Every graph contains a linearly sized induced subgraph with all degrees odd

Asaf Ferber∗ Michael Krivelevich†

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Abstract

We prove that every graph $G$ on $n$ vertices with no isolated vertices contains an induced subgraph of size at least $n/10000$ with all degrees odd. This solves an old and well-known conjecture in graph theory.

1 Introduction

We start with recalling a classical theorem of Gallai (see [2], Problem 5.17 for a proof):

Theorem 1 (Gallai’s Theorem). Let $G$ be any graph.

1. There exists a partition $V(G) = V_1 \cup V_2$ such that both graphs $G[V_1]$ and $G[V_2]$ have all their degrees even.

2. There exists a partition $V(G) = V_o \cup V_e$ such that the graph $G[V_e]$ has all its degrees even, and the graph $G[V_o]$ has all its degrees odd.

It follows immediately from 1. that every graph $G$ has an induced subgraph of size at least $|V(G)|/2$ with all its degrees even. This is easily seen to be tight by taking $G$ to be a path.

It is natural to ask whether we can derive analogous results for induced subgraphs with all degrees odd. Some caution is required here — an isolated vertex can never be a part of a subgraph with all degrees odd. Thus we restrict our attention to graphs of positive minimum degree.

Let us introduce a relevant notation: given a graph $G = (V, E)$, we define

$$f_o(G) = \max\{|V_0| : G[V_0] \text{ has all degrees odd.}\},$$

and set

$$f_o(n) = \min\{f_o(G) \mid G \text{ is a graph on } n \text{ vertices with } \delta(G) \geq 1\}.$$

The following is a very well known conjecture, aptly described by Caro already more than a quarter century ago [1] as “part of the graph theory folklore”:

∗Department of Mathematics, University of California, Irvine. Email:asaff@uci.edu. Research supported in part by NSF grants DMS-1954395 and DMS-1953799.

†School of Mathematical Sciences, Tel Aviv University, Tel Aviv 6997801, Israel. Email: krivelev@tauex.tau.ac.il. Research supported in part by USA–Israel BSF grant 2018267 and by ISF grant 1261/17.
Conjecture 2. There exists a constant \( c > 0 \) such that for every \( n \in \mathbb{N} \) we have \( f_o(n) \geq cn \).

Caro himself proved [1] that \( f_o(n) = \Omega(\sqrt{n}) \), resolving a question of Alon who asked whether \( f_o(n) \) is polynomial in \( n \). The current best bound, due to Scott [3], is \( f_o(n) = \Omega(n/\log n) \). There have been numerous variants and partial results about the conjecture, we will not cover them here.

Our main result establishes Conjecture 2 with \( c = 0.0001 \).

Theorem 3. Every graph \( G \) on \( n \) vertices with \( \delta(G) \geq 1 \) satisfies: \( f_o(G) \geq cn \) for \( c = \frac{1}{10000} \).

With some effort/more accurate calculations the constant can be improved but probably to a value which is still quite far from the optimal one; we decided not to invest a substantial effort in its optimization and just chose some constants that work.

A relevant parameter was studied by Scott [4]: given a graph \( G \) with no isolated vertices, let \( t(G) \) be the minimal \( k \) for which there exists a vertex cover of \( G \) with \( k \) sets, each spanning an induced graph with all degrees odd. Letting \( t(n) = \min \{ t(G) \mid G \text{ is a graph on } n \text{ vertices with } \delta(G) \geq 1 \} \), Scott proved (Theorem 4 in [4]) that \( \Omega(\log n) = t(n) = O(\log^2 n) \).

As indicated by Scott already, showing that \( f_o(n) \) is linear in \( n \) proves the following:

Corollary 4. \( t(n) = \Theta(\log n) \).

For completeness, we outline its proof here.

**Proof.** Let \( G \) be a graph on \( n \) vertices with \( \delta(G) \geq 1 \). By a repeated use of Theorem 3, we can find disjoint sets \( V_1, \ldots, V_t \) such that:

1. \( V_i \subseteq V(G) \setminus \left( \bigcup_{j=1}^{i-1} V_j \right) \), and
2. all the degrees in \( G[V_i] \) are odd, and
3. letting \( n_i \) be the number of non-isolated vertices in \( G \left[ V(G) \setminus \left( \bigcup_{j=1}^{i-1} V_j \right) \right] \), we have that \( |V_i| \geq n_i/10000 \).

