Sparse pancyclic subgraphs of random graphs

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Abstract

It is known that the complete graph $K_n$ contains a pancyclic subgraph with $n + (1 + o(1)) \cdot \log_2 n$ edges, and that there is no pancyclic graph on $n$ vertices with fewer than $n + \log_2(n - 1) - 1$ edges. We show that, with high probability, $G(n, p)$ contains a pancyclic subgraph with $n + (1 + o(1)) \log_2 n$ edges for $p \geq p^*$, where $p^* = (1 + o(1)) \ln n/n$, right above the threshold for pancyclicity.

1 Introduction

Say that a graph $G$ is pancyclic if $G$ contains a cycle of every length between 3 and $|V(G)|$. See monograph [6] for generic information on pancyclic graphs. In his influential paper on pancyclic graphs, Bondy [2] asked what is the minimum number of edges in a pancyclic $n$-vertex graph. This can be rephrased as the minimum number of edges in a pancyclic subgraph of $K_n$, which motivates the following definition.

Definition 1. Say that a pancyclic graph $G$ on $n$ vertices has pancyclicity excess $k$, and denote $\text{Pex}(G) = k$, if the minimum number of edges in a pancyclic subgraph of $G$ is $n + k$.

In other words, a pancyclic subgraph of $G$ achieving the minimum number of edges is formed by a Hamilton cycle and $\text{Pex}(G)$ additional chords. In his paper, Bondy stated that, for every $n$,

$$\log_2(n - 1) - 1 \leq \text{Pex}(K_n) \leq \log_2 n + \log^* n + O(1),$$

and did not provide a proof. Shi [10] later asserted the lower bound, by showing that an $n$-vertex graph with $n + k$ edges contains at most $2^{k+1} - 1$ distinct cycles, so every subgraph of $K_n$ with fewer than $n + \log_2(n - 1) - 1$ edges must have fewer than $2^{\log_2(n-1)} - 1 = n - 2$ cycles in total, regardless of their lengths. On the other hand, there are constructions for every $n$ of an $n$-vertex pancyclic graph with $\log_2 n + \log^* n + O(1)$ chords (see e.g. [6], Chapter 4.5), so $\text{Pex}(K_n) \leq \log_2 n + \log^* n + O(1)$. What is the exact value of $\text{Pex}(K_n)$ within this range is still an open question.

In this paper, we study the typical behaviour of $\text{Pex}(G)$, for $G \sim G(n, p)$. Cooper and Frieze [4] showed that, for $p \in [0, 1]$, the limiting probability of $G \sim G(n, p)$ being pancyclic is

$$\lim_{n \to \infty} \mathbb{P}(G(n, p) \text{ is pancyclic}) = \begin{cases} 1 & \text{if } np - \log n - \log \log n \to \infty; \\
e^{-e^{-c}} & \text{if } np - \log n - \log \log n \to c; \\
0 & \text{if } np - \log n - \log \log n \to -\infty. \end{cases}$$

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Here and later, if the base of the logarithm is not stated then it is the natural base. The above expression is also the limiting probability of \( G \) being Hamiltonian, and the limiting probability of \( \delta(G) \geq 2 \). In particular, the three properties have the same threshold.

Clearly, \( \text{Pex}(G) \geq \log_2(n - 1) - 1 \) for every pancyclic graph \( G \) on \( n \) vertices. On the other hand, Cooper [3] showed that if \( p \) is above the pancyclicity threshold, then with high probability \( G \sim G(n, p) \) is a so called \( 1\text{-pancyclic} \) graph, that is, it contains a Hamilton cycle \( H \) with the property that, for every \( \ell \in [3, n - 1] \), there is an edge \( e \in E(G) \) such that \( H \cup \{e\} \) contains a cycle of length \( \ell \) and a cycle of length \( n - \ell + 2 \). Observe that if \( G \) is a 1-pancyclic \( n \)-vertex graph then \( \text{Pex}(G) \leq \lceil \frac{n-3}{2} \rceil \). So Cooper’s result implies that \( \text{Pex}(G(n, p)) \leq \lceil \frac{n-3}{2} \rceil \) with high probability, for all \( p \) above the pancyclicity threshold.

Our result in this paper shows that, for \( G \sim G(n, p) \), the pancyclicity excess of \( G \) is, typically, very close to the above stated general lower bound.

**Theorem 1.** There is \( p^* = p^*(n) = (1 + \varepsilon(n)) \cdot \frac{\log n}{n} \), where \( \varepsilon(n) = O \left( \frac{1}{\log \log n} \right) \), such that, if \( p \geq p^* \) and \( G \sim G(n, p) \), then with high probability \( g \) is pancyclic with \( \text{Pex}(G) = (1 + o(1)) \cdot \log_2 n \).

It is worth noting that we did not attempt to optimize the error term \( \varepsilon(n) \), opting rather for a more simple proof. We therefore leave the question of whether \( \text{Pex}(G(n, p)) \) also typically satisfies \( \text{Pex}(G(n, p)) = (1 + o(1)) \cdot \log_2 n \) for all \( p \) above the pancyclicity threshold as an open question.

**Paper structure** In Section 2 we introduce definitions and notation required for the rest of the paper, as well as auxiliary results to be used in our proof. In Section 3 we introduce a construction of a subgraph of a given \( n \)-vertex graph, which, if successful, produces a subgraph with \( n + (1 + o(1)) \cdot \log_2 n \) edges. In Section 4 we show that, with high probability, the construction is possible in \( G(n, p) \) for \( p \geq p^* \), and in Section 5 we complete the proof of Theorem 1 by showing that the constructed subgraph is pancyclic.

## 2 Preliminaries

### 2.1 Definitions and notation

The following graph theoretic notation is used throughout the paper.

