Rainbow Hamilton cycles in random graphs and hypergraphs

Asaf Ferber and Michael Krivelevich

Abstract Let H be an edge colored hypergraph. We say that H contains a *rainbow* copy of a hypergraph S if it contains an isomorphic copy of S with all edges of distinct colors. We consider the following setting. A randomly edge colored random hypergraph $H \sim \mathscr{H}_c^k(n,p)$ is obtained by adding each k-subset of $[n]$ with probability p, and assigning it a color from $[c]$ uniformly, independently at random. As a first result we show that a typical $H \sim \mathscr{H}_c^2(n,p)$ (that is, a random edge colored graph) contains a rainbow Hamilton cycle, provided that $c = (1 + o(1))n$ and $p = \frac{\log n + \log \log n + \omega(1)}{n}$. This is asymptotically best possible with respect to both parameters, and improves a result of Frieze and Loh. Secondly, based on an ingenious coupling idea of McDiarmid, we provide a general tool for tackling problems related to finding "nicely edge colored" structures in random graphs/hypergraphs. We illustrate the generality of this statement by presenting two interesting applications. In one application we show that a typical $H \sim \mathscr{H}_c^k(n,p)$ contains a rainbow copy of a hypergraph S, provided that $c = (1 + o(1))|E(S)|$ and p is (up to a multiplicative constant) a threshold function for the property of containment of a copy of S. In the second application we show that a typical $G \sim \mathscr{H}_c^2(n,p)$ contains $(1 - o(1))np/2$ edge disjoint Hamilton cycles, each of which is rainbow, provided that $c = \omega(n)$ and $p = \omega(\log n/n)$.

Asaf Ferber

Department of Mathematics, Yale University, and Department of Mathematics, MIT, e-mail: asaf.ferber@yale.edu, ferbera@mit.edu.

Michael Krivelevich

School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, 6997801, Israel, Research supported in part by USA-Israel BSF grant 2010115 and by grant 912/12 from the Israel Science Foundation, e-mail: krivelev@post.tau.ac.il.

1 Introduction

We consider the following model of edge-colored random k-uniform hypergraphs. Let $p \in [0,1]$ and let c be a positive integer. Then we define $\mathscr{H}_c^k(n,p)$ to be the probability space of edge-colored k-uniform hypergraphs with vertex set $[n] := \{1,\ldots,n\}$, obtained by first choosing each k-tuple $e \in \binom{[n]}{k}$ to be an edge independently with probability p and then by coloring each chosen edge independently and uniformly at random with a color from the set $[c]$. For example, the case $k = 2$ reduces to the standard binomial graph $\mathscr{G}(n,p)$, whose edges are colored at random in c colors. In the special case where $c = 1$, we write $\mathscr{H}^k(n,p) := \mathscr{H}_c^k(n,p)$, and observe that this is just the standard binomial random hypergraph model. For $H \sim \mathscr{H}_c^k(n,p)$ and a hypergraph S, we say that H contains a *rainbow* copy of S if H contains an isomorphic copy of S with all edges in distinct colors. A frequent theme in recent research is to determine the conditions on p and c under which a random hypergraph $H \sim \mathscr{H}_c^k(n,p)$ contains, with high probability (w.h.p.), a rainbow copy of a given hypergraph S.

Let us first discuss the case $k = 2$ of binomial random graphs. Perhaps two of the most natural properties to address are when S is a perfect matching or S is a Hamilton cycle. Note that in these cases we need $c \geq n/2$ and $c \geq n$, respectively. Moreover, it is well known (see e.g., [6]) that a perfect matching (respectively a Hamilton cycle) starts to appear in a typical $G \sim \mathscr{G}(n,p)$ whenever $p = \frac{\log n + \omega(1)}{n}$ (respectively, $p = \frac{\log n + \log\log n + \omega(1)}{n}$), and therefore we can restrict ourselves to these regimes. In [7], Cooper and Frieze showed that for $p \approx 42\log n/n$ and $c = 21n$, a graph $G \sim \mathscr{G}_c(n,p)$ typically contains a rainbow Hamilton cycle. Later on, Frieze and Loh [11] improved this to $p = \frac{(1+o(1))\log n}{n}$ and $c = (1+o(1))n$, which is asymptotically optimal with respect to both of the parameters p and c. Recently, Bal and Frieze [3] obtained the optimal c by showing that for $p = \omega(\log n/n)$ and $c = n/2$ (respectively $c = n$), a graph $G \sim \mathscr{G}_c(n,p)$ w.h.p. contains a rainbow perfect matching (respectively a rainbow Hamilton cycle). For general graphs, Ferber, Nenadov and Peter [10] showed that for every graph S on n vertices with maximum degree $\Delta(S)$ and for $c = (1+o(1))e(S)$, a typical $G \sim \mathscr{G}_c(n,p)$ contains a rainbow copy of S, provided that $p = n^{-1/\Delta(S)}\text{polylog}(n)$ (here, as elsewhere, $e(S)$ denotes the number of edges in S). In this case, the number of colors c is asymptotically optimal, whereas the edge probability p is almost certainly not.

Our first main result improves the main theorem of Frieze and Loh from [11] to $p = \frac{\log n + \log\log n + \omega(1)}{n}$, which is clearly optimal. Our proof technique is completely different, resulting in a shorter proof than the one given in [11].

Theorem 1. *Let $\varepsilon > 0$, let $c = (1+\varepsilon)n$ and let $p = \frac{\log n + \log\log n + \omega(1)}{n}$. Then a graph $G \sim \mathscr{G}_c(n,p)$ w.h.p. contains a rainbow Hamilton cycle.*

Next, building upon an ingenious coupling idea of McDiarmid [18], we give a general statement regarding the problem of finding "nice" structures in randomly edge-colored random hypergraphs. Then, we exhibit its applications to derive interesting corollaries. Before doing so, let us introduce some useful notation. For an

integer c, suppose that $\mathscr{C} := \mathscr{C}(c,n,k)$ is a collection of edge-colored k-uniform hypergraphs on the same vertex set $[n]$, whose edge set is colored with colors from $[c]$. We say that $\mathscr{C}$ is ℓ-*rich* if for any $C \in \mathscr{C}$ and for any $e \in E(C)$ there are at least ℓ distinct ways to color e in order to obtain an element of $\mathscr{C}$. For example, consider the case where $k = 2$ and $c = n_1 \geq n$, and let $\mathscr{C}(c,n,k)$ be the set of all possible rainbow Hamilton cycles in K_n. Note that for each $C \in \mathscr{C}$ and for every $e \in E(C)$, there are $n_1 - n + 1$ ways to color e in order to obtain a rainbow Hamilton cycle. Therefore, $\mathscr{C}$ is $(n_1 - n + 1)$-rich. Now, given a collection of edge-colored hypergraphs $\mathscr{C}$, we define $\widetilde{\mathscr{C}}$ to be the set of all hypergraphs obtained by taking $C \in \mathscr{C}$ and deleting the colors from its edges. With this notation in hand we can state the following theorem:

Theorem 2. *Let $p := p(n) \in [0,1]$ and let ℓ, c be positive integers for which $q := \frac{cp}{\ell} \leq 1$. Let k be any positive integer and let $\mathscr{C} := \mathscr{C}(c,n,k)$ be any ℓ-rich set. Then, we have*

$$\Pr\left[H \sim \mathscr{H}^k(n,p) \text{ contains some } C \in \widetilde{\mathscr{C}}\right] \leq \Pr\left[H \sim H_c^k(n,q) \text{ contains some } C \in \mathscr{C}\right].$$

Next we present two interesting applications for Theorem 2, combining it with known results. First, we show that one can find a rainbow Hamilton cycles in a random hypergraph with an optimal (up to a multiplicative constant) edge density when working with the approximately optimal number of colors. Second, we present an application which is somewhat different in nature. We show that one can find "many" edge-disjoint Hamilton cycles, each of which is rainbow in a random graph. Before stating it formally, we need the following definition. Let H be a k-uniform hypergraph on n vertices. For $0 \leq \ell < k$ we define a Hamilton ℓ-cycle as a cyclic ordering of $V(H)$ for which the edges consist of k consecutive vertices, and for each two consecutive edges e_i and e_{i+1} we have $|e_i \cap e_{i+1}| = \ell$ (where we consider $n + 1 = 1$). It is easy to show that a Hamilton ℓ-cycle contains precisely $m_\ell := \frac{n}{k-\ell}$ edges and therefore we cannot expect to have one unless n is divisible by $k - \ell$. (Note that we can consider a perfect matching as a Hamilton 0-cycle.)