We continue the above process as long as \( n_i > 0 \). Clearly, the process terminates after \( t = O(\log n) \) steps. Moreover, letting \( U = V(G) \setminus \left( \bigcup_{i=1}^{t} V_i \right) \), we have that \( U \) is an independent set in \( G \). Finally, as shown in the proof of Theorem 4 in [4], every independent set in such \( G \) can be covered by \( O(\log n) \) odd graphs. This proves that \( t(n) = O(\log n) \).

To show a lower bound, we can use the following example due to Scott [4]: assume \( n \) is of the form \( n = s + \binom{s}{2} \). Let the vertex set of \( G \) be composed of two disjoint sets: \( A \) of size \( s \) associated with \([s]\), and \( B \) of size \( \binom{s}{2} \) associated with \( \binom{[s]}{2} \). The graph \( G \) is bipartite with the edges defined as follows: a pair \((i, j) \in B \) is connected to both \( i, j \in A \). Observe that if \( U \subseteq V(G) \) spans a subgraph of \( G \) with all degrees odd and containing \((i, j) \in B \), then \( U \) contains exactly one of \( i, j \in A \). Hence if \( \mathcal{U} = (U_1, \ldots, U_t) \) forms a cover of \( V(G) \) with subsets spanning odd subgraphs, then \( \mathcal{U} \) separates the set \( A \), and the minimum size of such a separating family is easily shown to be asymptotic to \( \log_2 s = \Omega(\log_2 n) \). \( \square \)
2 Auxiliary results

The following lemma appears as Theorem 2.1 in [1]. For the convenience of the reader we provide its simple proof.

**Lemma 2.1.** For every graph $G$ we have that $f_o(G) \geq \frac{\Delta(G)}{2}$.

**Proof.** Let $v \in V(G)$ be a vertex with $d_G(v) = \Delta(G)$, and let $U \subseteq N_G(v)$ be an odd subset of size $|U| \geq \Delta(G) - 1$. Apply Gallai’s Theorem to $G[U]$ to obtain a partition $U = V_e \cup V_o$, and observe that $V_o$ must be of an even size (so in particular, $|V_e|$ is odd). If $|V_o| \geq \Delta(G)/2$, then we are done. Otherwise, define $V^* = \{v\} \cup V_e$, and observe that $G[V^*]$ has all its degrees odd and is of size at least $\Delta(G)/2$ as required.

The next lemma appears as Theorem 1 in [3], and again, for the sake of completeness, we give its proof here.

**Lemma 2.2.** For every graph $G$ with $\delta(G) \geq 1$ we have that $f_o(G) \geq \frac{\alpha(G)}{2}$.

**Proof.** Let $I \subseteq V(G)$ be a largest independent set in $G$. Since $\delta(G) \geq 1$, every $u \in I$ has at least one neighbor in $V(G) \setminus I$.

Let $D \subseteq V(G) \setminus I$ be a smallest subset dominating all vertices in $I$. Observe that by the minimality of $D$ for every $w \in D$ there exists some $u_w \in I$ such that $N_G(u_w) \cap D = \{w\}$; let $I_D := \{u_w \mid w \in D\}$.

Let $D' \subseteq D$ be a subset of $D$ chosen uniformly at random, and let $I_0 \subseteq I \setminus I_D$ be a subset consisting of all elements $u \in I \setminus I_D$ that have an odd degree into $D'$.

Let

$$I_1 = \{u_w \in I_D \mid w \in D' \text{ and } w' \text{ has even degree in } D' \cup I_0\},$$

and observe that $G[I_0 \cup I_1 \cup D']$ is an induced subgraph of $G$ with all its degrees odd.

Finally, since $\Pr[u \in I_0] = \frac{1}{2}$, by linearity of expectation we have that

$$\mathbb{E}[|I_0 \cup I_1 \cup D'|] = \mathbb{E}[|I_0|] + \mathbb{E}[|I_1|] + \mathbb{E}[|D'|] \geq \frac{|I| - |D|}{2} + \frac{|D|}{2} = \frac{\alpha(G)}{2}.$$ 

Hence there exists a set $D'$ for which

$$|I_0| + |I_1| + |D'| \geq \frac{\alpha(G)}{2},$$

as desired.

