection of graphs, there are two such that one is a minor of the other (in other words, the set of finite graphs with the "minor" relation as a partial order is well-quasi-ordered; see, e.g. [21] for more information about the theory of well-quasi-ordering). An equivalent formulation is that every family of graphs closed with respect to taking minors can be characterized by a finite family of excluded minors. As a corollary Robertson and Seymour were able to derive that for every closed compact surface there is a finite list of graphs such that a graph G is embeddable in this surface if and only if it does not contain any of these as a minor. This is of course an extremely far-reaching generalization of the Kuratowski theorem. The Robertson-Seymour Structural Graph Theory is undoubtedly an admirable research effort and one of the crown achievements of Combinatorics, whose impact is truly immense. As our research in minors will proceed along rather different lines, we will not dwell on this wonderful theory anymore, referring the reader instead to a very nice survey of Lovász on the subject [26].

2.2 Expanding graphs

The second fundamental concept of this paper is expanding graphs. Informally, a graph G is said to be an expanding graph or an expander if every subset X of $V(G)$ has relatively many neighbors outside X. (This is what is usually called *vertex expansion*, sometimes an alternative notion of *edge expansion* is used, there every set X is required to be incident to many edges crossing between X and its complement in $V(G)$; for constant degree graphs these two notions are essentially equivalent). Of course, a formal definition is required here, firstly, to measure the expansion quantitatively, and secondly to distinguish between the expansion of small and large sets – obviously a set X containing half the vertices of V cannot have more than $|X|$ outside neighbors, while a much smaller set X can expand by a much larger factor. There are several definitions of expanders in common use, capturing sometimes rather different expansion properties. In this paper we find it much more important to look at the expansion of small sets, and for this reason we adopt the following formal definition of an expander.

Definition 2 *Let $t > 0$, $0 < \alpha < 1$. A graph $G = (V, E)$ is* (t, α)-expanding *if every subset $X \subset V$ of size $|X| \leq \alpha|V|/t$ has at least $t|X|$ external neighbors in G.*

Normally we will think of α as being an absolute constant. In this case, the above definition says that every set X of size $|X| = O(n/t)$ expands by a factor of at least t.

As the research in the last quarter century has convincingly shown, the notion of expanders is of utmost value in an amazing variety of fields, both in and outside of Discrete Mathematics.

Applications include design of efficient communication networks, error-correcting codes with efficient encoding and decoding, derandomization of randomized algorithms, study of metric embeddings, to mention just a few. Expanders are usually constructed much easier using probabilistic, existential arguments (see, e.g. [33]); explicit constructions of expander graphs are much harder to come by and range from classical papers of Margulis [28] and of Lubotzky, Phillips and Sarnak [27], to a relatively recent zig-zag product construction of Reingold, Vadhan and Wigderson [35].

Our viewpoint here will be somewhat different from the above mentioned papers. Instead of discussing ways to construct good expanders, we will concentrate on *properties* of expanders, and more specifically on the appearance of large minors in expanding graphs.

General information about expanders, their properties and applications can be found in a recent excellent survey of Hoory, Linial and Wigderson [15].

2.3 Pseudo-random graphs

A notion closely related to expanding graphs is that of pseudo-random graphs. As the name clearly suggests, pseudo-random graphs can be informally described as graphs resembling truly random graphs, most commonly the so called binomial random graphs $G_{n,p}$. We first remind the reader the definition of this probability space. Given two parameters n and $0 \leq p \leq 1$, the *random graph* $G_{n,p}$ is a probability space of all graphs on n vertices labeled $1, ..., n$, where for each pair $1 \leq i \neq j \leq n$, the probability that (i, j) is an edge is p, independently of all other pairs. Equivalently, $G_{n,p}$ is the probability spaces of all labeled graphs with vertex set $\{1, \ldots, n\}$, endowed with the probability measure $Pr[G] = p^{|E(G)|}(1-p)^{\binom{n}{2}-|E(G)|}$. In quite a few cases the edge probability p is in fact a function $p = p(n)$ of the number of vertices n, vanishing as n tends to infinity. We say that random graph possesses a property $\mathcal{P}$ *with high probability* , if the probability that $G_{n,p}$ satisfies $\mathcal{P}$ tends to 1 as n tends to infinity. This probability space is undoubtedly the most studied and the most convenient to work with probability distribution on graphs. When defining pseudo-random graphs, one usually tries to capture quantitatively their similarity to truly random graphs, in this aspect or another. Arguably the most important feature of random graphs is their edge distribution, and so it is quite natural to expect that a definition of a pseudo-random graph will address this property. For the probability space $G_{n,p}$, edge distribution is quite easy to handle – for a given subset $X \subseteq V(G)$, the number of edges spanned by X in $G_{n,p}$ is a binomially distributed random variable with parameters $\binom{|X|}{2}$ and p; applying standard bounds on the tails of the binomial distribution one can easily show that with high probability *all* sets X of cardinality k span indeed close to $\binom{k}{2}p$ edges in $G_{n,p}$, if k is not too small. This fact motivates the following definition of a pseudo-random graphs due to Thomason [39], [40]:

Definition 3 *A graph $G = (V, E)$ is* (p, β)-jumbled *if for every subset $X \subseteq V(G)$,*

$$\left| e_G(X) - \frac{p|X|^2}{2} \right| \leq \beta|X| \, .$$

Thus, if G is a (p, β)-jumbled graph, its edge density is around p, and its edge distribution is similar to that of the random graph $G_{n,p}$, where the degree of similarity (or rather of proximity to the expected number of edges) is controlled by the parameter β. Random graphs $G_{n,p}$ are easily shown to be $(p, O(\sqrt{np}))$-jumbled for all not too small values of the edge probability p. Moreover, one can show (see [13]) that if a graph G on n vertices is (p,β)-jumbled, then $\beta = \Omega(\sqrt{np})$; for this reason (p,β)-jumbled graphs G with $\beta = \Theta(\sqrt{np})$ are considered very good pseudo-random graphs.

Pseudo-random graphs is a central concept in modern Combinatorics, whose importance is derived in part from that of random graphs. Quite a few known constructions of pseudo-random graphs are deterministic, allowing thus to substitute somewhat elusive truly random graphs, defined through probabilistic, existential means, with quite accessible deterministic descriptions – a feature crucial in a variety of applications. Moreover, in certain applications one can utilize features of (carefully crafted) pseudo-random graphs, non-existent typically in random graphs of the same edge density.

As we have indicated, several alternative definitions of pseudo-random graphs are available; here we describe just one of them, based on graph spectrum. Given a graph $G = (V, E)$ with vertex set $V = \{v_1, \dots, v_n\}$, the *adjacency matrix* of G is an n-by-n matrix A of zeroes and ones, defined by: $a_{ij} = 1$ if and only if $(v_i, v_j) \in E(G)$, and $a_{ij} = 0$ otherwise. Observe that A is a symmetric real matrix, and therefore A has a full set of n real eigenvalues, denoted by $\lambda_1, \dots, \lambda_n$, customarily sorted in the non-increasing order $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and usually called the *eigenvalues* of the graph G itself. If G is a d-regular graph, then the first eigenvalue λ_1 is easily seen to be $\lambda_1 = d$ (with the corresponding eigenvector being the all-one vector), while all others satisfy $|\lambda_i| \leq d$, $i = 2, \dots, n$. Now, equipped with this terminology, we can give an alternative definition of a pseudo-random graph introduced by Alon. A graph $G = (V, E)$ is called an (n, d, λ)*-graph* if G has n vertices, is d-regular, and in addition all of its eigenvalues but the first one satisfy: $|\lambda_i| \leq \lambda$, $i = 2, \dots, n$. A very frequently used result from Spectral Graph Theory (see, e.g., Chapter 9 in [6]) postulates that if G is an (n, d, λ)-graph, then

$$\left| e_G(X) - \frac{d|X|^2}{2n} \right| \leq \lambda |X| \,,$$

for all subsets $X \subseteq V(G)$, implying that an (n, d, λ)-graph is $(d/n, \lambda)$-jumbled. Several constructions of (n, d, λ)-graphs with $\lambda = O(\sqrt{d})$ are available, they are based on a variety of algebraic and geometric properties. We would like to mention in passing that graph eigenvalues are frequently used to ensure graph expansion too.

The reader is advised to consult a survey [20] on pseudo-random graphs by the authors for an extensive coverage of pseudo-random graphs, their definitions and properties.

3 Extremal problems for minors

The subject of this paper can be classified as "Extremal problems for minors". Given the prominence of these two branches of Graph Theory (theory of minors and extremal graph theory), it is quite natural to expect the appearance of results combining these two subjects. And indeed, our paper is certainly not the first to address extremal problems for minors; in fact, this is already a well established part of Graph Theory, with a variety of results achieved. A recent survey of Thomason [42] on the subject describes several of its achievements.

Generally speaking, the motto of the extremal minor theory can be stated as finding sufficient conditions for the existence of a minor from given family, or a concrete minor (say, a clique minor of certain order) in a given graph. Here is an illustrative example of a result of this sort: every graph

G on n vertices with more than $3n-6$ edges contains a complete graph K_5 or a complete bipartite graph $K_{3,3}$ as a minor. This is of course nothing else but rephrasing of the Kuratowski-Wagner theorem combined with the classical fact that a planar graph on n vertices has at most $3n-6$ edges (which in turn follows easily from the celebrated Euler formula connecting the numbers of vertices, edges and faces in any planar embedding). As yet another illustration we can mention the famous Hadwiger Conjecture, suggesting that a graph that cannot be properly colored with k colors has a clique K_{k+1} as a minor; this notorious conjecture has been proven so far for very few initial values of k, see [44] for a survey of its status.

Here we will be mostly looking for results of the following sort: if a graph G is sufficiently dense, or has sufficiently large average degree (plus possibly additional conditions imposed), then G contains a large clique minor. Perhaps the best known result of this sort was proved independently by Kostochka [18] and by Thomason [38], who showed that there exists an absolute constant $c > 0$ such that every graph G with average degree $d = 2|E(G)|/|V(G)|$ contains a clique on $cd/\sqrt{\log d}$ vertices as a minor. Later the asymptotic value of c has been determined by Thomason [41].

Under certain additional conditions one can guarantee a clique minor of order (much) larger than $d/\sqrt{\log d}$ in a graph of average degree d. Several of our results are indeed of this type. When looking for large minors one should remember however that there is a limit of the size of a minor one can find in a graph. This limit is given by the following very simple yet very useful observation.

Proposition 1 *Let H be a minor of G. Then the number of edges of H does not exceed the number of edges of G.*

The above proposition immediately implies that a graph G on n vertices with average degree d cannot contain a graph Γ with average degree $k > \sqrt{nd}$ as minor. Indeed, the number of edges of G is $nd/2$, and thus if Γ is a minor of G then $\frac{k^2}{2} \leq \frac{nd}{2}$. We will repeatedly use this simple bound as a benchmark to measure the quality of our results.

In the rest of this section we survey a variety of known results in Extremal Minor Theory, having in mind our theorems and their comparison to the previously obtained results.

There are several results connecting between (the absence of) separators and minors in graphs. A *separator* S of a graph G is a set of vertices whose removal separates the graph into connected components, each of size at most $\frac{2}{3}|V(G)|$. Alon, Seymour and Thomas [5] proved that a graph of order n without a K_h minor has a separator of size $O(h^{3/2}n^{1/2})$. This was extended to large h by Plotkin, Rao and Smith [34] who proved that a graph without a K_h minor has a separator of size $O(h\sqrt{n\log n})$. The last result implies in particular that an expander graph of constant degree has a clique minor of size $\Omega(\sqrt{n/\log n})$. On the other hand, since every graph has trivially a separator of size $n/3$, one can only show the existence of a clique minor of order at most $O(\sqrt{n/\log n})$ using these results.