The problem of finding the threshold for the existence of Hamilton ℓ-cycles in random hypergraphs has drawn a lot of attention in the last decade. Among the many known results, it is worth mentioning the one of Johansson, Kahn and Vu [13], who showed that $p = \Theta(\log n/n^{k-1})$ is a threshold for the appearance of a Hamilton 0-cycle (that is, a perfect matching) in a typical $H \sim \mathscr{H}^k(n,p)$, provided that n is divisible by k. In general, for every $\ell < k$, the threshold for the appearance of a Hamilton ℓ-cycle in a typical $H \sim \mathscr{H}^k(n,p)$ (assuming that n is divisible by $k - \ell$) is around $p \approx \frac{1}{n^{k-\ell}}$ (in some cases an extra log factor appears). For more details we refer the reader to [8] and its references.

Now we are ready to state our next result:

Theorem 3. *Let $0 \leq \ell < k$ be two integers, and let $p \in [0,1]$ be such that*

$$\Pr\left[H \sim \mathscr{H}^k(n,p) \text{ contains a Hamilton } \ell\text{-cycle}\right] = 1 - o(1).$$

Then, for every $\varepsilon > 0$, letting $c = (1 + \varepsilon)m_\ell$ and $q = \frac{cp}{\varepsilon m_\ell + 1}$ we have

$$\Pr\left[H \sim \mathscr{H}_c^k(n,q) \text{ contains a rainbow Hamilton } \ell\text{-cycle}\right] = 1 - o(1).$$

Remark: Note that we allow to take ε to be a function of n (or even 0) in the statement above. Moreover, if we take ε to be a constant, then in particular we see that by losing a multiplicative constant in the threshold, a rainbow Hamilton ℓ-cycle w.h.p. exists. This for example reproves and extends the first result obtained by Cooper and Frieze [7] in a very concise way.

The second application is regarding the problem of finding many rainbow edge-disjoint Hamilton cycles in a typical $G \sim \mathscr{G}(n,p)$. The analogous problem without the rainbow requirements is well studied and quite recently, completing a long sequence of papers, Knox, Kühn and Osthus [14], Krivelevich and Samotij [16] and Kühn and Osthus [17] solved this question for the entire range of p. Combining these results with Theorem 2 we can in particular obtain the following:

Theorem 4. *For every* $0 < \varepsilon < 1$ *there exists* $C := C(\varepsilon) > 0$ *such that for every* $p \geq \omega(\log n/n)$ *and* $c = Cn$ *the following holds:*

$$\Pr\left[G \sim \mathscr{G}_c(n,p) \text{ contains } (1-\varepsilon)np/2 \text{ edge disjoint rainbow Hamilton cycles}\right] = 1 - o(1).$$

Notation. Our graph theoretic notation is quite standard and mainly follows that of [20]. For $p \in [0,1]$ we let $\mathscr{H}^k(n,p)$ denote the probability space of k-uniform hypergraphs on vertex set $[n]$, obtained by adding each possible k-subset of $[n]$ as an edge with probability p, independently at random. In the special case $k = 2$, we denote $\mathscr{G}(n,p) := \mathscr{H}^2(n,p)$, the well studied binomial random graph model. For an integer c, we let $\mathscr{H}_c^k(n,p)$ be the probability space of edge-colored k-uniform hypergraphs on vertex set $[n]$ obtained as follows. First, take $H \sim \mathscr{H}^k(n,p)$, and then, to each edge, assign a color from $\mathscr{C} := [c]$ uniformly, independently at random. As before, in the case $k = 2$ we write $\mathscr{G}_c(n,p) := \mathscr{H}_c^2(n,p)$. Given a subhypergraph H' of an edge-colored hypergraph H, we say that H is *rainbow* if all its edges receive distinct colors. For a vertex $v \in V(H)$ we denote by $d_H^c(v)$ its *color* degree, that is, the number of distinct colors appearing on edges incident to v. For an edge $e \in E(H)$, we let $c(e)$ denote its color. Given a subset of vertices $W \subseteq V(H)$ and a subset of colors $\mathscr{C}_0 \subseteq \mathscr{C}$, we let $H[W; \mathscr{C}_0]$ denote the subhypergraph of H on a vertex set W for which $e \in \binom{W}{k} \cap E(H)$ is an edge of $H[W; \mathscr{C}_0]$ if and only if $c(e) \in \mathscr{C}_0$. In case that G is a graph, given two disjoint subsets of vertices $S, W \subseteq V(G)$ and a subset of colors $\mathscr{C}_0 \subseteq \mathscr{C}$, we let $G[S, W; \mathscr{C}_0]$ denote the bipartite subgraph of G with parts S and W, and edges $sw \in E(G)$, where $s \in S$, $w \in W$ and $c(sw) \in \mathscr{C}_0$. Moreover, for two disjoint subsets $S, W \subseteq V(G)$ and an integer D, we say that G contains a D-matching from S to W if there exists a rainbow subgraph M of G such that $d_M(s, W) = D$ for every $s \in S$ and $d_M(w) \leq 1$ for every $w \in W$.

We will frequently omit rounding signs for the sake of clarity of presentation.

2 Tools

In this section we introduce some tools and auxiliary results to be used in our proofs.

2.1 Probabilistic tools

We will routinely employ bounds on large deviations of random variables. We will mostly use the following well-known bound on the lower and the upper tails of the binomial distribution due to Chernoff (see [2], [12]).

Lemma 1 (Chernoff's inequality). *Let $X \sim \mathrm{Bin}(n, p)$ and let $\mu = \mathbb{E}(X)$. Then*

- $\Pr[X < (1-a)\mu] < e^{-a^2\mu/2}$ *for every $a > 0$;*
- $\Pr[X > (1+a)\mu] < e^{-a^2\mu/3}$ *for every $0 < a < 3/2$.*

We also make use of the following approximation for the lower tail of a binomially distributed random variable.

Lemma 2. *Let $\frac{\log n}{n} \leq p \leq \frac{2\log n}{n}$, and let $0 < \delta < 1$ be such that $\left(\frac{e^2}{\delta}\right)^{\delta} e^{-1+\delta} \leq e^{-0.7}$. Then*

$$\Pr[\mathrm{Bin}(n, p) \leq \delta np] \leq n^{-2/3}.$$

Proof. Note that

$$\begin{aligned}
\Pr[\mathrm{Bin}(n, p) \leq \delta np] = \sum_{i=0}^{\delta np} \binom{n}{i} p^i (1-p)^{n-i} &\leq \delta np \left(\frac{e}{\delta}\right)^{\delta np} e^{-(1-\delta)np} \\
&\leq \delta np \left[\left(\frac{e}{\delta}\right)^{\delta} e^{-1+\delta}\right]^{np} \leq \delta np e^{-0.7np} \\
&\leq e^{-(2/3)\log n} = n^{-2/3}.
\end{aligned}$$

Before introducing the next tool to be used, we need the following definition.

Definition 1. Let $(A_i)_{i=1}^n$ be a collection of events in some probability space. A graph D on a vertex set $[n]$ is called a *dependency graph* for $(A_i)_i$ if A_i is mutually independent of all the events $\{A_j : ij \notin E(D)\}$.

We make use of the following asymmetric version of the Lovász Local Lemma (see, e.g. [2]).