Next we argue that if $G$ contains a semi-induced matching with “nice” expansion properties, then it also has a large induced subgraph with all degrees odd.

**Lemma 2.3.** Let $G$ be a graph and let $M$ be a matching in $G$ with parts $U$ and $W$, where every vertex $w \in W$ has only one neighbor in $G$ between the vertices covered by $M$. Assume that $|N_G(U) \setminus (W \cup N_G(W))| \geq k$. Then $f_o(G) \geq \frac{k}{4}$.

**Proof.** Let $X = N_G(U) \setminus (W \cup N_G(W))$ and recall that $|X| \geq k$. Let $U_0$ be a random subset of $U$ chosen according to the uniform distribution, and let

$$X_0 = \{x \in X : d_G(x, U_0) \text{ is odd}\}.$$
Since $\mathbb{E}|X_0| = |X|/2$, it follows that there exists an outcome $U_0 \subseteq U$ for which $|X_0| \geq |X|/2 \geq k/2$. Fix such $U_0$.

Next, apply Gallai’s theorem to $G[X_0]$ to find a subset $X_1 \subseteq X_0$ with $|X_1| \geq |X_0|/2 \geq k/4$ and all degrees in $G[X_1]$ even. Finally, for every $u \in U_0$ with $d_G(u, X_1)$ even, add an edge of $M$ containing $u$. Clearly, the obtained graph $G_1$ has size at least $|X_1| \geq |X|/4 \geq k/4$, and all its degrees are odd. This completes the proof.

The following simple lemma will be used several times below.

**Lemma 2.4.** Let $G$ be a bipartite graph with parts $A, B$ such that $d(b) > 0$ for every $b \in B$. Assume that $|A| \leq \alpha|B|$ for some $0 < \alpha \leq 1$. Then there is an edge $ab \in E(G)$ with $d(a) \geq \frac{d(b)}{\alpha}$.

**Proof.** We have:

$$\sum_{ab \in E(G)} \left( \frac{1}{d(b)} - \frac{1}{d(a)} \right) = \sum_{b \in B} d(b) \cdot \frac{1}{d(b)} - \sum_{a \in A, d(a) > 0} d(a) \cdot \frac{1}{d(a)} \geq |B| - |A| \geq (1 - \alpha)|B|.$$  

Hence there is $b \in B$ with

$$\sum_{a \in N_G(b)} \left( \frac{1}{d(b)} - \frac{1}{d(a)} \right) \geq 1 - \alpha.$$  

It follows that there is a neighbor $a$ of $b$ for which $\frac{1}{d(b)} - \frac{1}{d(a)} \geq (1 - \alpha)\frac{1}{d(b)}$, implying $d(a) \geq \frac{d(b)}{\alpha}$ as desired.

For a graph $G = (V, E)$ and $\beta > 0$, define

$$L = L(G; \beta) = \{ v \in V : \exists u \in V, uv \in E(G), |N(u) \setminus N(v)| \geq \beta|N(u) \cup N(v)| \}.$$  

We say that for $v \in L$, an edge $uv$ as above witnesses $v \in L$.

Set

$$\beta = \frac{1}{20},$$

$$\delta = \frac{1}{14},$$

$$\epsilon = \frac{1}{10}.$$  

The next lemma is a key part in the proof of our main theorem. We did not really pursue the goal of optimizing the constants in its statement.

**Lemma 2.5.** Let $G = (V, E)$ be a graph on $|V| = n$ vertices with $\delta(G) > 0$ and $|L(G; \beta)| \leq \delta n$. Then $f_0(G) \geq n/61$.

**Proof.** Define

$$V_1 = \{ v \in V \setminus L : d(v, L) \geq \epsilon d(v) \},$$

$$V_2 = V \setminus (V_1 \cup L).$$

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Suppose first that \(|V_1| \geq 12|L|\). Observe that \(d(v, L) \geq \epsilon d(v) > 0\) by the assumption \(\delta(G) > 0\). By Lemma 2.4 there exists \(uv \in E(G)\) with \(v \in V_1\) and \(u \in L\) such that \(d(u, V_1) \geq 12d(v, L) \geq 12\epsilon d(v)\). Therefore we have that

\[
|N(u) \setminus N(v)| \geq d(u) - d(v) \geq \frac{12\epsilon - 1}{12\epsilon + 1}(d(u) + d(v)) > \beta|N(u) \cup N(v)|,
\]

so in particular \(v\) should also be in \(L\) with \(uv\) witnessing it — a contradiction. We conclude that

\[
|V_1| < 12|L| \leq 12\delta n,
\]

and therefore \(|V_2| \geq (1 - \delta - 12\delta)n = \frac{n}{11} - \frac{L}{v}\).