Kleinberg and Rubinfeld addressed in [17] a connection between expansion and the existence of large minors. They used the following, rather weak, definition of expansion: a graph G is an α-expander if every set X of at most half of the vertices of G has at least $\alpha|X|$ outside neighbors in G. It is proven in [17] that for every fixed $\alpha > 0$ there is a constant $c > 0$ such that an α-expander graph of order n contains every graph H with at most $n/\log^c n$ vertices and edges as a minor. While this result is quite useful in finding large minors in sparse graphs (in particular those of constant maximum degree), it appears to be of rather limited value for the denser case and can not be used to show the existence of a clique minor of order larger than $\Omega(\sqrt{n/\log n})$.

Sunil Chandran and Subramanian [9] discussed a connection between spectral properties of a graph and its minors. They proved in particular that if G is a d-regular graph on n vertices whose second eigenvalue is at most λ, then G contains a clique minor on $\Omega\left(\left(\frac{n(d-\lambda)^2}{(3d-2\lambda)^2}\right)^{1/3}\right)$ vertices. Observe that this result can be used only to show the existence of clique minors of order up to $cn^{1/3}$, which is a relatively weak bound.

Another avenue of research in extremal problems in minors (also pursued in this paper) aims to prove the existence of large minors in graphs with excluded subgraphs. Kühn and Osthus proved in [23] that for all integers $2 \leq s \leq s'$ there exist constants $r_0 = r_0(s, s')$ and $c = c(s, s')$ such that every $K_{s,s'}$-free graph G of average degree $r \geq r_0$ contains a minor of average degree d satisfying

$$d \geq c\frac{r^{1+\frac{1}{2(s-1)}}}{(\log r)^{2+\frac{1}{s+1}}} .$$

They conjectured however that the logarithmic term is not needed in this bound and were able to verify this conjecture for the case when the graph G is assumed to be regular. Observe that after having obtained a minor of average degree d one can use the above mentioned results of Kostochka and Thomason [18], [38], [41] to derive the existence of a clique minor on $cd/\sqrt{\log d}$ vertices.

Another nice result of Kühn and Osthus guarantees the existence of large minors in graphs with large girth (i.e. without short cycles). They proved in [22] that for every odd integer $g \geq 5$ there exists a constant $c = c(g) > 0$ such that every graph G of average degree r and without cycles shorter than g (such a graph is said to have *girth* at least g) contains a minor with average degree at least $cr^{(g+1)/4}$. This result improves significantly a much earlier result of Thomassen [43] and a recently obtained result by Diestel and Rempel [11]. Observe that the assumption for the case $g = 5$ essentially amounts to forbidding a 4-cycle, or $K_{2,2}$; thus this result of Kühn and Osthus establishes their above mentioned conjecture for the case $s = s' = 2$.

Bollobás, Catlin and Erdős [8] analyzed the appearance of large minors in random graphs. They proved that for a constant edge probability p, $0 < p < 1$, the largest clique minor in a random graph $G_{n,p}$ is of order $n/\sqrt{\log n}$ (in fact, their result is more accurate – they were able to establish not only the asymptotic order of magnitude of the largest clique minor in $G_{n,p}$, but actually its asymptotic value). As a result, and taking into account a well known fact that the chromatic number of $G_{n,p}$ in this range is with high probability $O(n/\log n)$, Bollobás et al. were able to derive that almost every graph satisfies the Hadwiger conjecture. (It is worth mentioning here that the above stated results of Kühn and Osthus and some of the results of the current paper imply readily the validity of the Hadwiger conjecture for $K_{s,s'}$-free graphs and graphs with high girth, or with forbidden long enough cycle.) The argument of [8] can be used to show that with high probability the largest clique minor in $G_{n,p}$ has order of magnitude $\Theta\left(n\sqrt{p}/\sqrt{\log n}\right)$, for subconstant values of the edge probability $p(n)$ as well.

Much less is known in the case of pseudo-random (or jumbled) graphs. Thomason proved in [39] (see also [40]) that (p, β)-jumbled graphs with p constant and $\beta = O(n^{1-\epsilon})$ contain a clique minor of size at least $(1 + o(1))n/\sqrt{\log_b n}$, where $b = 1/(1 - p)$. For small p, this has the same order of magnitude $\frac{n\sqrt{p}}{\sqrt{\log n}}$ as the result for $G_{n,p}$.

Finally, we mention a recent result of Drier and Linial [12] who discussed minors in lifts of graphs. An ℓ-lift of a labeled graph $G = (V, E)$ is a graph with vertex set $V \times [\ell]$, whose edge set is the union of perfect matchings between $\{u\} \times [\ell]$ and $\{v\} \times [\ell]$ for each edge $(u, v) \in E$. In a random lift these matchings are selected uniformly at random. Drier and Linial proved that for $\ell \leq O(\log n)$ almost every lift of the complete graph K_n contains a clique minor of size $\Theta(n)$, and for $\ell > \log n$ it contains a clique minor of size at least $\Omega\left(\frac{n\sqrt{\ell}}{\sqrt{\log(n\ell)}}\right)$. The last result was shown to be tight in [12] as long as $\log n < \ell < n^{1/3-\epsilon}$.

4 Our results

In this section we present in full details the results of this paper. We also compare them with previously obtained results, surveyed in brief in Section 3, and discuss their tightness.

The first of our results is about minors in expanding graphs. We prove:

Theorem 4.1 *Let G be a (t, α)-expanding graph of order n and let $t \geq 10$. Then G contains a minor with average degree at least*

$$c\alpha^3 \frac{\sqrt{nt \log t}}{\sqrt{\log n}},$$

where $c > 0$ is some absolute constant independent of α.

This theorem together with the results of Kostochka [18] and Thomason [38] mentioned in Section 3 gives the following corollary.

Corollary 4.2 *Let G be a (t, α)-expanding graph of order n, and let $t \geq 10$. Then G contains a clique minor of size*

$$c\alpha^3 \frac{\sqrt{nt \log t}}{\log n},$$

where c is some absolute constant independent of α.

For $t \geq n^{\epsilon}$ this gives a clique minor of size $\Omega\left(\frac{\sqrt{nt}}{\sqrt{\log n}}\right)$. The random graph $G_{n,p}$ with $p = 10t/n$ can be easily shown with high probability to be $(t, 0.5)$-expanding in this range of t, and as we mentioned before its largest clique minor is typically of order $O\left(\frac{\sqrt{nt}}{\sqrt{\log n}}\right)$. This shows that our result is tight up to a constant factor. For small values of $t \leq \log n$ the result of this corollary can be slightly improved as follows:

Proposition 4.3 *If G is a (t, α)-expanding graph of order n and $t \geq 10$, then G contains a clique minor of size*

$$\Omega\left(\alpha^2 \sqrt{\frac{n \log t}{\log n}}\right).$$

Observe that the above results constitute a substantial extension of the results of Alon, Seymour and Thomas [5], Plotkin, Rao and Smith [34], and of Kleinberg and Rubinfeld [17], that cover basically the case of expansion by a constant factor $t = \Theta(1)$. Our results, though applicable also for the case $t = \Theta(1)$, enable to show the existence of larger minors whenever the expansion factor t becomes super-constant.

The next our result is about minors in pseudo-random (or jumbled) graphs. We prove:

Theorem 4.4 *Let G be a (p, β)-jumbled graph of order n such that $\beta = o(np)$. Then G contains a minor with average degree $\Omega(n\sqrt{p})$.*

This statement is an extension of the results of Thomason [39, 40], who studied the case of constant p. As a (p, β)-jumbled graph G on n vertices with $\beta = o(np)$ has average degree close to np and thus $\Theta(n^2p)$ edges, Proposition 1 shows that Theorem 4.4 is asymptotically tight, up to a constant factor. The above theorem also implies that an (n, d, λ)-graph G with $\lambda = o(d)$ has a minor of average degree $\Omega(\sqrt{nd})$. This can be used in particular to derive some of the results of Drier and Linial [12] on minors in random lifts. For example, when $\ell \ll n$ we have that with high probability every pair of vertices in a random ℓ-lift of the complete graph K_n has at most $(1 + o(1))n/\ell$ common neighbors. Using this one can easily show that this graph is an $(n\ell, n-1, \lambda)$-graph with $\lambda = o(n)$. Therefore a random ℓ-lift of K_n contains a minor with average degree $\Omega(n\sqrt{\ell})$ and thus a clique minor of order $\Omega\Big(\frac{n\sqrt{\ell}}{\sqrt{\log(n\ell)}}\Big)$, by applying again Kostochka-Thomason. For larger values of ℓ one can obtain similar lower bound on the size of the clique minor in a random ℓ-lift of the complete graph K_n by first proving that all subsets of order at most $O(\ell)$ in such graph expand by a factor of $\Omega(n)$ and then using Theorem 4.1.

The next group of results guarantees the existence of large minors in graphs with excluded subgraphs. First, we prove:

Theorem 4.5 *Let $2 \le s \le s'$ be integers. Let G be a $K_{s,s'}$-free graph with average degree r. Then G contains a minor with average degree $\Omega\left(r^{1+\frac{1}{2(s-1)}}\right)$.*

This confirms a conjecture of Kühn and Osthus from [23]. The result is asymptotically tight modulo a well known and widely accepted conjecture about the Turán numbers of complete bipartite graphs $K_{s,s'}$, saying that for constant $2 \le s \le s'$, there exists a $K_{s,s'}$-free graph G on n vertices with at least $\Omega\big(n^{2-1/s}\big)$ edges. Denoting the average degree of such a graph by r, we have then $r = \Omega(n^{1-1/s})$, and therefore by Proposition 1 a minor H of G has $O\big(r^{2+1/(s-1)}\big)$ edges, and hence the average degree of H is at most $O\Big(r^{1+\frac{1}{2(s-1)}}\Big)$. The latter conjecture has been settled for $s = 2, 3$ and all $s' \ge s$ (see, e.g., Chapter VI of [7]), furthermore, Alon, Rónyai and Szabó proved it [4] for $s' > (s-1)!$. The asymptotic tightness of Theorem 4.5 thus follows in all these cases.

Theorem 4.5 can be generalized somewhat to the case where an excluded graph H is a bipartite graph with bounded degrees at one side. The corresponding result is:

Theorem 4.6 *Let H be a bipartite graph of order h with parts A and B such that the degrees of all vertices in B do not exceed s. If G is an H-free graph with average degree r, then G contains a minor with average degree $\Omega\left(r^{1+\frac{1}{2(s-1)}}\right)$.*

Finally, we prove a minor-related result for C_{2k}-free graphs.

Theorem 4.7 *Let $k \geq 2$ and let G be a C_{2k}-free graph with average degree r. Then G contains a minor with average degree $\Omega\left(r^{\frac{k+1}{2}}\right)$.*

This generalizes a result of Kühn and Osthus [22], who proved such a theorem under the (much more restrictive) assumption that G has girth at least $2k+1$. Here too the asymptotic optimality of Theorem 4.7 relies on a well known conjecture from Extremal Graph Theory (see, e.g., [7], p. 164), postulating that for any fixed $k \geq 2$, there exists a graph G on n vertices without cycles of length up to $2k$ and with $\Omega(n^{1+1/k})$ edges. This conjecture has been proven so far for $k = 2, 3, 5$.