Lemma 3. *(Asymmetric Local Lemma) Let $(A_i)_{i=1}^n$ be a sequence of events in some probability space. Suppose that D is a dependency graph for $(A_i)_i$, and suppose that there are real numbers $(x_i)_{i=1}^n$ such that for every i the following holds:*

$$\Pr[A_i] \leq x_i \prod_{ij \in E(D)} (1-x_j).$$

Then, $\Pr[\bigcap_{i=1}^{n} \bar{A}_i] > 0$.

2.2 Properties of $\mathscr{G}_c(n, p)$

Here we gather fairly standard typical properties of sparse binomial random graphs. Given a graph $G = (V, E)$ with vertex set $V = [n]$ vertices, define the set of vertices $SMALL \subseteq V$ by

$$SMALL := \{v \in [n] : d_G(v) \le \delta \log n\},$$

where $\delta > 0$ is a small enough absolute constant.

Lemma 4. *Let* $0 < \beta, \varepsilon < 1$ *be absolute constants, let* $c = (1+\varepsilon)n$, *and let* $\frac{\log n}{n} \le p \le \frac{2\log n}{n}$. *Then, w.h.p. a graph* $G \sim \mathscr{G}_c(n, p)$ *satisfies the following properties.*

(P1) $\Delta(G) \le 10 \log n$.

(P2) $|SMALL| \le n^{0.4}$.

(P3) *For every* $v \in [n]$, $d_G^c(v) \ge d_G(v) - 2$.

(P4) *Let* $E_0 = \{e \in E(G) : e \cap SMALL \neq \emptyset\}$. *Then all the elements of* E_0 *are of distinct colors.*

(P5) *No two vertices* $x, y \in SMALL$ *(x and y might be the same) have a path of length at most* 4 *with x, y as its endpoints in G.*

(P6) *For every two disjoint subsets* X *and* Y *of size* $|X| = |Y| = \omega\left(\frac{n}{(\log n)^{1/2}}\right)$, *the number of colors appearing on the edges between* X *and* Y *is at least* $(1 + \varepsilon - o(1))n$.

(P7) *For every subset* $\mathscr{C} \subseteq [c]$ *of size* βn *and for every subset* $X \subseteq [n]$ *for which* $|X|^2 p = \omega(n)$ *we have that* $\frac{\beta}{3} |X|^2 p \le e(G[X; \mathscr{C}]) \le \beta |X|^2 p$.

(P8) *For every subset* $\mathscr{C} \subseteq [c]$ *of size* βn *and for every two disjoint subsets* $X, Y \subseteq [n]$ *such that* $|X||Y|p = \omega(n)$, *we have that* $\frac{\beta}{2} |X||Y|p \le e(G[X, Y; \mathscr{C}]) \le \beta |X||Y|p$.

(P9) *For every* $s \in [c]$, *the number of edges in* G *which are colored in* s *is at most* $10 \log n$.

(P10) *For every subset* $X \subseteq [n]$, *if* $|X| \le \frac{n}{\log^{4/3} n}$, *then* $e_G(X) \le 8|X|$.

(P11) *For every* $X \subseteq [n]$ *of size* $|X| \ge \frac{n}{\log^{4/3} n}$, *we have* $e_G(X) \le |X|^2 p \left(\frac{n}{|X|}\right)^{1/2}$.

Proof. For (P1), just note that given a vertex $v \in [n]$, since $d_G(v) \sim \text{Bin}(n-1, p)$, it follows that

$$\Pr[d_G(v) \ge 10 \log n] \le \binom{n}{10 \log n} p^{10 \log n}$$
$$\le \left(\frac{enp}{10 \log n}\right)^{10 \log n} = o(1/n).$$

Therefore, by applying the union bound we obtain that w.h.p. $\Delta(G) \le 10 \log n$.

For $(P2)$ note that by Lemma 2, the expected number of such vertices is at most $n^{1/3}$. Therefore, by applying Markov's inequality, $(P2)$ immediately follows.

For $(P3)$, aasume that $d_G^c(v) \leq d_G(v) - 3$. In particular, this means that there are 2 disjoint pairs $\{x_1, y_1\}$ and $\{x_2, y_2\}$ of neighbors of v such that for $i = 1, 2$ both vx_i and vy_i have the same color. The probability of this to happen is upper bounded by $(d_G(v))^4 c^{-2} = o(n^{-1})$. Therefore, by applying the union bound we obtain $(P3)$.

For $(P4)$, note that by $(P2)$ we have that $|SMALL| = o(n^{0.49})$. Now, since $\Delta(D) \leq 10 \log n$, it follows that $|E_0| = o(n^{1/2})$. Therefore,

$$\Pr\left[\exists e, e' \in E_0 \text{ with the same color}\right] \leq \binom{|E_0|}{2} c^{-1} = o(1).$$

For $(P5)$, note that, given two vertices x, y, the probability that there exists a path of length at most 4 between them is upper bounded by $p + np^2 + n^2 p^3 + n^3 p^4 \leq \frac{17 \log^4 n}{n}$. Now, since by $(P2)$ we have $|SMALL| = o(n^{-0.49})$, and since the events "$v \in SMALL$" are "almost independent", by applying the union bound we can easily obtain that the probability for having two such vertices x, y in $SMALL$ is $o(1)$.

For $(P6)$, let X and Y be two disjoint subsets. Note that for a set $\mathscr{C}' \subseteq \mathscr{C}$ of size t, the probability that none of the colors of $\mathscr{C}'$ appears on $E(X, Y)$ is upper bounded by:

$$(1 - p + p(1 - t/c))^{|X||Y|} \leq e^{-pt|X||Y|/c}.$$

Therefore, if $t = \gamma n$ for some fixed constant $\gamma > 0$, then by applying the union bound we obtain that the probability for having a subset $\mathscr{C}'$ of colors of size γn, and two disjoint subsets X and Y of sizes $|X| = |Y| = x = \frac{10n}{(\gamma \log n)^{1/2}}$ for which none of the edges in $E(X, Y)$ uses colors of $\mathscr{C}'$ is at most

$$\binom{n}{x}\binom{n}{x}\binom{(1+\varepsilon)n}{\gamma n} e^{-\gamma npx^2/c} \leq 8^n e^{-\frac{100\gamma pn^2}{c\gamma \log n}} = o(1).$$

For $(P7)$ just note that the expectation of the number of such edges is $\binom{|X|}{2} \cdot p \cdot \frac{\beta n}{c} = \omega(n)$, and therefore, by Chernoff's inequality and the union bound over all choices of X and of $\mathscr{C}$, we easily obtain the desired claim.

For $(P8)$, let $X, Y \subseteq [n]$ be such subsets. Note that since $\mathscr{C}$ is of size βn, the expected number of edges between X and Y which are assigned colors from $\mathscr{C}$ is $\frac{\beta}{1+\varepsilon}|X||Y|p = \omega(n)$. Therefore, the property follows easily from Chernoff's inequality and the union bound.

For $(P9)$, $s \in [c]$ be some color and let Y denote the random variable which counts the number of edges colored s in G. Clearly, $Y \sim \text{Bin}\left(e(G), \frac{1}{(1+\varepsilon)n}\right)$. Now, it is easy to show that w.h.p. $e(G) = (1 + o(1))\binom{n}{2}p \leq (1 + o(1))n \log n \leq 2n \log n$, and therefore,

$$\Pr[Y \geq 10\log n] \leq \binom{2n\log n}{10\log n} \left(\frac{1}{(1+\varepsilon)n}\right)^{10\log n}$$

$$\leq \left(\frac{2en\log n}{10(1+\varepsilon)n\log n}\right)^{10\log n} = o(1/n).$$

Next, by applying the union bound we obtain the desired claim.

We leave $(P10)$ and $(P11)$ as an exercise for the reader.

2.3 Finding rainbow star matchings between appropriate sets

In this subsection we describe the main technical lemma which will be used in the proof of Theorem 1. Informally speaking, this lemma ensures the existence of rainbow star matchings between sets of appropriate sizes.