Let \(v \in L\). Take an edge \(uv \in E(G)\). Then

\[
\max\{1, d(u) - d(v)\} \leq |N(u) \setminus N(v)| \leq \beta|N(u) \cup N(v)| \leq \beta(d(u) + d(v)),
\]
yielding:

\[
d(u) \leq \frac{1 + \beta}{1 - \beta} d(v),
\]

and

\[
\beta \left(\frac{1 + \beta}{1 - \beta} + 1\right) d(v) \geq 1.
\]

This shows that every vertex \(v \in V \setminus L\) has degree \(d(v) \geq \lceil \frac{1 - \beta}{2\beta} \rceil = 10\).

Let now \(uv \in E(G)\) with \(u, v \in V\). Then

\[
|N(u) \setminus N(v)|, |N(v) \setminus N(u)| \leq \beta|N(u) \cup N(v)|,
\]

and hence

\[
|N(u) \cap N(v)| \geq (1 - 2\beta)|N(u) \cup N(v)|. \tag{1}
\]

Since

\[
|N(u) \cap N(v)| \leq \min\{d(u), d(v)\} \quad \text{and} \quad |N(u) \cup N(v)| \geq \max\{d(u), d(v)\},
\]

it follows that

\[
(1 - 2\beta)d(u) \leq d(v) \leq \frac{d(u)}{1 - 2\beta} < (1 + 3\beta)d(u). \tag{2}
\]

Now, for all \(v \in L\) define \(R(v) = (\{v\} \cup N(v)) \setminus L\). Notice that as \(d(v) \geq 10\) we have \(|R(v)| \geq (1 - \epsilon)d(v) + 1 \geq 10\) for \(v \in V_2\). Suppose that \(R(u) \cap R(v) \neq \emptyset\) for some \(u \neq v\) where \(v \in V_2\) (note that it might be that \(u \in V_1\)). Then for \(w \in R(v) \cap R(u)\), by (1) we have

\[
|N(u) \cap N(w)| \geq (1 - 2\beta)|N(u) \cup N(w)| \quad \text{and} \quad |N(v) \cap N(w)| \geq (1 - 2\beta)|N(v) \cup N(w)|,
\]

which implies, by the identity \(|A \Delta B| = |A \cup B| - |A \cap B|\), that

\[
|N(u) \Delta N(w)| \leq \frac{2\beta}{1 - 2\beta}|N(u) \cap N(w)| < 3\beta|N(u) \cap N(w)|
\]

and

\[
|N(v) \Delta N(w)| \leq \frac{2\beta}{1 - 2\beta}|N(v) \cap N(w)| < 3\beta|N(v) \cap N(w)|.
\]
Therefore, we have
\[
\begin{align*}
|N(u) \cap N(v)| &\geq |N(u) \cap N(v) \cap N(w)| \\
&\geq |N(u) \cup N(v)| - |N(u) \triangle N(w)| - |N(v) \triangle N(w)| \\
&> |N(u) \cup N(v)| - 6\beta \max\{d(u), d(v)\} \\
&\geq (1 - 6\beta)|N(u) \cup N(v)|.
\end{align*}
\] (3)

Since \(v \in V_2\) we conclude that
\[
|R(u) \cap R(v)| \geq |N(u) \cap N(v)| - \epsilon d(v) \\
> (1 - 6\beta - \epsilon)|N(u) \cup N(v)| \\
\geq (1 - 6\beta - \epsilon)|N(u) \cup R(v)| \\
= (1 - 8\beta)|N(u) \cup R(v)|.
\] (4)

Next, let \(R_1, \ldots, R_k\) be a maximal by inclusion collection of non-intersecting sets \(R(v_i), v_i \in V_2\). Due to maximality, every \(v \in V_2\) has its set \(R(v)\) intersecting with at least one of the \(R_i\)'s; moreover, the above argument shows that it can intersect only one such set. Define now
\[
U_i = \{v \notin L : R(v) \cap R_i \neq \emptyset\}.
\]

Trivially we have \(R_i \subseteq U_i\). Also, \(V_2 \subseteq \bigcup_{i=1}^{k} U_i\) due to the maximality of the family \(R_1, \ldots, R_k\).