Of course the Kostochka-Thomason result can be utilized to convert minors with large average degree into clique minors, just as we have done several times already. Arguments similar to those used by Kühn and Osthus in [23] (see Proposition 15 there) can be used to show that the obtained lower bond on the size of a largest clique minor are asymptotically tight, up to a constant factor, assuming the validity of the above mentioned conjectures about the Turán numbers of $K_{s,s'}$ and of C_{2k}.

The alert reader has probably noticed that in all three results above the excluded fixed graph is bipartite. This is for a good reason – the complete bipartite graph $K_{r,r}$ is H-free for any non-bipartite graph H, and yet every minor H of $K_{r,r}$ has obviously average degree $O(r)$. This indicates that if one's aim is to force an untypically large minor by excluding a fixed graph H, H should better be bipartite.

The rest of the paper is organized as follows. In Section 5 we discuss minors in expanding graphs and prove Theorem 4.1 and Proposition 4.3. Section 6 is devoted to minors in pseudo-random graphs, there we prove Theorem 4.4. In Section 7 we derive Theorems 4.5 and 4.6 about minors in $K_{s,s'}$-free graphs and in H-free graphs. In Section 8 we prove Theorem 4.7 about large minors in C_{2k}-free graphs. Section 9 is devoted to concluding remarks.

5 Minors in expanding graphs

In this section we prove Theorem 4.1 and Proposition 4.3.

Observe first that if G is a (t, α)-expanding graph of order n, then every subset X of G of size $\alpha n/t \leq |X| \leq \alpha n/2$ has $|N(X)| \geq \alpha n/2$. Indeed, such X contains a subset Y of size exactly $\alpha n/t$, hence $|N(X)| \geq |N(Y)| - |X| \geq t|Y| - |X| \geq \alpha n/2$.

Lemma 5.1 *Let G be a connected (s, β)-expanding graph of order n. Then the diameter of G is at most $3\beta^{-1} \log n / \log s$.*

Proof. From the expansion of G we have that for every vertex v and integer q there are at least $\min\{s^q, \beta n\}$ vertices which are within distance at most q from v. Taking $q = \log n / \log s$ we obtain that there are at least βn vertices within distance at most $\log n / \log s$ from every vertex in G.

Now, suppose G contains a pair of vertices u, w such that the distance between them is at least $3\beta^{-1} \log n / \log s$. Then on a shortest path from u to w we can find vertices $v_1 = u, \ldots, v_k = w$ such that $k > 1/\beta$ and the distance between every pair v_i, v_j is at least $2\log n/\log s$. Denote by U_i the set of vertices which are at distance at most $\log n/\log s$ from v_i. These sets are disjoint, each has size at least βn and therefore the size of their union is larger than n. This contradiction completes the proof. □

Proof of Theorem 4.1. Let

$$p = \frac{\alpha^2}{100}\frac{\sqrt{nt\log t}}{\sqrt{\log n}} \quad \text{and} \quad q = 6\alpha^{-1}\frac{\sqrt{n\log n}}{\sqrt{t\log t}},$$

and consider the following iterative procedure which we will repeat p times. In the beginning of iteration $k+1$ we will have k disjoint sets $B_1, \dots, B_k$ each of size $|B_i| = q$, such that all induced subgraphs $G[B_i]$ are connected. We will construct a new subset B_{k+1}, also of size q, such that induced subgraph $G[B_{k+1}]$ is connected and there are at least $\alpha k/8$ indices $1 \le i \le k$ such that there is an edge from B_i to B_{k+1}. In the end of this algorithm if we contract all subsets B_i we will get a graph with average degree

$$\Omega(\alpha p) = \Omega\left(\alpha^3\frac{\sqrt{nt\log t}}{\sqrt{\log n}}\right).$$

Let $B = \cup_{i=1}^k B_i$ and note that $|B| = b \le pq = 0.06\alpha n$. Denote by $C = V(G) - B$ and by G' the subgraph of G induced by C. Let X be a subset of C such that $2b/t \le |X| \le \alpha n/t$ and $|N_{G'}(X)| < t|X|/2$. Then we have

$$|N_G(X)| \le |N_{G'}(X)| + |B| \le t|X|/2 + b \le t|X|,$$

which contradicts the assumption that G is (t,α)-expanding. Therefore there exists $X \subset C$ of size at most $2b/t$ such that the remaining set $D = C - X$ spans a subgraph of G in which every subset of size at most $\alpha n/t$ expands by a factor of at least $t/2$. Denote by G'' the subgraph of G induced by D. This graph might be disconnected, but as we will see next it must have few very large components that cover almost all its vertices.

Let Y be a subset of G'' such that $3b/t \le |Y| < \alpha n/2$. Then, by the remark in the beginning of this section, we have that $|N_G(Y)| \ge \min\{3b, \alpha n/2\} > |B| + |X|$. Hence Y has neighbors inside $D - Y$ and cannot be an isolated component of G''. Thus G'' contains a subset Y of size at most $3b/t$ such that all the vertices of $G'' - Y$ are contained in connected components of size at least $\alpha n/2$. Denote these connected components by $G_1, \dots, G_\ell$. Then clearly $\ell \le 2/\alpha$, and we also have that every subset of G_i of size at most $\alpha n/t$ expands by factor at least $t/2$. By Lemma 5.1 (with $\beta = \alpha/2$ and $s = t/2$) this implies that the diameter of each G_i is at most $7\alpha^{-1}\log n/\log t$.

Next we claim that there is an index i such that there are at least $r = \frac{k}{2\ell}$ sets B_j, each having at least $\frac{t|B_j|}{2\ell}$ neighbors in G_i. If this is not the case then we have $k - \ell\,\frac{k}{2\ell} = k/2$ sets B_j, each having at most $\ell\,\frac{t|B_j|}{2\ell} = tq/2$ neighbors inside $\cup_i G_i$. First suppose that $kq/2 \le \alpha n/t$. Then taking a union of $k/2$ such sets B_j we obtain a set of size $b/2$ with at most $(k/2)\cdot(tq/2) = tb/4$ neighbors in $\cup_i G_i$. On the other hand, by expansion the number of neighbors of this set in G is at least $tb/2$. Therefore the remaining $tb/4$ neighbors should be inside $X \cup Y \cup B$. Since $t \ge 10$, this set has size at most $b + 5b/t < tb/4$, a contradiction. If $kq/2 \ge \alpha n/t$ then we can take a union of $\alpha n/(tq)$ such sets B_j and obtain a set of size $\alpha n/t$ with at most $(tq/2)(\alpha n/(tq)) = \alpha n/2$ neighbors in $\cup_i G_i$. Again, by expansion, this set has at least αn neighbors in G, so at least $\alpha n/2$ of them are in $X \cup Y \cup B$. But the size of this set is not big enough, a contradiction (here we use the assumption $t \ge 10$).

Therefore, without loss of generality, we can assume that each of the first $r = \frac{k}{2\ell}$ sets $B_1, \dots, B_r$ has at least $t|B_j|/(2\ell) = tq/(2\ell)$ neighbors in G_1. Denote these sets of neighbors by $U_1, \dots, U_r$ respectively. Pick uniformly at random with repetition $\frac{|G_1|}{tq/(2\ell)}$ vertices of G_1 and denote this set by W. For every index $1 \le i \le r$, the probability that W does not intersect U_i is at most $\left(1 - \frac{|U_i|}{|G_1|}\right)^{|W|} \le 1/e$. Therefore the expected number of sets U_i which have non-empty intersection with W is at least $(1 - 1/e)r > r/2$. Hence there is a choice of W that intersects at least $r/2 \ge k/(4\ell) \ge \alpha k/8$ sets U_i (the last inequality is due to $\ell \le 2/\alpha$). Fix an arbitrary vertex $w_0 \in W$ and consider a collection of shortest paths in G_1 from w_0 to the remaining vertices in W. Since the diameter of G_1 is at most $7\alpha^{-1} \log n / \log t$ and

$$7\alpha^{-1}|W| \log n/ \log t \le 7\alpha^{-1} \frac{n}{tq/(2\ell)} \frac{\log n}{\log t} \le \frac{14}{3}\alpha^{-1} \frac{\sqrt{n \log n}}{\sqrt{t \log t}} < q,$$

by taking a union of these paths and adding extra vertices if necessary we can construct a connected subset of size q containing W. Denote this set by B_{k+1} and note that it is connected by an edge to at least $\alpha k/8$ sets U_i, $1 \le i \le k$. This completes the proof of the theorem. $\square$

Proof of Proposition 4.3. First we claim that if A is an arbitrary subset of G of size at most $\alpha n/8$, then $G - A$ contains a connected component of size at least $\alpha n/4$. Indeed, if all components of $G - A$ have size at most $\alpha n/4$, then by taking several of them together we can find a subset A' such that $\alpha n/4 \le |A'| \le \alpha n/2$ and A' has no neighbors in $G - A$, i.e., $N(A') \subseteq A$. On the other hand, by the remark in the beginning of the section, we have that $|N(A')| \ge \alpha n/2$. This contradiction proves our claim. Let

$$p = \frac{\alpha}{100}\sqrt{\frac{n \log t}{\log n}} \quad \text{and} \quad q = \sqrt{\frac{n \log n}{\log t}},$$

and note that $pq = \alpha n/100$. Hence , using the above claim, we can greedily find p disjoint sets $B_1, \dots, B_p$, each of size $|B_i| = q$, such that all induced subgraphs $G[B_i]$ are connected.

Let $B' = \cup_i B_i$, let B'' be an arbitrary subset of G of size at most $|B|'/10$ and let $B = B' \cup B''$. Then using the same argument as in the proof of Theorem 4.1 one can show that there exist a subset X of $G - B$ of size at most $5|B|/t \le |B|/2$ (recall that $t \ge 10$) such that the following holds.

- The graph $G' = G - X - B$ is a $(t/2, \alpha)$-expanding graph with at most $\ell = 2/\alpha$ connected components $G_1, \dots, G_\ell$, each of which has diameter at most $7\alpha^{-1} \log n/ \log t$.
- There exists an index $1 \le i \le \ell$ such that at least $p/(2\ell) \ge \alpha p/4$ sets B_j have neighbors in G_i.

In particular this implies that there is a collection of $\alpha p/4$ sets B_j, such that any pair of them can be connected by a path P of length at most $7\alpha^{-1} \log n/ \log t$. Moreover all vertices of P except endpoints are contained in $G - B' \cup B''$.

Now consider the following iterative procedure. In the beginning of each iteration we will have sets $B' = \cup_i B_i$ and $B'', |B''| \le |B|'/10$, where B'' is the set of vertices of disjoint paths that have been used at previous iterations to connect sets B_j. We stop when we will have at least $\alpha p/4$ sets B_j which are pairwise connected. Then the contraction of all these sets will give us a clique minor of size at least $\Omega\left(\alpha^2 \sqrt{\frac{n \log t}{\log n}}\right)$. By the above discussion, at each iteration we indeed can construct a path of length at most $7\alpha^{-1}\log n/\log t$ that does not use vertices from $B' \cup B''$ and connects two previously not connected sets B_j. Since the number of iterations is clearly at most $\binom{p}{2}$ we have that the size of the set B'' remains bounded by $\binom{p}{2}7\alpha^{-1}\log n/\log t \le |B'|/10$ (this is because $|B'| = pq = \frac{\alpha n}{100}$) during all iterations. □

6 Minors in pseudo-random graphs

Here we prove Theorem 4.4. Throughout this section we assume that np is at least a sufficiently large constant and p is smaller than a sufficiently small constant.