Lemma 5. *Let $\alpha, \varepsilon > 0$ be constants, let D be a fixed integer, let $c = (1+\varepsilon)n$ and let $\frac{\log n}{n} \leq p \leq \frac{2\log n}{n}$. Then, a graph $G \sim \mathscr{G}_c(n,p)$ is w.h.p. such that the following holds. Suppose that*

(i) $W \subseteq [n]$ *of size* $(1+o(1))\frac{n}{\log\log n}$, *and*
(ii) $S \subseteq [n]$ *of size* $\frac{n}{\log^{0.4} n} \leq |S| \leq \frac{2n}{\log^{0.4} n}$, *and*
(iii) $\mathscr{C}_1 \subseteq \mathscr{C} := [c]$ *of size* $|\mathscr{C}_1| = \alpha n$, *and*
(iv) *for every $s \in S$ there are at least $\frac{\log n}{(\log\log n)^2}$ edges $e = sw$ with $w \in W$ and $c(e) \in \mathscr{C}_1$.*

Then, there exists a rainbow D-matching from S to W, with all colors from $\mathscr{C}_1$.

The main ingredient in the proof of Lemma 5 is the following powerful tool due to Aharoni and Haxell [1], generalizing Hall's theorem to hypergraphs.

Theorem 5. *Let g and D be positive integers and let $\mathscr{H} = \{\mathscr{H}_1, \dots, \mathscr{H}_t\}$ be a family of g-uniform hypergraphs on the same vertex set. If, for every $I \subseteq [t]$, the hypergraph $\bigcup_{i \in I} \mathscr{H}$ contains a matching of size greater than $Dg(|I|-1)$, then there exists a function $f : [t] \times [D] \to \bigcup_{i=1}^{t} E(\mathscr{H}_i)$ such that $f(i,j) \in E(\mathscr{H}_i)$ for every i and j, and $f(i,j) \cap f(i',j') = \emptyset$ for $(i,j) \neq (i',j')$.*

When applying Theorem 5, we will distinguish between few cases according to the size of $I \subseteq [t]$. The following lemmas will make our life a bit easier with it.

Lemma 6. *Let $\varepsilon > 0$, let $c = (1+\varepsilon)n$, let $D \in \mathbb{N}$ and let $\frac{\log n}{n} \leq p \leq \frac{2\log n}{n}$. Then a graph $G \sim \mathscr{G}_c(n,p)$ is w.h.p. such that the following holds. For every collection of $j \leq \frac{n}{\log^{0.9} n}$ vertex disjoint stars, each of size $\log^{0.2} n$, the number of colors appearing on their edges is at least $2Dj$.*

Proof. Let $s := \log^{0.2} n$. We show that the probability of having a collection of $j \le \frac{n}{\log^{0.9} n}$ stars, each of which of size s whose union contains at most $2Dj$ colors is $o(1)$. The following expression is an upper bound for this probability:

$$\sum_{j=1}^{\frac{n}{\log^{0.9} n}} \binom{n}{j} \binom{n}{s}^j p^{js} \binom{(1+\varepsilon)n}{2Dj} \left(\frac{2Dj}{(1+\varepsilon)n} \right)^{js}$$

$$\le \sum_{j=1}^{\frac{n}{\log^{0.9} n}} \left(\frac{en}{j} \right)^j \left(\frac{e(1+\varepsilon)n}{2Dj} \right)^{2Dj} \left(\frac{2eDjnp}{s(1+\varepsilon)n} \right)^{js} = o(1).$$

Indeed, fix $j \le \frac{n}{\log^{0.9} n}$ and first choose j vertices to be the "centers" of the stars. For each of these vertices choose s neighbors and multiply by the probability of all these edge to appear. Next, choose a set of $2Dj$ colors from $c = (1+\varepsilon)n$ and multiply by the probability that all the edges of the stars are colored with these colors. This completes the proof of the lemma.

The following lemma may look at the first glance a bit complicated to understand, but its role will become clear during the proof of Lemma 5.

Lemma 7. *Let $0 < \alpha < \varepsilon < 1$, let $c = (1+\varepsilon)n$, let $D \in \mathbb{N}$ and let $\frac{\log n}{n} \le p \le \frac{2\log n}{n}$. Then a graph $G \sim \mathcal{G}_c(n, p)$ is w.h.p. such that for every*

(i) $\frac{n}{\log^{0.9} n} \le j \le \frac{2n}{\log^{0.4} n}$,
(ii) $W \subseteq [n]$ *of size* $|W| = (1+o(1))\frac{n}{\log\log n}$, *and*
(iii) $\mathcal{C}_1 \subseteq \mathcal{C} := [c]$ *of size* $|\mathcal{C}_1| = \alpha n$,

the following holds. The probability of having subsets $X \subseteq [n]$ of size j, $W' \subseteq W$ and $\mathcal{C}_2 \subseteq \mathcal{C}_1$ of sizes at most $2Dj$ such that for every edge $xw \in E_G(X, W)$ we have $c(xw) \in \mathcal{C}_2$ or $w \in W'$ or $c(xw) \notin \mathcal{C}_1$ is $o(1)$.

Proof. In order to prove the lemma, note that we can upper bound the probability by

$$\binom{n}{|W|} \binom{|W|}{|W'|} \binom{c}{|\mathcal{C}_1|} \binom{|\mathcal{C}_1|}{|\mathcal{C}_2|} \cdot \sum_{j=\frac{n}{\log^{0.9} n}}^{\frac{n}{\log^{0.4} n}} \left(1 - p + p \left(\frac{|\mathcal{C}_2| + |\mathcal{C} \setminus \mathcal{C}_1|}{|\mathcal{C}|} \right) \right)^{j|W \setminus W'|}$$

$$\le 16^n \sum_{j=\frac{n}{\log^{0.9} n}}^{\frac{n}{\log^{0.4} n}} \left(1 - p + p \frac{2Dj}{(1+\varepsilon)n} + p(1 - \alpha/(1+\varepsilon) + o(1)) \right)^{j|W \setminus W'|}$$

$$\le 16^n \sum_{j=\frac{n}{\log^{0.9} n}}^{\frac{n}{\log^{0.4} n}} \exp\left(-Cjp \left(\frac{n}{\log\log n} - Dj \right) \right) = o(1).$$

(C is some constant which depends on α). This completes the proof.

The following lemma shows that in a typical random graph $G \sim \mathscr{G}(n,p)$, any bipartite subgraph $B = (S \cup W, E') \subseteq G$ with all the vertices in S of "large" degree contains an s-matching from S to W, for an appropriate choice of parameters.

Lemma 8. *Let* $\frac{\log n}{n} \leq p \leq \frac{2\log n}{n}$. *Then, a graph* $G \sim \mathscr{G}(n,p)$ *is w.h.p. such that the following holds. Suppose that* $B = (S \cup W, E')$ *is a bipartite (not necessarily induced) subgraph of G, with*

(i) $|S| \leq \frac{n}{\log^{0.9} n}$, *and*

(ii) $|W| = (1 + o(1))\frac{n}{\log\log n}$, *and*

(iii) $d_B(v) \geq \frac{\log n}{(\log\log n)^2}$ *for every* $v \in S$.

Then there exists a $\log^{0.2} n$-*matching from S to W.*

Proof. Let $B = (S \cup W, E')$ be the subgraph of G as described above. In order to prove the lemma, we use the following version of Hall's Theorem (see, e.g., [20]). A bipartite graph $B = (S \cup W, E')$ contains an s-matching from S to W, if and only if the following holds:

$$\text{For every } X \subseteq S \text{ we have } |N_B(X)| \geq s|X|. \tag{1}$$

Suppose that (1) fails for B with $s = \log^{0.2} n$. Then, there exists a subset $X \subseteq S$ for which $|N_B(X) \cup X| \leq (s+1)|X|$. In particular, letting $Y = N_B(X) \cup X$, by (iii), we conclude that $e_B(Y) \geq |Y|\frac{\log n}{(s+1)(\log\log n)^2} \geq |Y|\frac{\log^{0.8} n}{2(\log\log n)^2}$. Since $|Y| \leq (s+1)|X| \leq \frac{2n}{\log^{0.7} n}$, and since $|Y|\log^{0.7} n > |Y|^2 p\,(n/|Y|)^{1/2}$ for every $|Y| \leq n/\log^{0.6} n$, we obtain a contradiction to $(P10)$ and $(P11)$.