Let \( G \) be a graph and \( U, W \subseteq V(G) \) vertex subsets. We denote by \( E_G(U, W) \) the set of edges of \( G \) with vertex in \( U \) and one vertex in \( W \), and \( e_G(U, W) = |E_G(U, W)| \). We let \( G[U] \) denote the subgraph induced by \( G \) on the vertex subset \( U \), by \( E_G(U) \) the set of edges in \( G[U] \), and by \( e_G(U) \) its size. We denote by \( N_G(U) \) the (external) neighbourhood of \( U \), that is, the set of vertices in \( V(G) \) \setminus U \) adjacent to a vertex of \( U \). The degree of a vertex \( v \in V(G) \), denoted by \( d_G(v) \), is the number of edges of \( G \) incident to \( v \).

We let \( \mathcal{L}(G) \) denote the set of cycle lengths found in \( G \), that is, \( \mathcal{L}(G) \) is the set of integers \( \ell \) such that \( G \) contains a cycle of length \( \ell \).

While using the above notation we occasionally omit \( G \) if the identity of the specific graph is clear from the context.

We occasionally suppress the rounding signs to simplify the presentation.

Finally, we require the following definition.

**Definition 2.** A graph \( G \) is called a \((k, \alpha)\)-expander if every subset \( U \subseteq V(G) \) with \(|U| \leq k \) satisfies \(|N_G(U)| \geq \alpha \cdot |U|\).
2.2 Auxiliary results

**Theorem 2.1** (Cycle lengths in $G(n,p)$, a corollary of Łuczak [9]). Let $p = p(n)$ be such that $np \to \infty$ and let $G \sim G(n,p)$. Then, with high probability, $[3, 0.99n] \subseteq L(G)$.

**Theorem 2.2** (Tree embeddings in expanders, a corollary of [7] as given in [1]). Let $N, \Delta$ be integers, and let $G$ be a graph. Assume that there exists an integer $k$ such that

1. For every $U \subseteq V(G)$ with $|U| \leq k$ we have $|N_G(U)| \geq \Delta \cdot |U| + 1$;

2. For every $U \subseteq V(G)$ with $k < |U| \leq 2k$ we have $|N_G(U)| \geq \Delta \cdot |U| + N$.

Then, for every $v \in V(G)$ and every rooted tree $T$ with at most $N$ vertices and maximum degree at most $\Delta$, the graph $G$ contains a copy of $T$ rooted in $v$.

**Lemma 2.1** (Hamiltonicity and expansion of $G(n,p)$, see e.g. [8], Section 4). Let $p = p(n)$ be such that $np - \log n - \log \log n \to \infty$, and let $G \sim G(n,p)$. Then, with high probability, there is a subset $S \subseteq V(G)$ of $\frac{n}{4}$ vertices, such that for every $s \in S$ there is a subset $T_s \subseteq V(G)$ of $\frac{n}{4}$ vertices, and for every $t \in T_s$ there is a Hamilton path between $s$ and $t$.

3 The constructed pancyclic subgraph

We emulate (an approximation of) the construction in [6].

**Definition 3.** Let $G$ be a graph and $H \subseteq G$ be a Hamilton cycle, and let $2 \leq \ell \leq n - 2$. We say that an edge $e \in E(G)$ is an $\ell$-shortcut with respect to $H$ if (at least) one of the two intervals on $H$ that connects the two endpoints of $e$ has length $\ell + 1$.

The motivation behind this definition is that by using $H$ and an $\ell$-shortcut we can find a cycle of length $n - \ell$ in $G$, by replacing an interval of length $\ell + 1$ with a single edge (the $\ell$-shortcut). In the construction described in [6], one creates a sparse pancyclic graph by taking an $n$-cycle $H$ and $K$ shortcuts $e_0, e_1, \ldots, e_K$, where $K$ is such that $\frac{1}{2}n \leq 2^{K+1} + K - 1 \leq n$ and $e_i$ is a $2^i$-shortcut. Additionally, these shortcuts are consecutive on the cycle, so that $e_i, e_{i+1}$ and their corresponding intervals intersect in a vertex $v_i$. By taking intervals from the cycle $H$ and a subset of shortcuts, one can now encode a cycle of every length between $n$ and $n - 2^{K+1} + 1$. Next, by adding the edge between the first vertex of $e_0$ and the second vertex of $e_K$, all cycle lengths between $K + 2$ and $2^{K+1} + K$ can be encoded. This leaves out only a subset of cycle lengths contained in $[5, K+1]$, and adding these lengths to the set of cycle lengths in the graph can be done by inserting $O(\log^* n)$ additional edges. For the full details of the construction, we refer the reader to [6] Chapter 4.5.

We approximate this construction by finding a Hamilton cycle and shortcuts to encode an interval of $L = \Omega\left(\frac{n}{\sqrt{\log n}}\right)$ consecutive cycle lengths. Like in the deterministic version, we will utilize binary encoding of the cycle lengths, so that the number of required shortcuts is $(1 + o(1)) \log_2 n$. Additionally, we will require the shortcuts to reside on a short interval of the cycle (where in the deterministic version they intersected each other in a vertex). Next, by adding certain edges to the subgraph we can add an interval of $L$ cycle lengths with each such added edge. If the said additional edges are chosen well (which we will show is possible to do with high probability), one can get a union of $O(\sqrt{\log n})$ of these intervals that covers all the lengths between some initial length $\ell^* = (1 + o(1)) \log_2 n$ and $n$. 

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To handle cycle lengths shorter than \( \ell^* \) we will show that, with high probability, almost all of them (that is, all but \( o(\log n) \) cycle lengths in \([3, \ell^*]\)) can be encoded by \( o(\log n) \) carefully chosen shortcuts, this time utilizing an encoding in base \( b = \lceil \log \log n \rceil \). The remaining unencoded cycle lengths, which constitute a subset of \([3, \ell^*]\) of size \( o(\log n) \), can now be added one-by-one by using at most \( o(\log n) \) additional edges, with high probability.