Lemma 6.1 *Let $G = (V, E)$ be a (p, β)-jumbled graph of order n such that $\beta = o(np)$. Then G contains an induced subgraph G' of order $n' = (1 - o(1))n$ such that the degree of every vertex in G' is $(1 + o(1))n'p$ and every subset X of G' satisfies*

$$e(X, V(G') - X) \ge (1 - o(1))p|X|(n' - |X|).$$

Proof Set $\epsilon = (4\beta/(np))^{1/3}$ and consider two disjoint subsets S and T both of size at least ϵn. Then $e(S, T) = e(S \cup T) - e(S) - e(T)$ and therefore

$$\begin{aligned} e(S,T) &\ge p\binom{|S|+|T|}{2} - \beta(|S|+|T|) - p\binom{|S|}{2} - \beta|S| - p\binom{|T|}{2} - \beta|T| \\ &= p|S||T| - 2\beta(|S|+|T|) \ge p|S||T| - 2\beta n \\ &= p|S||T| - \epsilon^3 n^2 p/2 \ge (1-\epsilon/2)p|S||T|. \end{aligned} \tag{1}$$

Similarly one can show that $e(S, T) \le (1 + \epsilon/2)p|S||T|$ for every two subsets S, T as above.

Let U be the set of vertices of G with degree at least $(1 + \epsilon)np$. If U has size at least ϵn then we have that

$$\begin{aligned} e(U, V-U) &= \sum_{v\in U} d(v) - 2e(U) \ge (1+\epsilon)np|U| - 2p\binom{|U|}{2} - 2\beta|U| \\ &\ge (1+\epsilon)np|U| - p|U|^2 - \epsilon^3 np|U| \\ &> (1+\epsilon/2)p|U|(n-|U|). \end{aligned}$$

This contradiction implies that there are less than ϵn vertices in G with degree at least $(1 + \epsilon)np$. Let $V_0 = V - U$, $n_0 = |V_0| > (1 - \epsilon)n$, and let G_0 be the subgraph induced by V_0.

Consider the following process. If at step i the graph G_{i-1} contains a subset X_i such that $|X_i| = x_i \le \epsilon n$ and $e(X_i, V(G_{i-1}) - X_i) < (1 - 4\epsilon)px_i(n_{i-1} - x_i)$ delete X_i from the graph, update $G_i = G_{i-1} - X_i$, $n_i = |V(G_i)|$, and continue. Consider the first time when we deleted at least ϵn vertices and let $Y = \cup_i X_i$. Then $\epsilon n \le |Y| \le 2\epsilon n < 3\epsilon n_0$ and

$$\begin{aligned} e(Y, V(G_0) - Y) &\le \sum_i e(X_i, V(G_{i-1}) - X_i) < (1-4\epsilon)p\sum_i x_i(n_{i-1} - x_i) \\ &\le (1-4\epsilon)pn_0\sum_i x_i = (1-4\epsilon)pn_0|Y| \\ &\le \frac{1-4\epsilon}{1-3\epsilon}p|Y|(n_0 - |Y|) \le (1-\epsilon/2)p|Y|(n_0 - |Y|). \end{aligned}$$

This contradicts (1). Therefore there is a subset Y of G_0 of size at most $2\epsilon n$ such that every subset X of the graph $G' = G_0[V_0 - Y]$ of size at most ϵn satisfies $e(X, V(G') - X) \geq (1-4\epsilon)p|X|(n' - |X|)$, where $n' = |V(G')|$. In particular, taking X to be a single vertex we have that the minimum degree in G' is at least $(1-4\epsilon)p(n'-1)$. By (1) we also have that every subset X with $\epsilon n \leq |X| \leq n'/2$ satisfies that $e(X, V(G') - X) \geq (1-4\epsilon)p|X|(n' - |X|)$. This inequality is satisfied by sets of size larger than $n'/2$ by symmetry. Since $n' \geq (1-3\epsilon)n$, by the above discussion, the maximum degree of G' is at most $(1+\epsilon)np \leq (1+5\epsilon)n'p$. Finally, note that ϵ tends to zero as np tends to infinity. Therefore G' satisfies the assertion of the lemma. $\square$

A *lazy random walk* on a graph G with vertex set $V(G) = \{1, \ldots, n\}$ is a Markov chain whose matrix of transition probabilities $P = (p_{i,j})$ is defined by

$$p_{i,j} = \begin{cases} \frac{1}{2d(i)} & \text{if } (i,j) \in E(G) \\ 1/2 & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases}$$

i.e., if at some step we are at vertex i then with probability $1/2$ we stay at i and with probability $\frac{1}{2d(i)}$ we move to a random neighbor of i. This Markov chain has the stationary distribution π defined by $\pi(i) = \frac{d(i)}{2e(G)}$. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of P. Then the largest eigenvalue $\lambda_1 = 1$ and since P is positive semidefinite, all other eigenvalues $\lambda_i, i \geq 2$, are non-negative. For more information about random walks on graphs we refer the interested reader to an excellent survey of Lovász [25].

Lemma 6.2 *Let $G = (V, E)$ be a graph of order n such that every vertex in G has degree $(1-o(1))np$ and every subset X satisfies $e(X, V - X) \geq (1-o(1))p|X|(n - |X|)$. Then for every subset U of size u the probability that a lazy random walk on G which starts from the stationary distribution π and makes ℓ steps does not visit U is at most $e^{-0.03u\ell/n}$.*

Proof. By the degree assumption we have that $2|E| = \sum_i d(i) = (1+o(1))n^2p$ and therefore the stationary distribution π satisfies $\pi(i) = d(i)/(2|E|) = (1+o(1))/n$. Thus for every subset S the measure of S with respect to π equals $\pi(S) = \sum_{i \in S} \pi(i) = (1+o(1))|S|/n$. Let

$$\Phi = \min_{\pi(S) \leq 1/2} \frac{\sum_{i \in S, j \in V-S} \pi(i)p_{i,j}}{\pi(S)\pi(V-S)},$$

be the *conductance* of G. By the properties of G we have that

$$\begin{aligned} \Phi &= \min_{|S| \leq n/2 + o(n)} (1+o(1)) \frac{1}{n} \frac{1}{2np} \frac{e(S, V-S)}{(|S|/n)(1-|S|/n)} \\ &\geq \min_{|S| \leq n/2 + o(n)} (1+o(1)) \frac{1}{2n^2p} \frac{p|S|(n-|S|)}{(|S|/n)(1-|S|/n)} \geq 1/2 + o(1). \end{aligned}$$

Let λ_2 be the second largest eigenvalue of the transition probabilities matrix P. Since all eigenvalues of P are non-negative we have that the *spectral gap* of this Markov chain is $\delta = \max_{i \geq 2} 1 - |\lambda_i| = 1 - \lambda_2$. Then by the result of Jerrum and Sinclair [16] (see also [25]), which provides a connection between the spectral gap and the conductance of the graph, we have that $\delta = 1 - \lambda_2 \geq \Phi^2/8 > 0.03$. To finish the proof we can now use well known estimates on the probability that a Markov chain stays inside certain sets (see, e.g., [1], [2], [6], [30]). In particular, the assertion of Theorem 5.4 in [30] implies that the probability that a lazy random walk on G which starts from the stationary distribution π and makes ℓ steps does not visit a subset $U, |U| = u$, is bounded from above by

$$\leq \big(1-\pi(U)\big)\big(1-\delta\pi(U)\big)^{\ell} \leq \left(1-(1+o(1))\frac{\delta|U|}{n}\right)^{\ell} \leq e^{-0.03u\ell/n}. \qquad \Box$$

Lemma 6.3 *Let $c > 0$ be an arbitrary constant. Let $G = (V, E)$ be a (p, β)-jumbled graph of order n such that $\beta = o(np)$. Then G contains a connected subset B of size $cp^{-1/2}$ having at least $3cn\sqrt{p}/5$ neighbors in G.*

Proof By Lemma 6.1, we can assume that the minimum degree of G is at least $(1 + o(1))np$. We construct B using the following greedy procedure. Suppose we have already constructed a connected set B of size $k < cp^{-1/2}$ which has at least $3knp/5$ neighbors in G. Let X be a subset of $3knp/5$ of these neighbors. Then the number of edges inside $X \cup B$ is at most $p|X\cup B|^2/2+\beta|X\cup B| < (np)|X|/6$. Therefore X contains a vertex v with at most $np/3$ neighbors inside $X \cup B$. By the minimum degree assumption v has more than $3np/5$ neighbors outside $X \cup B$. Since by the definition of X, v has also a neighbor in B, the set $B \cup \{v\}$ is connected. This set has size $k + 1$ and at least $|X| + 3np/5 = 3(k + 1)np/5$ neighbors in G. Repeating this process $cp^{-1/2}$ times we obtain a connected set B that satisfies the assertion of the lemma. $\Box$

Proof of Theorem 4.4. Note that by definition any induced subgraph of G on at least $n/2$ vertices is still (p, β)-jumbled. Therefore by starting from G and repeatedly applying Lemma 6.3 with $c = 50$ to the remaining subgraph $G-\cup_{i<j}B_i$ we can construct $s = 10^{-3}n\sqrt{p}$ disjoint connected sets $B_1, \ldots, B_s$ such that each B_i has size $50p^{-1/2}$ and has at least $25n\sqrt{p}$ neighbors in G. Let $D_1, \ldots, D_s$ be sets of size $25n\sqrt{p}$ such that every vertex in D_i has a neighbor in B_i. Consider the following iterative procedure that we repeat s times. In the beginning of iteration $k + 1$ we have connected sets $C_1, \ldots, C_k$ each of size $50p^{-1/2}$, such that all C_i and B_j are disjoint. We construct a new connected set C_{k+1} of size $50p^{-1/2}$ such that C_{k+1} is disjoint from all previous sets and there are at least $s/6$ indices $1 \leq j \leq s$ such that there is an edge from C_{k+1} to B_j. In the end of the procedure if we contract all the sets C_i, B_j we will get a graph with average degree $\Omega(s) = \Omega(n\sqrt{p})$.

Let $U = (\cup_{i=1}^{s} B_i) \cup (\cup_{j\leq k} C_j)$ and note that $|U| \leq n/10$. Then the induced subgraph $G[V \setminus U]$ is (p, β)-jumbled and therefore by Lemma 6.1 there is an induced subgraph G' of $G - U$ on $n' \geq (1-o(1))(n-|U|) \geq 8n/9$ vertices such that the degree of every vertex in G' is $(1+o(1))n'p$ and every subset X of G' satisfies $e(X, V(G') - X) \geq (1 - o(1))p|X|(n' - |X|)$. Let V' be the vertex set of G', $U' = V - V'$, and note that $|U'| \leq n/9$. Next, we claim that $\sum_i |D_i - U'| \geq 10n\sqrt{p}\cdot s$. Note that from every vertex of $D_i \cap U'$ there is an edge to one of the vertices in B_i. Since B_i are disjoint, each edge inside U' is counted at most twice in the summation $\sum_i |D_i - U'|$, therefore $\sum_i |D_i \cap U'| \leq 2e(U')$. This implies that

$$\begin{aligned}\sum_i |D_i - U'| &= \sum_i \big(|D_i| - |D_i \cap U'|\big) \geq 25n\sqrt{p}s - 2e(U')\\ &\geq 25n\sqrt{p}s - p|U'|^2 - 2\beta|U'|\\ &\geq 25n\sqrt{p}s - n^2p/81 - o(n^2p)\\ &\geq 10n\sqrt{p}s.\end{aligned}$$

Since $|D_i - U'| \le 25n\sqrt{p}$, we have that there are at least $s/5$ sets D_i such that $D'_i = D_i - U' = D_i \cap V'$ has size at least $6n\sqrt{p}$. Let I be the set of indices i such that $|D'_i| \ge 6n\sqrt{p}$.