Now we are ready to prove Lemma 5.

Proof (Proof of Lemma 5). Let $\alpha < \varepsilon$, let $D \in \mathbb{N}$ and let $W, S \subseteq [n]$ and $\mathscr{C}_1 \subseteq \mathscr{C}$ as described in the assumptions of the lemma. For every $s \in S$, we define a graph $\mathscr{H}_s$ with vertex set $W \cup \mathscr{C}_1$ in the following way. For every $w \in W$ and $x \in \mathscr{C}_1$, $wx \in E(\mathscr{H}_s)$ if and only if $sw \in E(G)$ and $c(sw) = x$. Consider the family $\mathscr{H} := \{\mathscr{H}_s : s \in S\}$, and note that in order to prove the lemma, we need to show that there exists a function $f : S \times [D] \to \bigcup_{s \in S} E(\mathscr{H}_s)$ such that $f(s,i) \in E(\mathscr{H}_s)$ for every s and i, and $f(s,i) \cap f(s',i') = \emptyset$ for $(s,i) \neq (s',i')$. To this end, we make use of Theorem 5. All we need to show is that for every $T \subseteq S$, the graph $\bigcup_{t \in T} E(\mathscr{H}_t)$ contains a matching of size greater than $2D(|T| - 1)$. We distinguish between two cases:

Case 1: $|T| \leq \frac{n}{\log^{0.9} n}$. Consider the bipartite graph $B = (T \cup W, E')$, where $E' := \{tw : t \in T, w \in W \text{ and } c(tw) \in \mathscr{C}_1\}$. By applying Lemma 8 to B, we conclude that there exists an s-matching from T to W in B, where $s = \log^{0.2} n$. Let M be such a matching, and note that by applying Lemma 6 to M, it follows that the number of colors appearing in M is at least $2D|T|$. Now, one can easily deduce that $\bigcup_{t \in T} E(\mathscr{H}_t)$ contains a matching of size at least $2D|T| > 2D(|T| - 1)$.

Case 2: $\frac{n}{\log^{0.9} n} \leq |T| \leq |S|$. Let M be a matching in $\bigcup_{t \in T} E(\mathscr{H}_t)$ of maximum size, let $\mathscr{C}_2 := \{x \in \mathscr{C}_1 : \exists w \in W \text{ s.t } wx \in M\}$ and let $W' := \{w \in W : \exists x \in \mathscr{C}_2 \text{ s.t } wx \in M\}$.

Suppose that $|M| \leq 2D(|T|-1) < 2D|T|$. In particular, it means that for every $v \in T$ and $w \in W$ we have $vw \notin E(G)$, or $c(vw) \in \mathscr{C}_2$, or $w \in W'$, or $c(vw) \notin \mathscr{C}_1$, which contradicts Lemma 7. This completes the proof.

2.4 Expansion properties of subgraphs of random graphs

In the following lemma we show, that given an edge colored graph G, one can find two subsets of colors $\mathscr{C}_1, \mathscr{C}_2$ and a vertex subset W which inherits some desired properties from G. The statement of the lemma is adjusted so as to facilitate its application in the proof of Theorem 1.

Lemma 9. *Let $0 < \alpha, \delta, \varepsilon < 1/2$ be constants and let n be an integer. Let G be an edge colored graph on $m \geq (1 - o(1))n$ vertices, and let $\mathscr{C}^*$ be its set of colors, of size $|\mathscr{C}^*| \geq (1 + \varepsilon/2)n$. Suppose that $\delta \log n \leq \delta(G) \leq \Delta(G) \leq 10 \log n$, that each color appears at most $10 \log n$ times in G, and that for each $v \in V(G)$, $d_G^c(v) \geq d_G(v) - 2$. Then one can find subsets $\mathscr{C}_0, \mathscr{C}_1 \subseteq \mathscr{C}^*$, and $W \subseteq V(G)$ satisfying the following properties:*

(i) $|W| = (1 + o(1)) \frac{n}{\log \log n}$, and

(ii) $\mathscr{C}_0$ and $\mathscr{C}_1$ are two disjoint subsets of sizes $(1 + o(1)) \alpha n$, and

(iii) for every $w \in W$, $d_{\mathscr{C}_0}(w, W) \in \left(\frac{\alpha d_G(w)}{2 \log \log n}, \frac{2 \alpha d_G(w)}{\log \log n} \right)$, and

(iv) for every $v \in V(G)$, $d_{\mathscr{C}_1}(v, W) \in \left(\frac{\alpha d_G(v)}{2 \log \log n}, \frac{2 \alpha d_G(v)}{\log \log n} \right)$, and

(v) for every $x \in \mathscr{C}_0$, x appears on at most $\frac{100 \log n}{(\log \log n)^2}$ edges in $G[W]$.

Proof. Let $\mathscr{C}_0, \mathscr{C}_1 \subseteq \mathscr{C}^*$ be two disjoint random subsets, obtained in the following way: for each element of $\mathscr{C}^*$ toss a coin with probability 2α to decide whether it belongs to $\mathscr{C}_0 \cup \mathscr{C}_1$, then, with probability $1/2$ decide to which of these sets it belongs. All these choices are independent. Let $W \subseteq V(G)$ be a random subset of vertices, obtained by picking each $v \in V(G)$ with probability $\frac{1}{\log \log n}$, independently at random. We wish to show that the obtained sets satisfy (i)-(v) with positive probability. In order to do so, we consider several types of events. First, let A_W denote the event "$|W| \notin (1 \pm o(1)) \frac{n}{\log \log n}$". Second, for each $i \in \{0, 1\}$, let C_i denote the event "$|\mathscr{C}_i| \notin (1 \pm o(1)) \alpha n$". Third, for each vertex $v \in V(G)$, let $\Gamma_i(v)$ ($i \in \{0, 1\}$) denote the event "$d_{\mathscr{C}_i}(v, W) \notin \left(\frac{\alpha d_G(v)}{2 \log \log n}, \frac{2 \alpha d_G(v)}{\log \log n} \right)$". Lastly, for each $x \in \mathscr{C}^*$, let B_x be the event "more than $\frac{100 \log n}{(\log \log n)^2}$ edges in $G[W]$ are colored x". With this notation at hand, we wish to show that

$$\Pr\left[\overline{A_W} \cap \overline{C_1} \cap \overline{C_2} \cap \left(\bigcap_{i \in \{0,1\}, v \in V(G)} \overline{\Gamma_i(v)} \right) \cap \left(\bigcap_{x \in \mathscr{C}^*} \overline{B_x} \right) \right] > 0.$$

First, define $\mathcal{E} := \{A_W, C_1, C_2, \Gamma_i(v), B_x : v \in V(G), i \in \{0,1\}, x \in \mathcal{C}^*\}$, and let us estimate the probabilities of each of the events $X \in \mathcal{E}$. By using Chernoff's bounds we trivially get

(a) $\Pr[A_W] = \exp(-\Theta(n/\log\log n))$,
(b) $\Pr[C_i] = \exp(-\Theta(\alpha n))$, and
(c) $\Pr[\Gamma_i(v)] = \exp(-C\alpha d_G(v)/\log\log n)$, where C is an absolute constant which does not depend on α.

For estimating $\Pr[B_x]$, note that since $d_G^c(v) \geq d_G(v) - 2$ for every $v \in V(G)$, it follows that each color class can be partitioned into at most four matchings, each of size at most $10\log n$ (the maximum number of edges with the same color in G). Fix such a partition (into matchings) for each color class x. It follows that if B_x fails, then in at least one of the four matchings, at least $\frac{25\log n}{(\log\log n)^2}$ edges have been chosen. Since in each matching these choices are independent, and since for a fixed edge e, the probability that $e \in W$ is $\frac{1}{(\log\log n)^2}$, it follows by Chernoff's bounds that

(d) $\Pr[B_x] \leq e^{-\frac{\log n}{2(\log\log n)^2}}$.