Let 
\[
p_1 = p_5 = \frac{2 \log \log n}{n}, \quad p_2 = p_3 = \frac{50 \log n}{n \cdot \log \log n}, \quad p_4 = \frac{\log n + 10 \sqrt{\log n}}{n},
\]
and let 
\[
p^* = p^*(n) = 1 - \frac{5}{\prod_{i=1}^{5} (1 - p_i)}.
\]

Letting \( \varepsilon(n) := \frac{n}{\log n} \cdot p^* - 1 \) we get that \( \varepsilon(n) = O\left(\frac{1}{\log \log n}\right) \), and since the property \( \text{Pex}(G) \leq k \) is monotone increasing, it suffices to prove that \( \text{Pex}(G) \leq (1 + o(1)) \log_2 n \) holds with high probability for \( G(n, p^*) \sim \bigcup_{i=1}^{5} G(n, p_i) \). We note that we did not attempt to optimize the value of \( \varepsilon(n) \) determined by \( p_1, \ldots, p_5 \), aiming rather for simplicity.

Denote 
\[
\ell_i := 2^i + 1,
\]
and 
\[
\beta = \beta(n) := \frac{2(\log \log n)^2}{\log n}, \quad d = d(n) = \lfloor \log_{(5\beta)^{-1}}(n/200) \rfloor.
\]

Note that 
\[
d = \lfloor \log_{(5\beta)^{-1}}(n/200) \rfloor = (1 + o(1)) \cdot \frac{\log(n/200)}{-\log(5\beta)} = (1 + o(1)) \cdot \frac{\log n}{\log \log n}.
\]

For \( 1 \leq i \leq 5 \) let \( G_i \sim G(n, p_i) \). We divide the construction into five steps, where in the \( i \)'th step we sample \( G_i \) to try and produce a subgraph \( H_i \subseteq \bigcup_{j=1}^{i} G_j \). If the construction is successful, the produced subgraph \( H_5 \) will be pancyclic with \( |E(H_5)| = n + (1 + o(1)) \cdot \log_2 n \). The steps of our construction are as follows.

1. Let 
\[
K_0 := \lfloor \log_2 \left( \frac{\log n}{6 \log \log \log n} \right) \rfloor,
\]
and 
\[
b := \lfloor \log \log n \rfloor, \quad t := \lfloor \log_6 \log n \rfloor.
\]

Find a set of vertex disjoint cycles \( C_0, \ldots, C_{K_0}, C_{\text{short}} \) in \( G_1 \) of respective lengths \( \ell_0 + 1, \ell_1 + 1, \ldots, \ell_{K_0} + 1, t \cdot b + 1 \). The first \( K_0 + 1 \) cycles will later become the first \( K_0 + 1 \) shortcuts, and their corresponding intervals, where the edges of \( C_{\text{short}} \) will become the shortcuts required to handle short cycles. For every \( 0 \leq i \leq K_0 \), choose an arbitrary edge \( e_i \in C_i \) to serve as the shortcut. Denote \( H_1 = C_{\text{short}} \cup \bigcup_{i=0}^{K_0} C_i \).

2. For every \( 0 \leq i \leq K_0 \), find a path of length \( d + 2 \) in \( G_2 \) between the second vertex of \( e_i \) and the first vertex of \( e_{i+1} \) (where for \( i = K_0 \) the path is between \( e_{K_0} \) and \( e_0 \)), so that the \( K_0 + 1 \) paths are pairwise vertex disjoint from each other, and internally vertex disjoint from \( V(H_1) \). Call the cycle formed by the union of the paths and the shortcuts \( C^* \) and denote \( \ell^* := e(C^*) \), \( H_2 := H_1 \cup C^* \). We have 
\[
\ell^* = (1 + o(1)) \cdot K_0 \cdot d = (1 + o(1)) \cdot \log_2 n.
\]
4. Construct a Hamilton cycle by connecting the vertex of \( e^* \) and the vertex of \( e_{\text{short}} \) by a path \( P^* \) of length \( d + 2 \), internally disjoint from all previous construction, and denote \( H_3 := H_2 \cup P^* \cup \{P_{i,j} | i, j \leq b \} \cup \bigcup_{i=K_0}^K C_i \).

Finally for this step, connect one vertex of \( e^* \) to one vertex of \( e_{\text{short}} \) by a path \( P^* \) of length \( d + 2 \), internally disjoint from all previous construction, and denote \( H_3 := H_2 \cup P^* \cup \{P_{i,j} | i, j \leq b \} \cup \bigcup_{i=K_0}^K C_i \).

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Finally for this step, connect one vertex of \( e^* \) to one vertex of \( e_{\text{short}} \) by a path \( P^* \) of length \( d + 2 \), internally disjoint from all previous construction, and denote \( H_3 := H_2 \cup P^* \cup \{P_{i,j} | i, j \leq b \} \cup \bigcup_{i=K_0}^K C_i \).

Then the constructed \( H_4 := H_3 \cup P \) contains the Hamilton cycle \( C_H \) and \( K + b \cdot t + 3 \) additional edges, and all cycle lengths in \([((d + b + 1) \cdot t + 1, \ell^* + L] \cup [n - L, n]\).
5. Let \( m := \lfloor n \cdot 2^{-K} \rfloor = o(\log n) \). For \( 3 \leq i \leq m \) find an \( \ell^*_i \)-shortcut \( f_i \) in \( G_5 \), where \( \ell^*_i \) is an integer such that \( |\ell^*_i - i \cdot 2^K| \leq n^{0.9} \), and such that the \((\ell^*_i + 1)\)-path accompanying \( f_i \) contains \( V \left( C^* \cup \bigcup_{i=0}^{K} C_i \right) \).