Consider a lazy random walk on G' which starts from the stationary distribution and makes $\ell = 50p^{-1/2}$ steps. By Lemma 6.2 the probability that this walk does not intersects a given $D'_i, i \in I$, is at most $e^{-0.03|D'_i|\ell/n'} \le 0.01$. Therefore by Markov's inequality with positive probability this walk intersects at least $0.9|I| \ge s/6$ sets D'_i. Choose one such walk and denote its vertex set by C_{k+1}. This gives a connected subset of size (at most) $50p^{-1/2}$, which by definition is disjoint from all previous sets B_i, C_j and has neighbors in at least $s/6$ sets B_i. □

7 Minors in H-free graphs

In this section we prove Theorems 4.5 and 4.6.

We start with proving Theorem 4.5. We assume that s, s' are fixed integers satisfying $2 \le s \le s'$.

Lemma 7.1 *Let G be a graph of order n with average degree $d \le r$. Let X, Y, Z be a partition of the vertex set of G into three disjoint sets such that $|Y| \le \frac{|X|}{2a}$ and $e(X,Z) \le \frac{r}{4a}|X|$ for some $a > 0$. Then $G \setminus X$ still has the average degree at least d, or the average degree of the subgraph induced by the set $X \cup Y$ is at least $d - \frac{r}{a}$.*

Proof. Let $|X| = \alpha n$ and suppose that the average degree of $G \setminus X$ is at most d, i.e., $e(G \setminus X) \le (1-\alpha)dn/2$. Let G' be the subgraph of G induced by the set $X \cup Y$. Then $|V(G')| = |X \cup Y| \le (1 + 1/(2a))\alpha n$ and

$$\begin{aligned} e(G') &\ge e(G) - e(G \setminus X) - e(X,Z) \\ &\ge dn/2 - (1-\alpha)dn/2 - \frac{r}{4a}\alpha n \\ &= \left(d - \frac{r}{2a}\right)\frac{\alpha n}{2}. \end{aligned}$$

Since $d \le r$, the average degree of G' is:

$$\frac{2e(G')}{|V(G')|} \ge \frac{(d - r/(2a))\alpha n}{(1 + 1/(2a))\alpha n} = \frac{2a}{2a+1}d - \frac{r}{2a+1} \ge d - \frac{r}{a}.$$ □

Lemma 7.2 *Let G be $K_{s,s'}$-free graph, $s' \ge s$, and let $X \subseteq V(G)$ such that $e(X, V - X) \ge d|X|$ for some $d > 0$. Then*

$$|N(X)| \ge \begin{cases} \frac{d|X|}{s'} & \text{if } |X| \le d^{1/(s-1)} \\ \frac{d^{s/(s-1)}}{s'} & \text{otherwise} \end{cases}$$

Proof. First note that we need only to consider the case when $|X| \le d^{1/(s-1)}$. Indeed if $|X| \ge d^{1/(s-1)}$ then by the averaging argument there exists $X' \subseteq X$ of size $|X'| = d^{1/(s-1)}$ such that $e(X', V - X) \ge d|X'|$.

Let $|X| \le d^{1/(s-1)}$. Assume by the way of contradiction that $|N(X)| < d|X|/s'$. Let Y be a subset of $d|X|/s'$ vertices of $V \setminus X$ containing $N(X)$. Then there are at least $d|X|$ edges between X

and Y in G. Let us count the number of pairs (y, S), where $y \in Y$, $S \subseteq X \cap N(y)$, $|S| = s$. Denote this quantity by A. Then

$$A = \sum_{y \in Y} \binom{d(y,X)}{s} \geq |Y| \binom{\frac{\sum_{y \in Y} d(y,X)}{|Y|}}{s} \geq |Y| \binom{\frac{d|X|}{|Y|}}{s} = \frac{d|X|}{s'} \binom{s'}{s} .$$

On the other hand, each S appears in at most $s' - 1$ pairs (y, S) as otherwise we get a copy of $K_{s,s'}$ with s vertices in S and s' vertices in Y. Therefore,

$$A \leq (s' - 1) \binom{|X|}{s} .$$

Comparing the above two estimates for A we get:

$$\frac{d|X|}{s'} \binom{s'}{s} \leq A \leq (s' - 1) \binom{|X|}{s} < (s' - 1) \frac{|X|^s}{s!} ,$$

implying:

$$\frac{s!}{s'(s'-1)} \binom{s'}{s} < \frac{|X|^{s-1}}{d} .$$

As $s' \geq s \geq 2$, the LHS of the inequality above is easily seen to be at least 1, while by the assumption $|X| \leq d^{1/(s-1)}$, the RHS is at most 1 – a contradiction. $\Box$

Lemma 7.3 *Let $c > 0$ be a constant and let G be a $K_{s,s'}$-free graph on $cr^{\frac{s}{s-1}}$ vertices with average degree r. Then G contains a minor with average degree at least $\Omega\big(r^{1+\frac{1}{2(s-1)}}\big)$.*

Proof. Since the average degree of G is at least r, it contains a subgraph G' with minimum degree at least $r/2$. Let X be a subset of G' of size at most $r^{\frac{1}{s-1}}/4$. Since the minimum degree is at least $r/2$, every vertex of X has at least $r/4$ neighbors outside X, i.e., $e_{G'}(X, V(G') - X) \geq \frac{r}{4}|X|$. Therefore by Lemma 7.2 we have that $|N_{G'}(X)| \geq \frac{r}{4s'}|X|$. This implies that G' is a (t, α)-expanding graph of order $n = c_1 r^{\frac{s}{s-1}}$, where $t = r/(4s')$ and $\alpha = \frac{1}{16s'c_1}$. (Observe that by the well known bounds on the so called Zarankiewicz problem, a $K_{s,s'}$-free graph of average degree r has $\Omega\left(r^{\frac{s}{s-1}}\right)$ vertices). Thus, by Theorem 4.1, it contains a minor with average degree at least

$$\Omega\left(\alpha^3 \frac{\sqrt{nt \log t}}{\sqrt{\log n}}\right) = \Omega\left(r^{1+\frac{1}{2(s-1)}}\right). \qquad \Box$$

Lemma 7.4 *Let $2 \leq s \leq s' \leq a$ and let G be a $K_{s,s'}$-free graph of order $n \leq e^{2a} r^{\frac{s}{s-1}}$ such that for every two disjoint subsets $X, |X| \leq n/2$, and $Y, |Y| \leq \frac{1}{3a^2}|X|$, we have that $e\big(X, V(G) - (X \cup Y)\big) \geq \frac{r}{4a^2}|X|$. Then the diameter of G is at most $33a^3$.*

Proof. By the above condition, G has minimum degree at least $\frac{r}{4a^2}$. If $\frac{r}{4a^2} > n/2$ we are done, since the diameter of G is at most two. Let v be an arbitrary vertex of G and let $X \subset N(v)$ be a subset of $\frac{r}{4a^2}$ neighbors of v. Our assumptions on G imply that $e_G(X, V - X) \geq \frac{r}{4a^2}|X|$. Since G is $K_{s,s'}$-free, $s \geq 2$ and $s' \leq a$, by Lemma 7.2 (with $d = \frac{r}{4a^2}$), we have that

$$|N(X)| \geq \min\left\{\frac{r}{4a^2 s'}|X|, \frac{1}{s'}\left(\frac{r}{4a^2}\right)^{\frac{s}{s-1}}\right\} \geq \frac{r^{\frac{s}{s-1}}}{16a^5}.$$

Therefore there are at least $\frac{1}{16a^5} r^{\frac{s}{s-1}}$ vertices within distance at most two from any vertex of G. We also have that every subset U of G of size at most $n/2$ satisfies $|U \cup N(U)| \geq \big(1 + \frac{1}{3a^2}\big)|U|$. Since $8a^5 e^{2a} < \big(1 + \frac{1}{3a^2}\big)^{16a^3}$ for $a \geq 2$, we conclude that there are more than

$$\frac{r^{\frac{s}{s-1}}}{16a^5}\left(1+\frac{1}{3a^2}\right)^{16a^3} > \frac{1}{2}e^{2a} r^{\frac{s}{s-1}} \geq n/2$$

vertices within distance at most $2 + 16a^3$ from any given vertex of G. This implies that the diameter of G is at most $2(2 + 16a^3) \leq 33a^3$. $\square$

Lemma 7.5 *Let $2 \leq s \leq s' \leq a \leq 2\log r$ and let G be a $K_{s,s'}$-free graph of order n such that $a^{14} r^{\frac{s}{s-1}} \leq n \leq e^{2a} r^{\frac{s}{s-1}}$ and for every two disjoint subsets $X, |X| \leq 0.7n$, and $Y, |Y| \leq \frac{1}{2a^2}|X|$, we have that $e\big(X, V(G) - (X \cup Y)\big) \geq \frac{r}{4a^2}|X|$. Then G contains a minor with average degree at least $cr^{1+\frac{1}{2(s-1)}}$, where $c > 0$ is a constant independent of r and a.*

Proof. Let

$$p = \frac{1}{10^3} a^2 r^{1+\frac{1}{2(s-1)}} \quad \text{and} \quad q = 33\frac{n}{a^4 r^{1+\frac{1}{2(s-1)}}},$$

and consider the following iterative procedure which we will repeat p times. In the beginning of iteration $k+1$ we will have k disjoint sets $B_1, \ldots, B_k$ each of size $|B_i| = q$ such that all induced subgraphs $G[B_i]$ are connected. We will construct a new subset B_{k+1}, also of size q, such that the induced subgraph $G[B_{k+1}]$ is connected and there are at least $k/(8a^2)$ indices $1 \leq i \leq k$ such that there is an edge from B_i to B_{k+1}. In the end of this algorithm, if we contract all subsets B_i we will get a graph with average degree $\Omega(\frac{p}{8a^2}) \geq cr^{1+\frac{1}{2(s-1)}}$.

Let $B = \cup_{i=1}^k B_i$ and note that $|B| \leq pq \leq \frac{n}{30a^2}$. Denote $C = V(G) - B$ and let G' be the subgraph of G induced by C. Let X_1 and Y_1 be two disjoint subsets of C such that $n/5 \leq |X_1| \leq 0.7n$, $|Y_1| \leq \frac{1}{3a^2}|X_1|$ and $e\big(X_1, C - (X_1 \cup Y_1)\big) < \frac{r}{4a^2}|X_1|$. Set $Y' = Y_1 \cup B$. Then we have

$$|Y'| \leq |Y_1| + |B| \leq \frac{1}{3a^2}|X_1| + \frac{n}{30a^2} \leq \frac{1}{2a^2}|X_1|$$

and $e\big(X_1, V(G) - (X_1 \cup Y')\big) < \frac{r}{4a^2}|X_1|$ which contradicts our assumption about G. As long as there are two disjoint sets X, Y of size $0 < |X| \leq n/5$ and $|Y| \leq |X|/(3a^2)$ such that $e\big(X, V(G) - (X \cup Y)\big) \geq \frac{r}{4a^2}|X|$ delete X and continue. By the above discussion the union X_1 of all deleted sets has at most $n/5$ vertices. This implies that there exist two disjoint (or empty) subsets $X_1, Y_1 \subset C$ such that $|X_1| \leq n/5$, $|Y_1| \leq \frac{1}{3a^2}|X_1|$, $e\big(X_1, C - (X_1 \cup Y_1)\big) \leq \frac{r}{4a^2}|X_1|$ and the remaining set $D = C - X_1$ spans a graph G'' in which for every two disjoint subsets $X, |X| \leq n/2$, and $Y, |Y| \leq \frac{1}{3a^2}|X|$, we have that $e\big(X, V(G'') - (X \cup Y)\big) \geq \frac{r}{4a^2}|X|$. The restriction $|X| \leq n/2 = 0.7n - 0.2n$ comes from the assumption of the lemma about sets of size up to $0.7n$ and the fact that the union of deleted X's has cardinality at most $0.2n$. Note that by Lemma 7.4, G'' has diameter at most $33a^3$.