Next, let us define a dependency graph D for $\mathcal{E}$, where the edges of D are as follows:

- All pairs $A_W X$, where $X \in \mathcal{E}$, and
- all pairs $C_i X$, where $i \in \{0,1\}$ and $X \in \mathcal{E}$, and
- all pairs $\Gamma_i(v)\Gamma_j(u)$, where $i \neq j$ and $v = u$, or $v \neq u$ and $N_G(v) \cap N_G(u) \neq \emptyset$, or if the same color $c \in \mathcal{C}^*$ appears on edges incident to both u and v, and
- all pairs $\Gamma_i(v)B_x$ for which there exists an edge uw such that $\{u,w\} \cap (\{v\} \cup N_G(v)) \neq \emptyset$ and $c(uw) = x$, and
- all pairs $B_x B_y$, for which there exist two edges e and f, of colors x and y, respectively, such that $e \cap f \neq \emptyset$.

Now, for some fixed constant $c_0 > 1$, define $x_W = \sqrt{\Pr[A_W]}$, $y_i = \sqrt{\Pr[C_i]}$ (where $i \in \{0,1\}$), $x_{i,v} = c_0 \Pr[\Gamma_i(v)]$ (where $i \in \{0,1\}$ and $v \in V(G)$), and $z_x = \sqrt{\Pr[B_x]}$ for $x \in \mathcal{C}^*$. Note that

$$\Pr[A_W] = \exp(-\Theta(n/\log\log n)) \leq x_W \prod_{i,v}(1 - x_{i,v}),$$

and

$$\Pr[C_i] = \exp(-\Theta(\alpha n)) \leq y_i \prod_{i,v}(1 - x_{i,v}),$$

and

$$\Pr[\Gamma_i(v)] = \exp(-C\alpha d_G(v)/\log\log n) \leq c_0 x_{i,v} \prod_{j,u:\Gamma_j(u)\Gamma_i(v)\in E(D)}(1 - x_{j,u}) \prod_{x\in N_G^c(v)}(1 - z_x),$$

and

$$\Pr[B_x] \leq z_x \prod_{x \in N_G^c(v),i} (1 - x_{i,v}) \prod_{y : B_x B_y \in E(D)} (1 - z_y).$$

(The last two inequalities hold because the corresponding "degrees of dependencies" are some $polylog(n)$). All in all, one can apply the Asymmetric Local Lemma (Lemma 3) and obtain the desired claim.

Now, let $\frac{\log n}{n} \leq p \leq \frac{2\log n}{n}$, let $c = (1 + \varepsilon)n$, and let $G \sim \mathscr{G}_c(n, p)$. We show that w.h.p. G is such that every (not necessarily induced) subgraph $G_1 \subseteq G$ on $(1 + o(1))n/\log\log n$ vertices with some degree constraints is also a very good expander.

Lemma 10. *Let $\alpha, \delta, \varepsilon > 0$, let $\frac{\log n}{n} \leq p \leq \frac{2\log n}{n}$, and let $c = (1 + \varepsilon)n$. Then a graph $G \sim \mathscr{G}_c(n, p)$ is w.h.p. such that the following properties hold. Suppose that*

(i) *$W \subseteq [n]$ is of size $(1 + o(1))n/\log\log n$, and*
(ii) *$\mathscr{C}_0 \subseteq \mathscr{C}$ is of size $(1 + o(1))\alpha n$.*

Then $H := G[W; \mathscr{C}_0]$ satisfies:

(a) *For every subset $X \subseteq W$, if $|X| \leq \frac{n}{\log^{4/3} n}$, then $e_H(X) \leq 8|X|$, and*

(b) *for every $X \subseteq W$ of size $\frac{n}{\log^{4/3} n} \leq |X| \leq |W|$, we have $e_H(X) \leq |X|^2 p \left(\frac{n}{|X|}\right)^{1/2}$, and*

(c) *for every $X \subseteq W$, if $|X| \geq n/\log^{0.4} n$, then $e_H(X) \leq \alpha |X|^2 p$, and*

(d) *for every two subsets $X, Y \subseteq W$ satisfying $|X||Y|p = \omega(n)$, we have $e_H(X, Y) \geq \frac{\alpha}{2}|X||Y|p$.*

Proof. All these properties follow from the properties in Lemma 4 and similar arguments, hence are left as an exercise for the reader.

Let us define the following useful notion of a (k, d)-expander.

Definition 2. *A graph G is called a (k, d)-expander if for every subset $X \subseteq V(G)$ of size at most k we have*

$$|N_G(X) \setminus X| \geq d|X|.$$

The following lemma is almost identical to Lemma 2.4 in [15] (although with few minor modifications). For the convenience of the reader, we briefly sketch the proof.

Lemma 11. *Let $0 < \varepsilon, \delta < 1$ and let $\alpha < \delta e^{-100}$ be constants. Let $\frac{\log n}{n} \leq p \leq \frac{2\log n}{n}$, and let $c = (1 + \varepsilon)n$. Then there exists $d_0 \in \mathbb{N}$ such that for every $d \geq d_0$, a graph $G \sim \mathscr{G}_c(n, p)$ is w.h.p. such that the following holds. Suppose that $W \subseteq [n]$ is of size $(1 + o(1))n/\log\log n$, $\mathscr{C}_0 \subseteq \mathscr{C}$ of size $(1 + o(1))\alpha n$ and $H := G[W; \mathscr{C}_0]$ is a subgraph of G satisfying $\frac{\alpha\delta\log n}{2\log\log n} \leq \delta(H) \leq \Delta(H) \leq \frac{20\alpha\log n}{\log\log n}$ and properties (a)–(d) of Lemma 10. Moreover, assume that no color from $\mathscr{C}_0$ appears in H more than $\frac{100\log n}{(\log\log n)^2}$ times. Then, there exists a subgraph $R \subseteq H$ satisfying the following:*

(a) *R is rainbow, and*

(b) R *is a* $(k, 100)$-*expander (where* $k = \alpha\delta|W|/100$*), and*
(c) $|E(R)| \leq d|W|$.

Proof. (Lemma 2.4 in [15].) Let d be a large enough integer. Condition on G satisfying all the properties of Lemma 4. Now, for every $w \in W$, let w choose d random edges of H incident with w (with repetitions), and let $\Gamma(w)$ be the set of the edges chosen by w. Let R be the graph whose edge set is $\bigcup_{w \in W} \Gamma(w)$. We wish to show that R satisfies $(a) - (c)$ with positive probability.

We consider few types of events. First, the events regarding the rainbow part. For every two edges e_1, e_2 of the same color, let us denote by $A(e_1, e_2)$ the event "both e_1 and e_2 are in R" (in case $e_1 \neq e_2$), and "e_1 is chosen in more than one trial" (in case $e_1 = e_2$). Define

$$\mathscr{A} := \{A(e_1, e_2) : e_1 \text{ and } e_2 \text{ have the same color}\}.$$

Second, we consider the events ensuring the expansion of sets. For a set $X \subseteq W$, let $B(X)$ denote the event that $e_R(X) \geq \frac{d}{101}|X|$, and for every $\frac{n}{\log^{4/3} n} \leq x \leq k$, let

$$\mathscr{B}_x = \{B(X) : |X| = 101x\}.$$

Clearly, if none of the events in $\mathscr{A}$ happens, then (a) and (c) hold. Now, assume in addition that none of the events $\mathscr{B}_x$ happens, and we wish to show that (b) holds. Let $X \subseteq V(H)$ be a subset of size at most k, and we wish to show that $|N_H(X) \setminus X| \geq 100|X|$. Assume otherwise, we obtain a subset X for which $|X \cup N_H(X)| < 101|X|$. Since none of the events in $\mathscr{A}$ holds, it follows that $|E_H(X \cup N_H(X))| \geq d|X| > d|X \cup N_H(X)|/101$. Now, if $|X \cup N_H(X)| \leq \frac{n}{\log^{4/3} n}$, then for a large enough d it contradicts (a) of Lemma 10. If $\frac{n}{\log^{4/3} n} \leq |X \cup N_H(X)| \leq 101k$, then it contradicts the event $B(X \cup N_H(X))$.