We now have that \( H_4 \cup \{ f_i \} \) contains all cycle lengths in \([\ell^*_i + 2 - L, \ell^*_i + 2]\). Since \( \ell^*_i \geq \ell^*_{i+1} - L \) for all \( i \), and \( \ell^* + L \geq \ell^*_3 + 2 - L, \ell^*_m \geq n - L \), we get that \( H_4 \cup \{ f_3, ..., f_m \} \) contains all cycle lengths in \([((d + b + 1) \cdot t) + 1, n]\).

Finally, add the remaining at most \( (d + b + 1) \cdot t = o(\log n) \) cycle lengths by finding in \( G_5 \) an edge \( g_\ell \) that constitutes an \((\ell - 2)\)-shortcut with respect to \( C_H \), for every \( \ell \in [3, (d + b + 1) \cdot t] \).

This step adds at most \( m + (d + b + 1) \cdot t \) edges to the constructed subgraph \( H_5 := H_4 \cup \{ f_3, ..., f_m \} \cup \{ g_3, ..., g_{(d+b+1)t} \} \).

Observe that the resulting subgraph \( H_5 \) is a union of the Hamilton cycle \( C_H \) and an additional set of edges

\[
\{ e_0, ..., e_K, e^*, e_{\text{short}} \} \cup \{ e_{i,j} \mid 0 \leq i \leq t - 1, 0 \leq j \leq b - 1 \} \cup \{ f_3, ..., f_m \} \cup \{ g_3, ..., g_{(d+b+1)t} \}.
\]

Therefore \( H_5 \) contains at most

\[
n + K + b \cdot t + m + (d + b + 1) \cdot t = n + (1 + o(1)) \cdot \log_2 n
\]

edges. In Section 4 we prove that the construction of \( H_5 \) we described is possible with high probability in \( G(n, p^*) \). In Section 5 we prove that \( H_5 \), if it exists as a subgraph of \( G \), is indeed pancyclic.
4 Finding the subgraph in $G(n, p)$

We follow the steps described in Section 3, and show that, in each step, the desired substructure of the respective random graph $G_i$, $i = 1, 2, 3, 4, 5$, exists with high probability.

We will denote the subgraph output by the $i$'th step of the construction (if successful) by $H_i$.

Step 1

Recall the notation

$$K_0 := \lceil \log_2 \left( \frac{\log n}{6 \log \log \log n} \right) \rceil, \; \ell_i := 2^i + 1$$

and

$$b := \lceil \log \log n \rceil, \; t := \lceil \log_2 n \rceil.$$

By Theorem 2.1, with high probability, $G_1 \sim G(n, p_1)$ (where $p_1 = \frac{2 \log \log n}{n}$) contains a sequence of cycles $C_0, C_1, ..., C_{K_0}$, of respective lengths $\ell_0 + 1, ..., \ell_{K_0} + 1, b \cdot t + 1$. The following lemma implies that these cycles are also typically vertex disjoint.

**Lemma 4.1.** With high probability, no two cycles of length at most $\ell_{K_0} + 1$ in $G_1$ intersect each other.

**Proof.** Using the union bound we can show that, with high probability, $G_1$ does not contain a subgraph with at most $2 \ell_{K_0} + 1 \leq \frac{\log n}{2 \log \log \log n}$ vertices and more edges than vertices, which implies the lemma. Indeed, the probability that such a subgraph exists is at most

$$2 \ell_{K_0} + 1 \sum_{k=4}^{2 \ell_{K_0} + 1} \binom{n}{k} \cdot \binom{\binom{k}{2}}{k + 1} \cdot p_1^{k+1} \leq \sum_{k=4}^{2 \ell_{K_0} + 1} (e^2 n p_1)^k \cdot k \cdot p_1 \leq \log^2 n \cdot (2e^2 \log \log n) \cdot \log \frac{n}{\ell_{K_0}} \cdot p_1 = o(1).$$

Step 2

Recall that $G_2 \sim G(n, p_2)$, where $p_2 = \frac{50 \log n}{n \log \log n}$. For each $0 \leq i \leq K_0$ let $\{s_i, t_i\} := e_i \in E(C_i)$ be an arbitrary edge of $C_i$.

Recall that $\beta := \frac{2(\log \log n)^2}{\log n}$ and $d = \lfloor \log_{\log(2\beta)-1}(n/200) \rfloor = (1 + o(1)) \cdot \frac{\log n}{\log \log n}$.  

**Lemma 4.2.** With high probability $G_2$ contains paths $Q_0, ..., Q_{K_0}$ such that

1. $Q_i$ is a path between $t_i$ and $s_{i+1}$ for $0 \leq i \leq K_0 - 1$, and $Q_{K_0}$ is between $t_{K_0}$ and $s_0$;
2. $Q_0, ..., Q_{K_0}$ all have length $d + 2$;
3. $Q_0, ..., Q_{K_0}$ are vertex disjoint, and are internally vertex disjoint from $V(H_1)$.

Recall that $K = \lceil \log_2 \left( \frac{n}{\sqrt{\log n}} \right) \rceil$ and $L = 2^K + 1$. Before getting to the proof of Lemma 4.2, we show the following claim.

**Claim 4.3.** With high probability, for every vertex subset $U \subseteq V(G)$ with $|U| \geq n - 2L$, there is a vertex subset $U^* \subseteq U$, with $|U^*| \geq (1 - \beta) \cdot n$, such that the induced subgraph $G_2[U^*]$ is a $(\beta n, 1/3\beta)$-expander.
Proof. First, observe that, with high probability, for every $U, W \subseteq V(G)$ disjoint subsets with $|U| = |W| = \beta n$ there is an edge in $G_2$ between $U$ and $W$. Indeed, the probability that there are such subsets with no edge between them is at most
\[
\left( \frac{n}{\beta n} \right)^2 \cdot (1 - p_2)^{2n^2} \leq \left( \frac{e^2}{\beta^2} \cdot \exp(-\beta np_2) \right)^{\beta n} \leq \left( \frac{\log^2 n}{\log n} \cdot \exp(-100 \log \log n) \right)^{\omega(1)} = o(1).
\]

Now, assume that $G_2$ has the aforementioned property. We reiterate an argument from [5] and show that, in this case, for every such $U$ there is $U^* \subseteq U$ with the desired properties.