Consider all sets B_j that satisfy $e(B_j, D) \geq \frac{r}{4a^2}|B_j|$. Without loss of generality, we can assume that the first m sets $B_1, \ldots, B_m$ have this property. We claim that m is at least $\frac{k}{4a^2}$. If this is not the case then denote $Y_2 = \cup_{j=1}^m B_j$, and $X_2 = \cup_{j=m+1}^k B_j$. By definition $|Y_2| \leq \frac{m}{k-m}|X_2| \leq \frac{1}{3a^2}|X_2|$ and

$$e(X_2, D) = \sum_{j=m+1}^{k} e(B_j, D) < \sum_{j=m+1}^{k} \frac{r}{4a^2}|B_j| = \frac{r}{4a^2}|X_2|.$$

Define $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$. Then $|X| \le n/5 + |B| \le n/4$,

$$|Y| \le |Y_1| + |Y_2| \le \frac{1}{3a^2}|X_1| + \frac{1}{3a^2}|X_2| < \frac{1}{2a^2}|X|,$$

and also

$$\begin{aligned} e\big(X, V(G) - (X \cup Y)\big) &\le e(X_1, D - Y_1) + e(X_2, D) \\ &< \frac{r}{4a^2}\big(|X_1| + |X_2|\big) = \frac{r}{4a^2}|X|. \end{aligned}$$

This contradicts the properties of G. Therefore we have that the first $m = \frac{k}{4a^2}$ sets $B_1, \ldots, B_m$ satisfy that $e(B_j, D) \ge \frac{r}{4a^2}|B_j|$.

Denote by $U_j, 1 \le j \le m$, the set of neighbors of B_j in D. Since $n \ge a^{14} r^{\frac{s}{s-1}}$, $s' \le a$, $|B_j| = q$ and $a = r^{o(1)}$, by Lemma 7.2 (with $d = \frac{r}{4a^2}$), we have that

$$|U_j| \ge \min\left\{\frac{r}{4a^2 s'}|B_j|, \frac{1}{s'}\left(\frac{r}{4a^2}\right)^{\frac{s}{s-1}}\right\} \ge a^7 r^{1+\frac{1}{2(s-1)}}.$$

Pick uniformly at random with repetition $n/(a^7 r^{1+\frac{1}{2(s-1)}})$ vertices of G'' and denote this set by W. For every index $1 \le i \le m$ the probability that W does not intersect U_i is at most $\left(1 - \frac{|U_i|}{|G''|}\right)^{|W|} \le 1/e$. Therefore the expected number of sets U_i which have non-empty intersection with W is at least $(1 - 1/e)m > m/2$. Hence there is a choice of W that intersects at least $m/2 \ge k/(8a^2)$ sets U_i. Fix an arbitrary vertex $w_0 \in W$ and consider a collection of shortest paths in G_1 from w_0 to the remaining vertices in W. Since the diameter of G'' is at most $33a^3$ and $33a^3|W| \le q$, by taking union of these paths and adding extra vertices if necessary we can construct a connected subset of size q containing W. Denote this set by B_{k+1} and note that it is connected by an edge to at least $k/(8a^2)$ sets $U_i, 1 \le i \le k$. This completes the proof of the lemma. $\square$

Lemma 7.6 *Let G be a $K_{s,s'}$-free graph of average degree r and at most $r^{4+\frac{s}{s-1}}$ vertices. Then G contains a minor with average degree at least*

$$\Omega\left(r^{1+\frac{1}{2(s-1)}}\right).$$

Proof. Let $\{a_i, i \ge 0\}$ be an increasing sequence defined by $a_0 = 20s'$ and $a_{i+1} = e^{a_i/7}$. Note that $a_{i+1}^{14} = e^{2a_i}$ and let ℓ be the first index such that $e^{2a_\ell} > r^4$. Then there is some $0 \le i \le \ell$ so that the order n of our graph G satisfies

$$a_i^{14} r^{\frac{s}{s-1}} \le n < e^{2a_i} r^{\frac{s}{s-1}}.$$

If G has the property that for every two disjoint subsets $X, |X| \le 0.7n$, and $Y, |Y| \le \frac{1}{2a_i^2}|X|$, we have that

$$e\big(X, V(G) - (X \cup Y)\big) \ge \frac{r}{4a_i^2}|X|,$$

then by Lemma 7.5 it contains a minor with average degree $\Omega\left(r^{1+\frac{1}{2(s-1)}}\right)$ and we are done. Otherwise, there are two sets X, Y as above for which $e\big(X, V(G) - (X \cup Y)\big) < \frac{r}{4a_i^2}|X|$. Then, by Lemma 7.1, we have that the average degree of the graph $G - X$ is at least that of G, or the average degree of the subgraph induced by $X \cup Y$ drops by at most $\frac{r}{a_i^2}$. In the first case let $G_1 = G - X$ and in the second let $G_1 = G[X \cup Y]$. Note that the number of vertices n_1 of new graph is strictly smaller than that of G. Moreover if the average degree of G_1 is smaller than that of G we know that $n_1 = |X \cup Y| \le 3n/4$. Continue this process until we either find a minor with average degree at least $\Omega\left(r^{1+\frac{1}{2(s-1)}}\right)$, or arrive at a graph G' with n' vertices such that $n' \le a_0^{14} r^{\frac{s}{s-1}}$.

In the first case we are obviously done. In the second case we claim that the average degree of G' is still at least $r/2$. Note that if at some stage the order of our graph G_j satisfied

$$a_i^{14} r^{\frac{s}{s-1}} < |V(G_j)| \le e^{2a_i} r^{\frac{s}{s-1}},$$

then the average degree of the new graph G_{j+1} could decrease only by at most r/a_i^2. In this case the order of G_{j+1} drops as well so that $|G_{j+1}| \le 3|G_j|/4$. Since $(3/4)^4 < e^{-1}$, we have that this can happen only at most $8a_i$ times, before the order of the remaining graph will become smaller than $a_i^{14} r^{\frac{s}{s-1}} = e^{2a_{i-1}} r^{\frac{s}{s-1}}$. Since $a_{i+1} = e^{a_i/7} \ge 2a_i$, we have that during all iterations the average degree of the resulting graph can decrease by at most

$$\sum_i 8a_i \cdot \frac{r}{a_i^2} = r \sum_i \frac{8}{a_i} \le \frac{16}{a_0} r < r/2.$$

Hence the final graph G' has average degree at least $r/2$ and at most $O\left(r^{\frac{s}{s-1}}\right)$ vertices. Therefore, by Lemma 7.3, it contains a minor with average degree $\Omega\left(r^{1+\frac{1}{2(s-1)}}\right)$. □

Proof of Theorem 4.5. Let G be a $K_{s,s'}$-free graph with average degree r and let n be the number of vertices of G. By Lemma 7.6, we can assume that $n > r^5$. Suppose that G contains a subset $X, |X| \le 0.7n$, such that $|N(X)| \le \frac{|X|}{2\log^2 n}$. If the average degree of $G - X$ is at least as large as that of G, set $G_1 = G - X$ and let n_1 be the number of vertices in G_1. Otherwise, let G_1 be the subgraph induced by the set $X \cup N(X)$. In the second case, by Lemma 7.1, the average degree of G_1 is at least $r - \frac{r}{\log^2 n}$. Note that in both cases we obtain a smaller graph. Moreover if the average degree of G_1 is smaller than that of G we know that $n_1 = |X \cup N(X)| \le 3n/4$. Continue this process until we obtain a subgraph G' of G on n' vertices such that one of the following holds. Either $n' \le r^5$ or every subset X of G' of size $|X| \le 0.7n'$ has $|N(X)| \ge \frac{|X|}{2\log^2 n'}$. Note that in the second case the graph G' does not have a separator of size $\Theta\left(\frac{n'}{\log^2 n'}\right)$. Since $n' > r^5$, by a result of Plotkin, Rao and Smith [34] mentioned in Section 3, G' has a clique minor of size

$$\Omega\left(\frac{n'/\log^2 n'}{\sqrt{n' \log n'}}\right) = \Omega\left(\frac{\sqrt{n'}}{\log^{5/2} n'}\right) \ge r^{5/2-o(1)} \gg r^{1+\frac{1}{2(s-1)}}.$$

In the first case, when $n' \le r^5$ we claim that the average degree of G' is still at least $r/2$. Indeed, let $n \ge x_0, x_1, \dots, x_\ell \ge r^5$ be the sequence of orders of graphs that we had during the process when the average degree decreased. Then we know that $x_{i+1} \le 3x_i/4$ and the decrease in the average degree at the corresponding step was at most $r/\log^2 x_i$. Let $y_i = \log x_{\ell-i}$, then $y_0 \ge 5\log r$ and $y_{i+1} \ge y_i + \log(4/3) \ge y_i + 1/4$. Therefore

$$\sum_i \frac{1}{y_i^2} \le \sum_i \frac{1}{(y_0 + i/4)^2} \le 16 \sum_{i=0}^{\infty} \frac{1}{(4y_0 + i)^2} \le \frac{4}{y_0 - 1} \ll 1/2,$$

and we conclude that the average degree of G' is at least $r\big(1-\sum_i 1/\log^2 x_i\big) \geq r/2$. Therefore we can find in G' a minor with average degree $\Omega\left(r^{1+\frac{1}{2(s-1)}}\right)$ using Lemma 7.6. This completes the proof of the theorem. $\Box$

The proof of Theorem 4.6 is very similar to that of Theorem 4.5. The only (relatively) substantial difference in the proof of Theorem 4.6 compared to that of 4.5 lies in the proof of Lemma 7.2. Instead, we have:

Lemma 7.7 *Let H be a bipartite graph of order h with parts A and B such that the degrees of all vertices in B do not exceed s. Let G be H-free graph and let $X \subseteq V(G)$ such that $e(X, V-X) \geq (2dh)|X|$ for some $d > 0$. Then*

$$|N(X)| \geq \begin{cases} d|X| & \textit{if } |X| \leq d^{1/(s-1)} \\ d^{s/(s-1)} & \textit{otherwise} \end{cases}$$

Proof. Similarly to the proof of Lemma 7.2 we need to consider only the case when $|X| \leq d^{1/(s-1)}$. Then the result follows from a variant of the dependent random choice argument utilized in particular in [3]. If $|N(X)| \leq d|X|$ then pick a random vertex v in $N(X)$. Let the random variable Y count the number of neighbors of v in X, and let the random variable Z be the number of s-tuples of vertices in $N(v) \cap X$ that have at most $h-1$ common neighbors. Then the expected value of Y is at least $\frac{e(X,V-X)}{|N(X)|} \geq \frac{2dh|X|}{|N(X)|}$, while the expected value of Z is at most $\binom{|X|}{s}\frac{h-1}{|N(X)|}$. It thus follows that

$$\begin{aligned} \mathbb{E}[Y-Z] &= \mathbb{E}[Y]-\mathbb{E}[Z] \geq \frac{2dh|X|}{|N(X)|} - \binom{|X|}{s}\frac{h-1}{|N(X)|} \\ &> \frac{h}{|N(X)|}\left(2d|X| - \binom{|X|}{s}\right) \\ &> \frac{h|X|}{|N(X)|}\left(2d - \frac{|X|^{s-1}}{s!}\right) \\ &\geq \frac{dh|X|}{|N(X)|} \geq h\ . \end{aligned}$$

Therefore there exists a vertex $v \in N(X)$ so that $Y - Z \geq h$. Fix such a vertex, denote by A_0 its neighborhood in X, and for each s-tuple S in A_0 with less than h common neighbors, delete an arbitrary vertex from S. Denote the obtained set by A_1. Then $|A_1| \geq h$, and every s-tuple in A_1 has at least h common neighbors. We then can embed a copy of H in G by first embedding the side A of H one-to-one into A_1, and then embedding the vertices of B, the other side of H, vertex by vertex. As every s-tuple in A_1 has at least h common neighbors and the degree of every vertex in B is at most s, we will be always able to find a required vertex. $\Box$

Repeating the proof of Theorem 4.5 and using the above lemma instead of Lemma 7.2, we can prove Theorem 4.6.