We wish to show that all \(U_i\) are disjoint and that there are no edges in between different \(U_i\)'s. (This will add to the above stated fact that the family of \(U_i\)'s forms a cover of \(V_2\).)

To prove the latter claim, suppose that there exists an edge \(w_1w_2 \in E(G)\) for some \(w_1 \in U_i, w_2 \in U_j, 1 \leq i \neq j \leq k\). We will obtain a contradiction by showing that \(R_i \cap R_j \neq \emptyset\). Since both \(w_1, w_2 \notin L\), by (1) and (2) we conclude that
\[
|N(w_1) \cap N(w_2)| \geq (1 - 2\beta)|N(w_1) \cup N(w_2)| \quad \text{and} \quad |N(w_1)| \in (1 \pm 3\beta)|N(w_2)|.
\]

Moreover, by (3) we have
\[
|N(w_1) \cap N(v_i)| > (1 - 6\beta)|N(w_1) \cup N(v_i)|, \quad \text{and} \quad |N(w_2) \cap N(v_j)| \geq (1 - 6\beta)|N(w_2) \cup N(v_j)|.
\]

Since \(v_i, v_j \in V_2\), the above inequalities imply that
\[
|N(w_1) \cap R_i| > (1 - 6\beta - \epsilon)|N(w_1) \cup R_i|, \quad \text{and} \quad |N(w_2) \cap R_j| > (1 - 6\beta - \epsilon)|N(w_2) \cup R_j|.
\]

It follows that
\[
|N(w_1) \cap R_i| > (1 - 6\beta - \epsilon)|N(w_1)|
\]
and
\[
|N(w_2) \cap R_j| > (1 - 6\beta - \epsilon)|N(w_2)|,
\]
and recalling that
\[
|N(w_1) \cap N(w_2)| \geq (1 - 2\beta)|N(w_1) \cup N(w_2)|,
\]
we conclude that \(R_i \cap R_j \neq \emptyset\) — a contradiction. In a similar way we can show that \(U_i \cap U_j = \emptyset\).

Next, suppose that \(|U_i| \geq (1 + 19\beta)|R_i|\). Then by looking at the auxiliary bipartite graph between \(R_i\) and \(U_i\) \((v \in R_i, u \in U_i\) are connected by an edge if \(uv \in E(G)\)) and by applying Lemma 2.4 to this
We derive that there are \( v \in R_i, u \in U_i \) with \( d(v) \geq (1 + 19\beta)d(u, R_i) \). Since \( uv \in E(G) \) and both \( u, v \not\in L \), it follows that

\[
d(v) < (1 + 3\beta)d(u) .
\]

Moreover, since \( u \in U_i \) we have:

\[
d(u, R_i) \geq (1 - 8\beta)d(u) .
\]

All in all, since \( d(v) \geq (1 + 19\beta)d(u, R_i) \) we conclude that

\[
(1 + 3\beta)d(u) > d(v) \geq (1 + 19\beta)d(v, R_i) > (1 + 19\beta)(1 - 8\beta)d(u) > (1 + 3\beta)d(u) ,
\]

a contradiction.

Therefore, we can assume that \( |U_i| \leq (1 + 19\beta)|R_i| \) for all \( 1 \leq i \leq k \). Looking at the induced subgraph \( G[U_i] \), we note that it has vertex \( v_i \) of degree \( |R_i| - 1 \geq \frac{9|R_i|}{10} \geq \frac{9}{10(1 + 19\beta)}|U_i| \). By applying Lemma 2.1 to \( G[U_i] \) we find an induced odd subgraph \( O_i \) of \( G[U_i] \) of size at least \( \frac{9}{20(1 + 19\beta)}|U_i| = \frac{9|U_i|}{39} \).