It thus suffices to show that with positive probability none of these events occurs. To this end we make use of the Local Lemma. We estimate the probabilities of each event above, define a dependency graph D and estimate its degrees.

The family $\mathscr{A}$: For a fixed pair $e_1, e_2 \in E(G)$ of the same color we have

$$\Pr[A(e_1, e_2)] \leq \left(\frac{4d \log\log n}{\alpha\delta \log n}\right)^2.$$

Define $x = c_0 \left(\frac{4d \log\log n}{\alpha\delta \log n}\right)^2$ for some constant $c_0 > 1$. For the "degree of dependency" within $\mathscr{A}$, note that $A(e_1, e_2)$ depends on $A(f_1, f_2)$ if and only if $e_1 \cup e_2$ intersects $f_1 \cup f_2$. Now, recall that $\Delta(H) \leq \frac{20\alpha \log n}{\log\log n}$, and that each color class contains at most $\frac{100 \log n}{(\log\log n)^2}$ edges. Therefore, the number of events in $\mathscr{A}$ which are neighbors of $A(e_1, e_2)$ in the dependency graph is at most $4\Delta(H) \frac{100 \log n}{(\log\log n)^2} \leq \frac{8000\alpha \log^2 n}{(\log\log n)^3}$.

The family $\mathscr{B}_t$: For a fixed set X of size $101t$, similarly to [15], one can show that

$$\Pr[B(X)] \le \binom{e_H(X)}{dt} \cdot (2d)^{dt} \cdot \left(\frac{2\log\log n}{\alpha\delta\log n} \right)^{dt},$$

where for $t \le n/(101\log^{0.4} n)$, by assumption (b) of Lemma 10 we have $e_H(X) \le 101^2 t^2 p \left(\frac{n}{101t} \right)^{1/2}$, and therefore,

$$\Pr[B(X)] \le \left(10^9 \left(\frac{t}{n} \right)^{1/2} \log\log n / \alpha\delta \right)^{dt}.$$

For $t \le n/(101\log^{0.4} n)$, define $y_t = \left(10^9 e \left(\frac{t}{n} \right)^{1/2} \log\log n / \alpha\delta \right)^{dt/2}$ and note that for an appropriately large d we have

$$\sum_{t=n/(\log^{4/3} n)}^{n/\log^{0.4} n} |\mathcal{B}_t| \cdot y_t = o(1).$$

Now, for $n/(101\log^{0.4} n) \le t \le k$, we use (c) of Lemma 10 and obtain

$$\Pr[B(X)] \le \left(\frac{10^9 t \log\log n}{\delta n} \right)^{dt}.$$

Define $y_t = e^{C_1 t} \left(\frac{4et\log\log n}{\delta n} \right)^{dt}$ for some constant $C_1 > 0$ and note that for appropriate choices of δ, d and C_1 we obtain

$$\sum_{t=n/(101\log^{0.4} n)}^{k} |\mathcal{B}_t| \cdot y_t = \sum_{t=n/(101\log^{0.4} n)}^{\alpha\delta n/\log\log n} \binom{n/\log\log n}{101t} e^{C_1 t} \left(\frac{4et\log\log n}{\delta n} \right)^{dt} = o(1).$$

To compute the "degree of dependency" with members of $\mathcal{A}$ in D, note that $B(X)$ is correlated with an event $A(e_1, e_2)$ if $(e_1 \cup e_2) \cap X \ne \emptyset$. Therefore, since the maximum degree of H is at most $\frac{20\alpha\log n}{\log\log n}$ and since each color appears at most $\frac{100\log n}{(\log\log n)^2}$ times, the number of events $A(e_1, e_2)$ correlated with $B(X)$ is upper bounded by

$$|X| \frac{20\alpha\log n}{\log\log n} \frac{100\log n}{(\log\log n)^2} = |X| \frac{2000\alpha\log^2 n}{(\log\log n)^3}.$$

In order to apply the Asymmetric Local Lemma (Lemma 3) we need to show that the following inequalities hold.

$$\Pr[A(e_1, e_2)] \le x \cdot (1-x)^{\frac{8000\alpha\log^2 n}{(\log\log n)^3}} \left(\prod_t (1 - y_t)^{|\mathcal{B}_t|} \right),$$

$$\Pr[B(X)] \le y_t \cdot (1-x)^{|X| \frac{2000\alpha\log^2 n}{(\log\log n)^3}} \cdot \left(\prod_t (1 - y_t)^{|\mathcal{B}_t|} \right).$$

For the first inequality, note that since $x = c_0 \left(\frac{4d \log \log n}{\alpha \delta \log n} \right)^2$, it follows that

$$x \cdot (1-x)^{\frac{8000\alpha \log^2 n}{(\log \log n)^3}} \left(\prod_t (1-y_t)^{|\mathscr{B}_t|} \right) \geq c_0 \left(\frac{4d \log \log n}{\alpha \delta \log n} \right)^2 e^{-2c_0 \left(\frac{4d \log \log n}{\alpha \delta \log n} \right)^2 \frac{8000\alpha \log^2 n}{(\log \log n)^3}} e^{-2 \sum_t |\mathscr{B}_t| y_t}$$

$$= (1 + o(1))c_0 \left(\frac{4d \log \log n}{\alpha \delta \log n} \right)^2$$

$$\geq \Pr\left[A(e_1, e_2) \right].$$

(we used the facts that $\sum |\mathscr{B}_t| \cdot y_t = o(1)$ and that for small values of x we have $1 - x \geq e^{-2x}$).

The second inequality is even easier to verify and is left as an exercise for the reader.

2.5 Finding a long rainbow path

In this section we state the following lemma, which follows almost identically from the proof of Lemma 4.4 in [4]. Before doing so, we introduce the following definition:

Definition 3. A graph G on n vertices whose set of edges is colored is called *k-rainbow-pseudorandom*, if for every two disjoint subsets of vertices $A, B \subseteq V(G)$ of size $|A| = |B| = k$, the number of colors appearing on the edges of G between A and B is at least n.

In the following lemma we show that a graph $G \sim \mathscr{G}_c(n, p)$ is *k*-rainbow-pseudorandom in a "robust" way.

Lemma 12. *Let $0 < \alpha < \varepsilon < 1$ be two constants, let $c = (1+\varepsilon)n$ and let $\frac{\log n}{n} \leq p \leq \frac{2\log n}{n}$. Then a graph $G \sim \mathscr{G}_c(n, p)$ is w.h.p. such that the following holds. For every subset $\mathscr{C}^* \subseteq [c]$ of size $|\mathscr{C}^*| \geq (1+\alpha)n$, the graph $G[[n]; \mathscr{C}^*]$ is k-rainbow-pseudorandom for $k = \frac{n}{\log^{0.49} n}$.*

Proof. Follows immediately from Property $(P6)$ of Lemma 4.

Now we state a modification of Lemma 4.4 from [4].

Lemma 13. *Let G be an edge-colored graph on n vertices which is k-pseudorandom. Then G contains a rainbow path of length at least $n - 2k + 1$.*

Proof. The proof is more or less identical to the proof of Lemma 4.4 in [4] with some minor changes which are left to the reader.

2.6 Expander graphs

Here we show that the union of few expander graphs yields an expander graph as well.

First, we show that given an expander graph, by adding vertex disjoint stars to it one cannot harm the expansion properties too much.