For a given $U$, construct $U^*$ as follows. Set $U_0 = U$. For $i \geq 0$, if $|U_i| \geq (1 - \beta)n$ and there is $W_i \subseteq U_i$ with $|W_i| \leq \beta n$ and $|N_{G^*}[U_i](W_i)| \leq \frac{1}{\beta^3}|W_i|$, set $U_{i+1} = U_i \setminus W_i$. Otherwise, terminate the process with $U^* = U_i$.

Clearly, either the resulting $G_2[U^*]$ is a $(\beta n, 1/3\beta)$-expander, or $|U^*| < (1 - \beta) \cdot n$. In fact, in the latter case, it must be that $(1 - 2\beta) \cdot n \leq |U^*| < (1 - \beta) \cdot n$, since at most $\beta n$ vertices are removed in every step of the process. Suppose that this is the case, and denote $W := U \setminus U^*$. Then $|N_{G^*}(W)| \leq \frac{1}{\beta^3} |W| + |V(G) \setminus U| \leq \frac{2}{3} n$.

We therefore have that $W$ and $V(G) \setminus N_{G^*}(W)$ are subsets of size at least $\beta n$ with no edges between them, a contradiction to our assumption. $\square$

With Claim 4.3 at hand we are now able to prove Lemma 4.2 by appealing to Theorem 2.2.

Proof of Lemma 4.2. Assume that $G_2$ has the property in the assertion of Claim 4.3, and suppose that $Q_0, \ldots, Q_{i-1}$ have already been constructed. We attempt to construct $Q_i$.

Let $U := V(G) \setminus \left( V(H_1) \cup \bigcup_{j=0}^{i-1} V(Q_j) \right)$, so that
\[
|U| \geq n - 2b \cdot t - 2^{K_0 + 1} - K_0 \cdot (d + 2) \geq n - 2L,
\]
and let $U^* \subseteq U$ be a subset of size at least $(1 - \beta) \cdot n$ such that $G_2[U^*]$ is a $(\beta n, 1/3\beta)$-expander. Observe that $G_2[U^*]$ satisfies the conditions of Theorem 2.2 for $\Delta = \frac{1}{4\beta}, N = \frac{1}{50} n, k = \frac{1}{2} \beta n$.

Observe that, at this point, the edges of $G_2$ between $\{s_{i+1}, t_i\}$ ($s_0$ in the case $i = K_0$) and $U^*$ have not been sampled yet. The probability that $s_{i+1}$ does not have a neighbour in $U^*$ is at most $(1 - p_2)^{(1 - \beta)n} = o(K_0^{-1})$. Assume that there is such a neighbour, say $u$. By Theorem 2.2, $G_2[U^*]$ contains a complete $\frac{1}{4\beta}$-ary tree of depth $d$ rooted in $u$. This tree has at least $\frac{\beta}{10} \cdot n$ leaves. The probability that none of these leaves is a neighbour of $t_i$ in $G_2$ is at most
\[
(1 - p_2)^{\frac{\beta n}{40}} \leq \exp \left( -\frac{1}{40} \cdot \frac{50 \log n}{n \cdot \log \log n} \cdot \frac{2(\log n)^2}{\log n} \cdot n \right) = o(K_0^{-1}).
\]

Now, if indeed $t_i$ has a neighbour among the tree’s leaves, say $w$, the path from $s_{i+1}$ to $u$, down the the tree to $w$, and from $w$ to $t_i$ is a path of length $d + 2$ that intersects $V(C_{\text{short}}) \cup \bigcup_{j=0}^{K_0} V(C_j) \cup \bigcup_{j=0}^{K_0 - 1} V(Q_j)$ only in $\{s_{i+1}, t_i\}$.

Finally, for every $i$ we showed that the probability that such a path $Q_i$ does not exist is at most $o(K_0^{-1})$, and therefore, by the union bound, a sequence $Q_0, \ldots, Q_{K_0}$ as required exists with high probability. $\square$

Now $\bigcup_{i=0}^{K_0} Q_i$ is a cycle, denote it by $C^*$. We have
\[
|C^*| = (K_0 + 1) \cdot (d + 3) = (1 + o(1)) \cdot \log_2 n.
\]
Step 3

Recall that $G_3 \sim G(n, p_3)$, with $p_3 = \frac{50 \log n}{n \log \log n}$. Let $e_{K_0+1}, ..., e_K, e^*$ be distinct edges of $C^* \setminus \{e_0, ..., e_{K_0}\}$, such that $e^*$ is vertex disjoint from $e_{K_0+1}, ..., e_K$, and denote $e_i = \{s_i, t_i\}$ such that $s_i$ is the predecessor of $t_i$ on $C^*$ for all $i$, according to an arbitrary orientation of $C^*$.

As a preparation for a proof that the construction in Step 3 is possible with high probability, observe that $G_3$ and $G_2$ are drawn from the same distribution, and therefore Claim 4.3 also holds for $G_3$. That is, we have that, with high probability, for every $U \subseteq V(G)$ with $|U| \geq n - 2L$, there is $U^* \subseteq U$, with $|U^*| \geq (1 - \beta) \cdot n$, such that $G_3[U^*]$ is a $(\beta n, 1/3\beta)$-expander. In the proofs of the following two lemmas, we assume that indeed $G_3$ has this property.

**Lemma 4.4.** With high probability $G_3$ contains paths $Q_{K_0+1}, ..., Q_K$ such that

1. $Q_i$ is a path between $s_i$ and $t_i$ for $K_0 + 1 \leq i \leq K$;
2. $Q_i$ has length $\ell_i + 1$ for $K_0 + 1 \leq i \leq K$;
3. $Q_{K_0+1}, ..., Q_K$ are internally vertex disjoint from each other and from $V(H_2)$.