8 Minors in C_{2k}-free graphs

Here we prove Theorem 4.7. In the rest of this section we may and will assume that $k \geq 3$ is fixed ($k = 2$ follows from Theorem 4.5) and r is sufficiently large compared to k.

Lemma 8.1 *Let G be C_{2k}-free graph on n vertices with average degree d. Then $n \geq \left(\frac{d}{16k}\right)^k$.*

Proof. It was proved in [45] that the number of edges in a C_{2k}-free graph on n vertices is at most $8kn^{1+\frac{1}{k}}$. Therefore we have that $nd/2 \leq 8kn^{1+\frac{1}{k}}$, which implies that $n \geq \left(\frac{d}{16k}\right)^k$. □

Lemma 8.2 *Let $k \geq 3$ and let G be a C_{2k}-free graph. If $X \subseteq V(G)$ satisfies that $e(X, V - X) \geq d|X|$ for some $d \geq 50k^2$, then*

$$|N(X)| \geq \begin{cases} \frac{d|X|}{4k^2} & \text{if } |X| \leq d^{\frac{k-1}{2}} \\ \frac{d^{1/2}|X|}{4k^2} & \text{if } |X| \leq d^{\frac{k+1}{2}} \\ 3|X| & \text{if } |X| \leq \left(\frac{d}{6k}\right)^k \end{cases}.$$

Proof. These estimates can be easily deduced from a result of Naor and Verstraëte [32], who proved that the number of edges in a C_{2k}-free bipartite graph with parts X and Y is bounded by

$$e(X, Y) \leq (2k - 3)\Big((|X||Y|)^{\frac{k+1}{2k}} + |X| + |Y|\Big).$$

Indeed, we will have a contradiction with this inequality if $e(X, N(X)) \geq d|X|$ and the size of $N(X)$ is less than in the assertion of the lemma. □

Lemma 8.3 *Let $k \geq 3, \alpha \geq 1, \rho \geq 3$ and let G be a C_{2k}-free graph of order $n \leq \rho r^k$ such that for every two disjoint subsets $X, |X| \leq n/2$, and $Y, |Y| \leq \frac{1}{3\alpha}|X|$, we have that $e\big(X, V(G) - (X \cup Y)\big) \geq \frac{r}{4\alpha}|X|$. Then every subset $W \subset G$ of size at least $r^{k/2-1}\log r$ is contained in a connected subgraph of G on at most $(40k^2\alpha^{3/2}\log\rho)|W|$ vertices.*

Proof. By the above condition and Lemma 8.2, G has minimum degree at least $\frac{r}{4\alpha}$ and every subset of G of size at most $\left(\frac{r}{24k\alpha}\right)^k$ expands at least three times. Therefore for every vertex v there are at least $\left(\frac{r}{24k\alpha}\right)^k$ vertices which are within distance at most $k\log r$ from v. We also have that every subset U of G of size at most $n/2$ satisfies $|U \cup N(U)| \geq \left(1+\frac{1}{3\alpha}\right)|U|$. Since $\rho\big(24k\alpha\big)^k < \left(1+\frac{1}{3\alpha}\right)^{4\alpha\log\rho+8k^2\alpha^{3/2}}$, we conclude that there are more than

$$\frac{r^k}{(24k\alpha)^k}\left(1+\frac{1}{3\alpha}\right)^{4\alpha\log\rho+8k^2\alpha^{3/2}} > \frac{1}{2}\rho r^k \geq n/2$$

vertices within distance at most $k\log r + 4\alpha\log\rho + 8k^2\alpha^{3/2}$ from any given vertex of G. This implies that the diameter of G is at most $2k\log r + 8\alpha\log\rho + 16k^2\alpha^{3/2}$.

Similarly, by repeatedly applying the first estimate of Lemma 8.2, there are at least $\left(\frac{r}{16k^2\alpha}\right)^{\frac{k+1}{2}}$ vertices within distance at most $\frac{k+1}{2}$ from every vertex of G, and therefore (this time from the second estimate of Lemma 8.2) the number of vertices within distance at most $\frac{k+1}{2}+1 \le k$ is at least

$$\frac{(r/(4\alpha))^{1/2}}{4k^2}\left(\frac{r}{16k^2\alpha}\right)^{\frac{k+1}{2}} \geq \frac{r^{k/2+1}}{(16k^2\alpha)^k}.$$

Since $\rho\big(16k^2\alpha\big)^k < \big(1+\frac{1}{3\alpha}\big)^{4\alpha\log\rho+8k^2\alpha^{3/2}}$, we conclude that there are more than

$$\frac{r^{k/2+1}}{(16k^2\alpha)^k}\left(1+\frac{1}{3\alpha}\right)^{4\alpha\log\rho+8k^2\alpha^{3/2}} > \rho r^{k/2+1}$$

vertices within distance at most $k+4\alpha\log\rho+8k^2\alpha^{3/2} \le 4\alpha\log\rho+9k^2\alpha^{3/2}$ from any given vertex of G.

Let W be a subset of $V(G)$ of size at least $r^{\frac{k}{2}-1}\log r$ and consider the following iterative process that constructs a connected subgraph G' of G containing W. At the beginning the vertex set of G' is W. At every step if there are two connected components of G' such that the distance between them in G is at most $8\alpha\log\rho+18k^2\alpha^{3/2}$, connect them by a shortest path and add the vertices of this path to G'. We perform this step at most $|W|$ times until the distance between every two remaining connected components of G' is larger than $8\alpha\log\rho+18k^2\alpha^{3/2}$. Then the balls of radius $4\alpha\log\rho+9k^2\alpha^{3/2}$ around each component are disjoint. By the above discussion, each such ball contains at least $\rho r^{k/2+1}$ vertices so the number of components is at most $\frac{n}{\rho r^{k/2+1}} \le r^{k/2-1}$. Now fix one component of G' and connect it to every other component by a path whose length is bounded by the diameter of G. This gives a connected subgraph of G that contains W and has altogether at most

$$\big(8\alpha\log\rho+18k^2\alpha^{3/2}\big)|W| + r^{k/2-1}(2k\log r+8\alpha\log\rho+16k^2\alpha^{3/2}) \le \big(40k^2\alpha^{3/2}\log\rho\big)|W|$$

vertices. This completes the proof of the lemma. □

Lemma 8.4 *Let $1 \le \alpha \le \log^2 r, 3 \le \rho \le r^2$, and let G be a C_{2k}-free graph of order $n \le \rho r^k$ such that for every two disjoint subsets $X, |X| \le 0.7n$, and $Y, |Y| \le \frac{1}{2\alpha}|X|$, we have that $e\big(X, V(G)-(X\cup Y)\big) \ge \frac{r}{4\alpha}|X|$. Then G contains a minor with average degree at least*

$$c\frac{r\cdot n^{\frac{k-1}{2k}}}{\alpha^{\frac{15k+3}{4k}}\log^{\frac{k+1}{2k}}\rho},$$

where $c>0$ is a constant independent of r, ρ and α.

Proof. Let

$$q = \frac{1000k^3\alpha}{r}\big(n\alpha^{3/2}\log\rho\big)^{\frac{k+1}{2k}} \quad \text{and} \quad p = \frac{n}{30\alpha q},$$

and consider the following iterative procedure which we will repeat p times. In the beginning of iteration $t+1$ we have t disjoint sets $B_1,\dots,B_t$, each of size $|B_i| = q$, such that all induced subgraphs $G[B_i]$ are connected. We will construct a new subset B_{t+1}, also of size q, such that the induced subgraph $G[B_{t+1}]$ is connected, and there are at least $t/(8\alpha)$ indices $1 \le i \le t$ such that there is an edge from B_i to B_{t+1}. In the end of this algorithm if we contract all subsets B_i we get a graph with average degree

$$\Omega\Big(\frac{p}{8\alpha}\Big) \ge \Omega\left(\frac{r \cdot n^{\frac{k-1}{2k}}}{\alpha^{\frac{15k+3}{4k}} \log^{\frac{k+1}{2k}} \rho}\right).$$

Let $B = \cup_{i=1}^{t} B_i$ and note that $|B| \le pq = \frac{n}{30\alpha}$. Repeating the argument of the proof of Lemma 7.5 we obtain a subset D such that the subgraph G'' induced by D has the following properties.

- For every two disjoint subsets $X, |X| \le n/2$, and $Y, |Y| \le \frac{1}{3\alpha}|X|$, of G'' we have that
$$e\big(X, V(G'') - (X \cup Y)\big) \ge \frac{r}{4\alpha}|X|.$$
- At least $m = \frac{t}{4\alpha}$ sets B_j satisfy that $e(B_j, D) \ge \frac{r}{4\alpha}|B_j|$.

Without loss of generality, we can assume that $B_1, \ldots, B_m$ satisfy: $e(B_j, D) \ge \frac{r}{4\alpha}|B_j|$. Let U_j be the set of neighbors of B_j in D. Since our graph is C_{2k}-free we have, by the result of Naor and Verstraëte [32], that

$$e(B_j, U_j) \le (2k-3)\Big((|B_j||U_j|)^{\frac{k+1}{2k}} + |B_j| + |U_j|\Big).$$

This inequality together with $k \ge 3$ and $|B_j| = q$ implies that $|U_j| \ge (40k^2\alpha^{3/2}\log\rho)n/q$. (Here and later we use that the minimum degree is at least $r/(4\alpha)$ and therefore by Lemma 8.1 $n \ge (r/(64k\alpha))^k$, we also assume that r is large enough.) Pick uniformly at random with repetition $q/(40k^2\alpha^{3/2}\log\rho) > r^{k/2-1}\log r$ vertices of G'' and denote this set by W. For every index $1 \le i \le m$ the probability that W does not intersect U_i is at most $\Big(1 - \frac{|U_i|}{|G''|}\Big)^{|W|} \le 1/e$. Therefore the expected number of sets U_i that have a non-empty intersection with W is at least $(1-1/e)m > m/2$. Hence there is a choice of W that intersects at least $m/2 \ge t/(8\alpha)$ sets U_i. By Lemma 8.3, G'' contains a connected subgraph on $\le (40k^2\alpha^{3/2}\log\rho)|W| \le q$ vertices that contains W. By adding extra vertices if necessary we can construct a connected subset B_{t+1} of size q that contains W and hence is connected by an edge to at least $t/(8\alpha)$ sets $U_i, i \le t$. This completes the proof of the lemma. □

Substituting in the above lemma $\alpha = a^2$ and $\rho = e^{2a}$ we obtain the following corollary.