Finally, since all \( U_i \)'s are disjoint, there are no edges between any two such \( U_i \)'s and since \( V_2 \subseteq \bigcup U_i \), we conclude that \( O = \bigcup_{i=1}^k O_i \) is an induced odd subgraph of size at least \( \frac{9|V_2|}{39} > n/61 \). This completes the proof.

### 3 Proof of Theorem 3

The main plan is as follows. We will grow edge by edge a matching \( M \) with sides \( U, W \) so that every \( v \in W \) has exactly one neighbor between the vertices covered by \( M \) (which is of course its mate \( u \) in the matching). Moreover, the set \( U \) has “many” neighbors outside of \( M \) not connected to \( W \). If the set of such neighbors is substantially large, then we will be able to apply Lemma 2.3 to get a large induced subgraph with all degrees odd. Otherwise we will show that either there exists a large subset of vertices \( V' \) such that \( \delta(G[V']) \geq 1 \) with small \( L(G[V']; 1/20) \) (and then we are done by Lemma 2.5), or that we can extend the matching while enlarging substantially the set of neighbors outside \( M \) not connected to \( W \). The details are given below.

We start with \( M_0 = \emptyset \), and given \( M_i, i \geq 0 \), we define

\[
X_i = N(U_i) \setminus (W_i \cup N(W_i)) ,
\]

\[
V_i = V \setminus N(U_i \cup W_i) .
\]

In particular, we initially have \( X_0 = \emptyset \) and \( V_0 = V \). We will run our process until the first time we have \( |V_i| < n/2 \) (in particular, we may assume throughout the process that \( |V_i| \geq n/2 \)). Now, fix \( \beta = 1/20 \) and \( \delta = 1/14 \) (same parameters as set before Lemma 2.5). Our goal is to show that \( f_0(G) \geq \frac{n}{7} \), where \( T = 10000 \). We will maintain \( |X_i| \geq \frac{|V_i|}{40} \). If at some point we reach \( |X_i| \geq \frac{4n}{T} \) then we are done by Lemma 2.3. Hence we assume \( |X_i| \leq \frac{4n}{T} = \frac{2n}{2500} \). Moreover, if \( G[V_i] \) has at least \( 2n/T \) isolated vertices, then since this set induces an independent set in \( G \), by Lemma 2.2 we are done as well. Therefore, letting \( V'_i \subseteq V_i \) be the set of all non-isolated vertices in \( G[V_i] \), since \( |V_i| \geq n/2 \) we obtain that \( |V'_i| \geq (1 - 4/T)|V_i| \geq |V_i|/2 \). We can further assume \( |L(G[V'_i]; \beta)| \geq \delta |V'_i| \geq \delta n/4 \), as otherwise by Lemma 2.5 we obtain an odd subgraph of size at least \( |V'_i|/61 \geq n/244 \). Our goal now is to show that under these assumptions we can add an edge to \( M_i \) while maintaining \( |X_{i+1}| \geq \frac{|V_i|}{40} \).

Consider first the case where every \( v \in L := L(G[V'_i]; \beta) \) satisfies \( d(v, X_i) \geq d(v, V_i)/40 \). By Lemma 2.4 applied to the bipartite graph between \( X_i \) and \( L \), using the fact that

\[
|X_i| \leq \frac{4n}{T} = \frac{n}{2500} \leq \frac{|L|}{44} ,
\]

we conclude that there are \( v \in R_i, u \in U_i \) with \( d(v) \geq (1 + 19\beta)d(u, R_i) \). Since \( uv \in E(G) \) and both \( u, v \not\in L \), it follows that

\[
d(v) < (1 + 3\beta)d(u) .
\]

Moreover, since \( u \in U_i \) we have:

\[
d(u, R_i) \geq (1 - 8\beta)d(u) .
\]

All in all, since \( d(v) \geq (1 + 19\beta)d(u, R_i) \) we conclude that

\[
(1 + 3\beta)d(u) > d(v) \geq (1 + 19\beta)d(v, R_i) > (1 + 19\beta)(1 - 8\beta)d(u) > (1 + 3\beta)d(u) ,
\]

a contradiction.