**Proof.** Suppose that $Q_{K_0+1}, ..., Q_{i-1}$ were found, and attempt to construct $Q_i$.

Here, as in Lemma 4.2, we will appeal to Theorem 2.2.

Let $U := V(G) \setminus (V(H_2) \cup \bigcup_{j=0}^{i-1} V(Q_j))$ and observe that

$$|U| \geq n - 2^{K+1} - K_0 \cdot (d+2) \geq n - 2L.$$

Let $U^* \subseteq U$ be a subset with at least $(1 - \beta) \cdot n$ vertices such that $G_3[U^*]$ is a $(\beta n, 1/3\beta)$-expander.

As in the proof of Lemma 4.2, $G_3[U^*]$ satisfies the conditions of Theorem 2.2 for the same parameters $\Delta = \frac{1}{2\beta}, N = \frac{1}{50} n, k = \frac{1}{2} \beta n$. Recall that $d = \lfloor \log_{(5\beta)^{-1}}(n/200) \rfloor$, and let $T$ be the tree consisting of two complete $\frac{1}{50}n$-ary trees of depth $d$, whose roots are connected by a path of length $\ell_i - 2d - 1$ (which is positive for $i > K_0$). By Theorem 2.2, $U^*$ contains a copy of $T$ (rooted at an arbitrary vertex).

Let $L_{s_i}, L_{t_i} \subseteq U^*$ be the sets of leaves of the embedding of $T$ that correspond to the first and the second subtrees of $T$ that are connected by a path. By the definition of $T$ we have that $|L_{s_i}| = |L_{t_i}| \geq \frac{\beta}{50} n$.

Observe that $s_i$ and $t_i$ each belong to at most one other path among $Q_{K_0+1}, ..., Q_{i-1}$. For $v \in \{s_i, t_i\}$ do the following. If $v \notin V(Q_j)$ for $K_0 < j \leq i - 1$, then choose an arbitrary subset of $L_v$ of size $\frac{1}{2} |L_v|$ and connect $v$ to one of the vertices in the subset by an edge from $E(G_3)$, if there is a neighbour of $v$ in the subset. If $v \in V(Q_j)$ for some $K_0 < j \leq i - 1$, connect $v$ to a vertex of $L_v$ by a previously unexposed edge from $E(G_3)$, if such an edge exists. In both cases, at least $\frac{1}{2} |L_v| \geq \frac{\beta}{50} n$ edges are considered. Therefore, the probability that there is no edge between $v$ and (the subset of) $L_v$ is at most

$$(1 - p_3)^{\beta n / 80} \leq \exp \left( -\frac{1}{80} \cdot \frac{50 \log n}{n \cdot \log \log n} \cdot \frac{2(\log \log n)^2}{\log n} \cdot n \right) = \exp \left( -\frac{5}{4} \log \log n \right) = o(K^{-1}).$$

In the case that an edge is found, denote it by $e_v$.

Now, $e_{s_i}, e_{t_i}$ along with the path of length $\ell_i - 1$ in $T$ between the two leaves connected to $s_i$ and $t_i$ constitute a path between $s_i$ and $t_i$ of length $\ell_i + 1$, which is internally contained in $U$, and therefore, by the definition of $U$, is internally vertex disjoint from $V(H_2), Q_{K_0}, ..., Q_{i-1}$. Call this path $Q_i$.

The probability that there is $K_0 + 1 \leq i \leq K$ for which we did not manage to find a path $Q_i$ in this way is at most the probability that $G_3$ does not have the property in the assertion of Claim 4.3, or $s_i$ or $t_i$ did not have a leaf neighbour in the embedding of $T$ for some $i$, both of which are of order $o(1)$. 

\[ \Box \]
For $K_0 + 1 \leq i \leq K$, denote by $C_i$ the cycle $Q_i \cup \{e_i\}$.

Let $v_1, \ldots, v_{bt+1}$ be the vertices of $C_{short}$ according to their order on the cycle, let $\sigma : \{0, 1, \ldots, t - 1\} \times \{0, 1, \ldots, b - 1\} \to [tb]$ be a bijection and denote $e_{i,j} = \{v_{\sigma(i,j)}, v_{\sigma(i,j)+1}\}$ and $e_{short} = \{v_1, v_{bt+1}\}$.

**Lemma 4.5.** With high probability $G_3$ contains paths $\{P_{i,j} \mid 0 \leq i \leq t - 1, 0 \leq j \leq b - 1\}$ such that

1. $P_{i,j}$ is a path between $v_{\sigma(i,j)}$ and $v_{\sigma(i,j)+1}$, for all $i$ and $j$;
2. $P_{i,j}$ has length $d + 2 + j \cdot b^i$, for all $i$ and $j$;
3. $\{P_{i,j} \mid 0 \leq i \leq t - 1, 0 \leq j \leq b - 1\}$ are internally vertex disjoint from each other and from $V(H_2) \cup \bigcup_{i=K_0+1}^K V(C_i)$.

**Proof.** The proof follows similar steps to the proofs of Lemma 4.2 and Lemma 4.4 by appealing to Theorem 2.2. Assume that $P_{\sigma^{-1}(1)}, \ldots, P_{\sigma^{-1}(k-1)}$ have already been found, and let $(i, j) = \sigma^{-1}(k)$. Let $U^* \subseteq U := V(G) \setminus \left(V(H_2) \cup \bigcup_{r=K_0+1}^K V(C_r) \cup \bigcup_{r=1}^{k-1} V(P_{\sigma^{-1}(r)})\right)$ be such that $|U^*| \geq (1 - \beta) \cdot n$ and $G_3[U^*]$ is a $(\beta n, 1/3\beta)$-expander.