Corollary 8.5 *Let $1 \le a \le \log r$, and let G be a C_{2k}-free graph of order n such that $a^{26}r^k \le n \le e^{2a}r^k$ and for every two disjoint subsets $X, |X| \le 0.7n$, and $Y, |Y| \le \frac{1}{2a^2}|X|$, we have that $e\big(X, V(G) - (X \cup Y)\big) \ge \frac{r}{4a^2}|X|$. Then G contains a minor with average degree at least $cr^{\frac{k+1}{2}}$, where $c > 0$ is a constant independent of r and a.*

Lemma 8.6 *Let $\rho \ge 3$ be a constant and let G be a C_{2k}-free graph on $n = \rho r^k$ vertices with average degree r. Then G contains a minor with average degree at least $\Omega\big(r^{\frac{k+1}{2}}\big)$.*

Proof. Set $\alpha = 8(k\log(32k) + \log\rho)$ and note that it is a constant independent of r. Consider the following process. If we already have a graph G with average degree $\Omega(r)$ which has the property that for every two disjoint subsets $X, |X| \le 0.7n$ and $Y, |Y| \le \frac{1}{2\alpha}|X|$ we have that

$$e\big(X, V(G) - (X \cup Y)\big) \ge \frac{r}{4\alpha}|X|,$$

then by Lemma 8.4 it contains a minor with average degree $\Omega\left(rn^{\frac{k-1}{2k}}\right)$. Since by Lemma 8.1 every C_{2k}-free graph with average degree $\Omega(r)$ has at least $\Omega(r^k)$ vertices we are done. Otherwise, there are two sets X, Y as above for which $e\big(X, V(G)-(X\cup Y)\big) < \frac{r}{4\alpha}|X|$. Then, by Lemma 7.1, we have that the average degree of the graph $G-X$ is at least r, or the average degree of the subgraph induced by $X\cup Y$ is at least $r-\frac{r}{\alpha}$. In the first case let $G_1 = G-X$ and in the second let $G_1 = G[X\cup Y]$. Note that the number of vertices n_1 of the new graph is strictly smaller than that of G. Moreover if the average degree of G_1 is smaller than that of G we know $n_1 = |X\cup Y| \le 3n/4$. Continue this process until we either find a graph with average degree $\Omega(r)$ which satisfies the assumption of Lemma 8.4 and therefore contains a minor with average degree at least $\Omega\left(r^{\frac{k+1}{2}}\right)$, or we have at least $\alpha/2$ steps at which the average degree of the new graph decreases. In the second case, let G' be the resulting graph and n' be the number of its vertices.

Since the degree decreased exactly $\alpha/2$ times we know that the average degree of G' is at least $r-(\alpha/2)\frac{r}{\alpha} \ge r/2$ and the number of its vertices satisfies

$$n' \le \left(\frac{3}{4}\right)^{\alpha/2} n < e^{-k\log(32k)-\log\rho} n = \frac{n}{\rho(32k)^k} \le \left(\frac{r}{32k}\right)^k.$$

As G' is C_{2k}-free, it contradicts the assertion of Lemma 8.1. This shows that the second case is in fact impossible and our process always outputs a minor of average degree at least $\Omega\left(r^{\frac{k+1}{2}}\right)$. $\square$

Lemma 8.7 *Let G be a C_{2k}-free graph of with average degree r and at most r^{k+2} vertices. Then G contains a minor with average degree at least $\Omega\left(r^{\frac{k+1}{2}}\right)$.*

Proof. Let $\{a_i, i\ge 0\}$ be an increasing sequence defined by $a_0 = 65$ and $a_{i+1} = e^{a_i/13}$. Note that $a_{i+1}^{26} = e^{2a_i}$ and let ℓ be the first index such that $e^{2a_\ell} \ge r^2$. Then there is some $0 \le i \le \ell$ such that the order n of our graph G satisfies $a_i^{26} r^k \le n < e^{2a_i} r^k$. If G has the property that for every two disjoint subsets $X, |X| \le 0.7n$ and $Y, |Y| \le \frac{1}{2a_i^2}|X|$ we have that

$$e\big(X, V(G)-(X\cup Y)\big) \ge \frac{r}{4a_i^2}|X|,$$

then by Corollary 8.5 it contains a minor with average degree $\Omega\left(r^{\frac{k+1}{2}}\right)$ and we are done. Otherwise, there are two sets X, Y as above for which $e\big(X, V(G)-(X\cup Y)\big) < \frac{r}{4a_i^2}|X|$. Then, by Lemma 7.1, we have that the average degree of graph $G-X$ is at least as large as that of G, or the average degree of the subgraph induced by $X\cup Y$ drops by at most $\frac{r}{a_i^2}$. In the first case let $G_1 = G-X$ and in the second let $G_1 = G[X\cup Y]$. Note that the number of vertices n_1 of new graph is strictly smaller than that of G. Moreover if the average degree of G_1 is smaller than that of G we know $n_1 = |X\cup Y| \le 3n/4$. Continue this process until we either find a minor of with average degree at least $\Omega\left(r^{\frac{k+1}{2}}\right)$ or we arrive to a graph G' with n' vertices such that $n' \le a_0^{26} r^k$.

In the first case we are clearly done. In the second case we claim that the average degree of G' is still at least $r/2$. Note that if at some stage the order of our graph G_j satisfied $a_i^{26} r^k \le |V(G_j)| < e^{2a_i} r^k$, then the average degree of the new graph G_{j+1} could decrease only by at most r/a_i^2. In this case the order of G_{j+1} drops as well so that $|G_{j+1}| \le 3|G_j|/4$. Since $(3/4)^4 < e^{-1}$, we have that this can happen only at most $8a_i$ times, before the order of the remaining graph will become smaller than $a_i^{26} r^k = e^{2a_{i-1}} r^k$. Since $a_{i+1} = e^{a_i/13} \geq 2a_i$, we have that during all iterations the average degree of the resulting graph can decrease by at most

$$\sum_i 8a_i \cdot \frac{r}{a_i^2} = r \sum_i \frac{8}{a_i} \leq \frac{16}{a_0} r < r/2.$$

Hence the final graph G' has average degree at least $r/2$ and at most $O(r^k)$ vertices. Therefore, by Lemma 8.6, it contains a minor with average degree $\Omega\left(r^{\frac{k+1}{2}}\right)$. $\square$

Proof of Theorem 4.7. Let G be a C_{2k}-free graph with average degree r and let n be the number of vertices of G. By Lemma 8.7, we can assume that $n > r^{k+2}$. Suppose that G contains a subset $X, |X| \leq 0.7n$, such that $|N(X)| \leq \frac{|X|}{2\log^2 n}$. If the average degree of $G - X$ is at least r, set $G_1 = G - X$ and let n_1 be the number of vertices in G_1. Otherwise, let G_1 be the subgraph induced by the set $X \cup N(X)$. In the second case, by Lemma 7.1, the average degree of G_1 is at least $r - \frac{r}{\log^2 n}$. Note that in both cases we obtain a smaller graph. Moreover if the average degree of G_1 is smaller than that of G we know that $n_1 = |X \cup N(X)| \leq 3n/4$. Continue this process until we obtain a subgraph G' of G on n' vertices such that one of the following holds. Either $n' \leq r^{k+2}$ or every subset X of G' of size $|X| \leq 0.7n'$ has $|N(X)| \geq \frac{|X|}{2\log^2 n'}$. Note that in the second case the graph G' does not have a separator of size $\Theta\left(\frac{n'}{\log^2 n'}\right)$. Since $n' > r^{k+2}$, by the result of Plotkin, Rao and Smith [34], G' has a clique minor of size

$$\Omega\left(\frac{n'/\log^2 n'}{\sqrt{n' \log n'}}\right) = \Omega\left(\frac{\sqrt{n'}}{\log^{5/2} n'}\right) \geq r^{\frac{k+2}{2} - o(1)} \gg r^{\frac{k+1}{2}}.$$

In the first case, when $n' \leq r^{k+2}$ we claim that the average degree of G' is still at least $r/2$. Indeed, let $n \geq x_0, x_1, \ldots, x_\ell \geq r^{k+2}$ be the sequence of orders of graphs that we had during the process when the average degree decreased. Then we know that $x_{i+1} \leq 3x_i/4$ and the decrease in the average degree at the corresponding step was at most $r/\log^2 x_i$. Let $y_i = \log x_{\ell - i}$, then $y_0 \geq (k+2)\log r$ and $y_{i+1} \geq y_i + \log(4/3) \geq y_i + 1/4$. Therefore

$$\sum_i \frac{1}{y_i^2} \leq \sum_i \frac{1}{(y_0 + i/4)^2} \leq 16 \sum_{i=0}^{\infty} \frac{1}{(4y_0 + i)^2} \leq \frac{4}{y_0 - 1} \ll 1/2,$$

and we conclude that the average degree of G' is at least $r\big(1 - \sum_i 1/\log^2 x_i\big) \geq r/2$. Therefore we can find in G' a minor with average degree $\Omega\left(r^{\frac{k+1}{2}}\right)$ using Lemma 8.7. This completes the proof of the theorem. $\square$

9 Concluding remarks

In this paper we proved that if G is an expander graph than it contains a large clique minor. Moreover our results on H-free graphs suggest that already local expansion may be sufficient to derive results of this sort. This leads to the following general question which we think deserves further study. Let G be a graph of order n such that for every subset of vertices X of size at most s we have that

$|N(X)| \geq t|X|$. Denote by $f(s,t)$ the size of the largest clique minor which such graph must always contain. What is the asymptotic behavior of this function? Note that we already know the behavior of f in the two extremal cases when $s = 1$ and $s = \Theta(n/t)$. Indeed, if $s = 1$ we just have that the minimum degree of G is at least t and therefore it contains a clique minor of order $\Omega\big(t/\sqrt{\log t}\big)$ by Kostochka-Thomason. In the second case we have by Corollary 4.2 that our graph has clique minor of order $\Omega\big(t\sqrt{s\log t}/\log(st)\big)$.

One related and quite attractive question which remains unsettled is the asymptotic behavior of the largest clique minor size in sparse random graphs $G_{n,p}$. While for the case of constant edge probability p, Bollobás, Catlin and Erdős [8] showed this quantity to behave asymptotically as $\Theta\big(n/\sqrt{\log n}\big)$, their method is apparently insufficient to resolve the question for (much) smaller values of $p(n)$, and in particular, for the the rather intriguing case $p = c/n$, $c > 1$ is a constant, where a largest clique minor can be shown to be with high probability between $c_1\sqrt{n/\log n}$ and $c_2\sqrt{n}$.

Another interesting direction of future study can be to find sufficient conditions for ensuring a minor of a non-complete graph Γ (rather then just a clique K_k) in an expanding graph G. The first step in this direction has been made by Myers and Thomason [31] who derived an analog of the Kostochka-Thomason result for a general Γ.

Finally, it would be quite nice to obtain algorithmic analogs of our main results (see, e.g. [10] for a recent contribution to algorithmic graph minor theory), providing efficient, deterministic algorithms for finding large minors, matching our existential statements.

Note added in proof. Recently, the order of the largest clique minor in a sparse random graph was determined by Fountoulakis, Kühn and Osthus [14].

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