Therefore, we can assume that \( |U_i| \leq (1 + 19\beta)|R_i| \) for all \( 1 \leq i \leq k \). Looking at the induced subgraph \( G[U_i] \), we note that it has vertex \( v_i \) of degree \( |R_i| - 1 \geq \frac{9|R_i|}{10} \geq \frac{9}{10(1 + 19\beta)}|U_i| \). By applying Lemma 2.1 to \( G[U_i] \) we find an induced odd subgraph \( O_i \) of \( G[U_i] \) of size at least \( \frac{9}{20(1 + 19\beta)}|U_i| = \frac{9|U_i|}{39} \).

Finally, since all \( U_i \)'s are disjoint, there are no edges between any two such \( U_i \)'s and since \( V_2 \subseteq \bigcup U_i \), we conclude that \( O = \bigcup_{i=1}^k O_i \) is an induced odd subgraph of size at least \( \frac{9|V_2|}{39} > n/61 \). This completes the proof. \( \square \)
we derive that there is an edge \( xv \) with \( x \in X_i \) and \( v \in L \) and \( d(x, L) \geq 44d(v, X_i) \geq 1.1d(v, V_i) > 0 \). Then we can define \( M_{i+1} \) by adding \( xv \) to \( M_i \) and setting \( U_{i+1} := U_i \cup \{x\} \) and \( W_{i+1} := W_i \cup \{v\} \). By doing so we obtain that

\[
|X_{i+1}| = |N(U_{i+1}) \setminus (W_{i+1} \cup N(W_{i+1})| \\
\geq |N(U_i) \setminus (W_i \cup N(W_i))| + |N(x, V_i)| - |N(v, V_i)| - |N(v, X_i)| \\
= |X_i| + d(x, V_i) - d(v, X_i) - d(v, V_i) \\
\geq |X_i| + d(x, V_i) \left(1 - \frac{1}{44} - \frac{10}{11}\right) \\
> |X_i| + \frac{3d(x, V_i)}{44}.
\]

Moreover, since we clearly have that

\[
|V_{i+1}| \geq |V_i| - d(x, V_i) - d(v, V_i) \geq |V_i| - \frac{21d(x, V_i)}{11},
\]

it follows that at least \( \frac{3/44}{21/11} > \frac{1}{40} \) proportion of the vertices deleted from \( V_i \) go to \( X_{i+1} \).

In the complementary case there exists a vertex \( v \in L \) with \( d(v, X_i) \leq d(v, V_i)/40 \). Let \( uv \) be an edge in \( G[V_i'] \) witnessing \( v \in L \) (that is, \( |N(u, V_i) \setminus N(v, V_i)| \geq \beta |N(u, V_i) \cup N(v, V_i)| \)). Then we can define \( M_{i+1} \) by adding \( uv \) to \( M_i \), and set \( U_{i+1} := U_i \cup \{u\} \) and \( W_{i+1} := W_i \cup \{v\} \). In this case we have:

\[
|X_{i+1}| = |N(U_{i+1}) \setminus (W_{i+1} \cup N(W_{i+1})| \\
\geq |N(U_i) \setminus (W_i \cup N(W_i))| + |N(u, V_i) \setminus N(v, V_i)| - |N(v, X_i)| \\
= |X_i| + |N(u, V_i) \setminus N(v, V_i)| - |N(v, X_i)| \\
\geq |X_i| + \beta |N(u, V_i) \cup N(v, V_i)| - |N(v, X_i)| \\
\geq |X_i| + (\beta - \frac{1}{40})|N(u, V_i) \cup N(v, V_i)|.
\]

Moreover, since we have \( |V_{i+1}| \geq |V_i| - |N(u, V_i) \cup N(v, V_i)| \), at least \( \beta - \frac{1}{40} = \frac{1}{40} \) proportion of the vertices deleted from \( V_i \) go to \( X_{i+1} \).

All in all, in each step, either we find an odd subgraph of size at least \( \frac{n}{8} \) (in case that we have “many” isolated vertices, or that \( |X_i| \geq \frac{40n}{7} \), or that \( L(G[V_i']); \beta \)) is “large”), or we can keep \( X_i \) of size at least \( \frac{|V| \setminus V_i}{40} \). In particular, if the latter case holds until \( |V_i| < n/2 \), we obtain that \( |X_i| \geq \frac{n}{80} \) and we are done by Lemma 2.3. This completes the proof.

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\section*{References}