Let $s_{k+1}$ be a neighbour of $v_{k+1}$ from among the first $\frac{n}{\log \log n}$ vertices of $U^*$. The edges between $v_{k+1}$ and $U^*$ in $G_3$ have not been sampled yet, and the probability that no such neighbour exists is at most $(1 - p_3) \frac{n}{\log \log n} = o(1/tb)$.

As in the previous proofs, by Claim 4.3 and Theorem 2.2 we have that $G_3[U^*]$ contains a tree which consists of a complete $\frac{1}{5\beta}$-ary tree of depth $d$ with a path of length $j \cdot b^i$ (this can possibly be 0, in which case the path is just a vertex) attached to its root, and such that the other end of the path is $s_{k+1}$. Let $L_k$ be the set of leaves in the tree, so that $|L_k| \geq \frac{\beta}{40} n$. At most $\frac{n}{\log \log n}$ of the leaves were considered as neighbours of $v_k$ in previous steps, and therefore the probability that $v_k$ does not have a neighbour $t_k$ from among the remaining leaves is at most $(1 - p_3)^{\frac{n}{\log \log n}} = o(1/tb)$. Now the path from $v_{k+1}$, through $s_{k+1}$, along the $(j b^i)$-path, down the tree to $t_k$ and then to $v_k$, satisfies all the requirements to be $P_{i,j}$.

The probability that for some $k$ one of the vertices $s_{k+1}, t_k$ was not found is of order $o(1)$, and therefore, with high probability, this construction ends successfully.

We remain with finding a path between $e_{short}$ and $e^*$, which is done in the following lemma.

**Lemma 4.6.** With high probability $G_3$ contains a path $P^*$ of length $d + 2$ between a vertex of $e_{short}$ and a vertex of $e^*$, which is internally disjoint from $V(H_2) \cup \bigcup_{i=K_0+1}^K V(C_i) \cup \bigcup_{i,j} V(P_{i,j})$.

**Proof.** As in previous constructions in this step, let $U = V(G) \setminus \left(V(H_2) \cup \bigcup_{i=K_0+1}^K V(C_i) \cup \bigcup_{i,j} V(P_{i,j})\right)$ and let $U^*$ be a large subset spanning an expander. At least $\frac{1}{2} n$ vertices of $U^*$ have not yet been considered as neighbours of $v_{bt+1}$, and with high probability at least one of them is, denote it by $s$. By Claim 4.3 and Theorem 2.2, $G_3[U^*]$ contains a complete $\frac{1}{5\beta}$-ary tree of depth $d$ rooted in $s$. As none of the edges in $G_3$ of the vertices of $e^*$ have been sampled yet, with high probability there is an edge between one of them and the tree’s at least $\frac{\beta}{40} n$ leaves, which together with the path from the leaf to $s$ and with $\{s, v_{bt+1}\}$ forms a path $P^*$ satisfying the conditions.

**Step 4**

Denote $X = V(G) \setminus V(H_3)$, and let $s_H \in e^*, t_H \in e_{short}$ be the vertices of $e^*, e_{short}$ not already connected by $P^*$. 


Claim 4.7. With high probability there is a path \( P \) in \( G_4 \) between \( s_H \) and \( t_H \), whose vertex set is \( V(P) = X \cup \{s_H, t_H\} \).

Proof. Consider the induced subgraph \( G_4[X] \sim G(|X|, p_4) \). We have

\[
|X| \cdot p_4 \geq (n - 2L) \cdot \left( \frac{\log n + 10\sqrt{\log n}}{n} \right) \geq \log n \cdot \left( 1 - \frac{4}{\sqrt{\log n}} \right) \cdot \left( 1 + \frac{10}{\sqrt{\log n}} \right) = \log n + \omega(\log \log n) = \log |X| + \omega(\log \log |X|)
\]

Therefore, by Lemma 2.1, there is a set \( S \subseteq X \) with \( |S| = \frac{1}{3} |X| \geq \frac{1}{5} n \), and for every \( s \in S \) there is a subset \( T_s \subseteq X \) with \( |T_s| \geq \frac{1}{3} |X| \geq \frac{1}{5} n \), such that there is a Hamilton path in \( G_4[X] \) between \( s \) and \( t \) for every \( t \in T_s \).

The set \( E_{G_4}(s_H, X) \) has not yet been sampled. The probability that \( s_H \) has no neighbour in \( S \) is at most \( (1 - p_4)^{n/5} = o(1) \). Assume that there is one, and denote it by \( s \). Similarly, the probability that \( t_H \) has no neighbour in \( T_s \) is at most \( (1 - p_4)^{n/5} = o(1) \), denote such a neighbour by \( t \). Now the Hamilton path in \( G_4[X] \) between \( s \) and \( t \), along with the edges \( \{s_H, s\}, \{t_H, t\} \), constitute a path \( P \) with \( V(P) = X \cup \{s_H, t_H\} \), as desired.

Denote the obtained Hamilton cycle \( H_3 \cup P \setminus (\{e_0, ..., e_K, e^*, e_{\text{short}}\} \cup \{e_{i,j}\}_{i,j}) \) by \( C_H \).

Step 5

Let \( m := |n \cdot 2^{-K}| \).

Lemma 4.8. With high probability \( G_5 \) contains edges \( f_3, ..., f_m \), such that the following hold for every \( i \).

1. There is \( \ell_i^* \in [i \cdot 2^K - n^{0.9}, i \cdot 2^K + n^{0.9}] \) such that \( f_i \) is an \( \ell_i^* \)-shortcut with respect to \( C_H \);
2. The \( (\ell_i^* + 1) \)-path on \( C_H \) that connects the vertices of \( f_i \) contains \( V(C^* \cup \bigcup_{i=0}^{K} C_i) \).

Proof. For every \( 3 \leq i \leq m \) there is a set of at least \( \frac{1}{5} \cdot n^{1.8} \) potential edges that satisfy the conditions. The probability that there is \( 3 \leq i \leq m \) for which none of these edges appears in \( G_5 \) is at most

\[
m \cdot (1 - p_5)^{n^{1.8}/5} = o(1).
\]

Lemma 4.9. With high probability \( G_5 \) contains an \( (\ell - 2) \)-shortcut with respect to \( C_H \) for every \( \ell \in [3, (d + b) \cdot t] \).

Proof. For a given \( \ell \in [3, (d + b) \cdot t] \), the probability that such an \( (\ell - 2) \)-shortcut does not exist is \( (1 - p_5)^n = o(1/\log n) \), and by the union bound we obtain the lemma, as \((d + b)t = o(\log n)\).
5 Proof of Theorem 1

The following lemma, referring to the subgraph $H_5 \subseteq G$ described in Section 3 and whose construction is shown to be possible with high probability in Section 4, completes the proof of Theorem 1.

**Lemma 5.1.** The subgraph $H_5$ is pancyclic.

*Proof.* Let $\ell \in [3, n]$. We show that $H_5$ contains a cycle of length $\ell$. We divide the proof into cases based on a subinterval of $[3, n]$ that $\ell$ resides in. The subintervals are covering $[3, n]$ but not necessarily disjoint, so $\ell$ may be covered by more than one subinterval.

- If $\ell \in [3, (d+b+1) \cdot t]$, then $g_\ell$ is an $(\ell-2)$-shortcut with respect to $C_H$, so that $g_\ell$ and its accompanying $(\ell-1)$-path form an $\ell$-cycle.

- If $\ell \in [(d+b+1) \cdot t + 1, (d+b+1) \cdot t + b^t]$ and $k = \ell - (d+b+1) \cdot t - 1$, so $0 \leq k \leq b^t - 1$ can be encoded in base $b$ using $t$ digits. Let $(k_{t-1}, k_{t-2}, \ldots, k_1, k_0)$ be its encoding, that is, $0 \leq k_i \leq b - 1$ for all $i$, and $k = \sum_{i=0}^{t-1} k_i b^i$. Then
  \[
  \{ e_{\text{short}} \} \cup \bigcup_{j=k_i} P_{i,j} \cup \bigcup_{j \neq k_i} \{ e_{i,j} \}
  \]
is a cycle of length $\ell$ in $H_5$. Indeed, it is a cycle, since it is the result of replacing a subset of the edges of $C_{\text{short}}$ with internally disjoint paths, and its length is
  \[
  1 + (b - 1) \cdot t + \sum_{i=0}^{t-1} e(P_{i,k_i}) = 1 + (b - 1) \cdot t + \sum_{i=0}^{t-1} (d + 2 + k_i b^i) = 1 + (d + b + 1) \cdot t + k = \ell.
  \]

- If $\ell \in [\ell^*, \ell^* + L]$, let $k = \ell - \ell^*$, so $0 \leq k \leq 2^{K+1} - 1$ can be encoded by $K + 1$ binary digits, say $k = \sum_{i=0}^{K} k_i 2^i$, where $k_i \in \{0, 1\}$. Then
  \[
  \left( C^* \cup \bigcup_{i,k_i=1} C_i \right) \setminus \{ e_i \mid k_i = 1 \}
  \]
is a cycle in $H_5$ with length $\ell$. It is a cycle because it is the result of replacing edges of $C^*$ with internally disjoint paths. The length is indeed
  \[
  \ell^* + \sum_{i=0}^{K} k_i (e(C_i) - 1) = \ell^* + \sum_{i=0}^{K} k_i 2^i = \ell^* + k = \ell.
  \]

- If $\ell \in \left[ \left( \frac{1}{2} i - \frac{4}{5} \right) \cdot L, \left( \frac{1}{2} i - \frac{1}{5} \right) \cdot L \right]$, where $3 \leq i \leq m$, then in particular $\ell \in [\ell^*_i + 2 - L, \ell^*_i + 2]$. Similarly to the previous case, let $\ell^*_i + 2 - \ell = k = \sum_{i=0}^{K} k_i 2^i$, where $k_i \in \{0, 1\}$. Denote by $C^*_i$ the cycle of length $\ell^*_i + 2$ comprised of $f_i$ and its accompanying $(\ell^*_i + 1)$-path. Then
  \[
  \left( C^*_i \setminus \bigcup_{i,k_i=1} C_i \right) \cup \{ e_i \mid k_i = 1 \}
  \]
is a cycle of length $\ell^* + 2 - k = \ell$. 


If $\ell \in [n - L, n]$ then for $n - \ell = k = \sum_{i=0}^{K} k_i 2^i, k_i \in \{0, 1\}$, we get a cycle

$$\left( C_H \setminus \bigcup_{i: k_i = 1} C_i \right) \cup \{ e_i \mid k_i = 1 \}$$

of length $n - k = \ell$.

Observe that

$$(d + b + 1) \cdot t + b^t \geq b^{\log_b n} = \log n \geq \ell^*;$$

$$\ell^* + L \geq \left( \frac{1}{2} \cdot 3 - \frac{4}{5} \right) \cdot L;$$

$$\left( \frac{1}{2} \cdot i - \frac{1}{5} \right) \cdot L \geq \left( \frac{1}{2} \cdot (i + 1) - \frac{4}{5} \right) \cdot L;$$

$$\left( \frac{1}{2} \cdot m - \frac{1}{5} \right) \cdot L \geq \frac{1}{2} (n \cdot 2^{-K} - 1) \cdot (2^{K+1} - 1) - \frac{1}{5} L \geq n - \frac{1}{2} L - \frac{1}{5} L - O(\sqrt{\log n}) \geq n - L;$$

and therefore the subintervals indeed cover $[3, n]$, and so $H_5$ is pancyclic.

References