Lemma 14. *Let G be a graph, let $m > 1$, and let k be a positive integer. Let $S \subseteq V(G)$ be a subset of vertices for which there exists an m-matching from S to $V(G) \setminus S$. If $G[V(G) \setminus S]$ is a (k,m)-expander, then G is a $(k,(m-1)/2)$-expander.*

Proof. Let $X \subseteq V(G)$ of size $|X| \leq k$, and we wish to show that $|N_G(X) \setminus X| \geq \frac{m-1}{2}|X|$. Let us distinguish between the following two cases:

Case I: $|X \cap S| \leq |X|/2$. In this case, since $G[V(G) \setminus S]$ is a (k,m), it follows that $X \setminus S$ expands by a factor of c and therefore $|N_G(X) \setminus X| \geq \frac{(m-1)|X|}{2}$.

Case II: $|X \cap S| > |X|/2$. In this case, since there exists an m-matching from S to $V(G) \setminus S$, hence $|N_G(X) \setminus X| \geq |N_G(X \cap S) \cap (V(G) \setminus S)| - |X \setminus S| \geq m|X \cap S| - \frac{|X|}{2} \geq \frac{(m-1)|X|}{2}$.

The following simple lemma is from [5] (Claim 2.8).

Lemma 15. *Let G be a graph, let $m > 0$, and let k be a positive integer. Let $U \subseteq V(G)$ be a subset for which $d_G(u) \geq m - 1$ for every $u \in U$, and, moreover, there is no path of length at most 4 in G whose (possibly identical) endpoints lie in U. If $G[V(G) \setminus U]$ is a (k,m)-expander, then G is a $(k,m-1)$-expander.*

2.7 Boosters

In the proof of Theorem 1 we need to find a Hamilton path between two designated vertices x' and y' in a sparse expander subgraph G_1 of a typical $G \sim \mathcal{G}_c(n,p)$. Moreover, we need such a Hamilton path to be rainbow within a prescribed subset of colors. In this section we show how to achieve such a goal.

A routine way to turn a non-Hamiltonian expander graph G_1 into a Hamiltonian graph is by using *boosters*. A booster is a non-edge e of G_1 such that the addition of e to G_1 decreases the number of connected components of G_1, or creates a path which is longer than a longest path of G_1, or turns G_1 into a Hamiltonian graph. In order to turn G_1 into a Hamiltonian graph, we start by adding a booster e of G_1. If the new graph $G_1 \cup \{e\}$ is not Hamiltonian then one can continue by adding a booster of the new graph. Note that after at most $2|V(G_1)|$ successive steps the process must terminate and we end up with a Hamiltonian graph. The main point using this method is that it is well-known (for example, see [19]) that a non-Hamiltonian graph G_1 with "good" expansion properties has many boosters. However, our goal is a bit different. We wish to turn G_1 into a graph that contains a rainbow Hamilton path with x' and y' as its endpoints. In order to do so, we add one (possibly) fake edge

$x'y'$ to G_1, color it with a new color (which does not belong to $\mathscr{C}$) and try to find a rainbow Hamilton cycle that contains the edge $x'y'$. Then, the path obtained by deleting this edge from the Hamilton cycle will be the desired path. For that we need to define the notion of *e-boosters*.

Given a graph G_1 and a pair $e \in \binom{V(G_1)}{2}$, consider a path P of $G_1 \cup \{e\}$ of maximal length which contains e as an edge. A non-edge e' of G_1 is called an e-booster if $G_1 \cup \{e, e'\}$ has fewer connected components than $G_1 \cup \{e\}$ has, or contains a path P' which passes through e and which is longer than P, or that $G_1 \cup \{e, e'\}$ contains a Hamilton cycle that uses e. The following lemma is from [9] and shows that every connected and non-Hamiltonian graph G_1 with "good" expansion properties has many e-boosters for every possible e.

Lemma 16. *Let G_1 be a connected graph for which $|N_{G_1}(X) \setminus X| \geq 2|X| + 2$ holds for every subset $X \subseteq V(G_1)$ of size $|X| \leq k$. Then, for every pair $e \in \binom{V(G_1)}{2}$ such that $G_1 \cup \{e\}$ does not contain a Hamilton cycle which uses the edge e, the number of e-boosters for G_1 is at least $(k+1)^2/2$.*

Remark 1. The proof of Lemma 16 can be easily modified (in fact, the same proof holds) for the following case. G_1 is a graph obtained by adding not too many (say, at most *polylogn*) vertices of degree 2, any two of them are far apart (say, of distance at least 3), to a $(k, 3)$ expander, and e is not incident with any of these vertices.

As another remark, note that for a $(k, 2)$-expander, we trivially have that each connected component is of size larger than k, and therefore, if the graph is not connected, then there are at least $(k+1)^2$ boosters which decrease the number of connected components.

Note that in order to turn a rainbow graph G_1 into a graph that contains a rainbow Hamiltonian cycle passing through e, one should repeatedly add e-boosters, one by one, every time adding a booster with an unused color, at most $2|V(G_1)|$ times. Therefore, we wish to show that a graph $G \sim G_c(n, p)$ typically contains "many" e-boosters of "many" colors for every sparse expander subgraph G_1 and every pair $e \in \binom{V(G_1)}{2}$.

Lemma 17. *Let $0 < \varepsilon < 1, \beta > 0$, let $c = (1 + \varepsilon)n$ and let $\frac{\log n}{n} \leq p \leq \frac{2\log n}{n}$. Then a graph $G \sim \mathscr{G}_c(n, p)$ is w.h.p. such that the following holds. Suppose that*

(i) *$G_1 \subseteq G$ is any subgraph with $\frac{n}{\log\log n} \leq |V(G_1)| \leq \frac{2n}{\log\log n}$ and $|E(G_1)| = \Theta(n/\log\log n)$ which is an $(\beta|V(G_1)|, 2)$-expander, and*

(ii) *$e \in \binom{V(G_1)}{2}$ is any pair which is not incident with vertices of degree 2 in G_1, and*

(iii) *$\mathscr{C}_2 \subseteq [c]$ is a subset of size at least $\varepsilon n/100$,*

then, G contains e-boosters for G_1 assigned with colors from $\mathscr{C}_2$.

Proof. Note first that by Remark 1 after Lemma 16, there are at least $\beta^2|V(G_1)|^2/2 \geq \frac{\beta^2 n^2}{2(\log\log n)^2}$ e-boosters for every such G_1. Fix a subset $\mathscr{C}_2 \subseteq [c]$ of size at least $\varepsilon n/100$,

and observe that the probability of $E(G)$ not to contain any e-booster which is assigned with a color from $\mathscr{C}_2$ is at most

$$\left(1-p+p\frac{1+0.99\varepsilon n}{(1+\varepsilon)n}\right)^{\frac{\beta^2 n^2}{2(\log\log n)^2}} \leq (1-p)^{\frac{\varepsilon\beta^2 n^2}{300(\log\log n)^2}} \leq \exp\left(-\frac{\varepsilon\beta^2 n\log n}{300(\log\log n)^2}\right).$$

Now, taking the union bound over all subsets $V(G_1) \subseteq [n]$ of size $\frac{n}{\log\log n} \leq |V(G_1)| \leq \frac{2n}{\log\log n}$ and over all subgraph G_1 of G on vertex set $V(G_1)$ with at most $\frac{Cn}{\log\log n}$ many edges (where C is some fixed constant), and over all subsets of colors $\mathscr{C}_2 \subseteq [c]$ of size at least $\varepsilon n/5$ we obtain that the probability of having a counterexample is upper bounded by

$$\sum_{t=n/\log\log n}^{2n/\log\log n} 2^c \binom{n}{t} \binom{\binom{t}{2}}{Ct} p^{Ct} \exp\left(-\frac{\varepsilon\beta^2 n\log n}{300(\log\log n)^2}\right)$$

$$\leq \frac{2n}{\log\log n} 2^{(1+\varepsilon)n} 2^n \left(\frac{enp}{C\log\log n}\right)^{2Cn/\log\log n} \exp\left(-\frac{\varepsilon\beta^2 n\log n}{300(\log\log n)^2}\right)$$

$$\leq 8^n \exp\left(C\frac{2n}{\log\log n}\log\frac{enp}{C\log\log n}\right) \exp\left(-\frac{\varepsilon\beta^2 n\log n}{300(\log\log n)^2}\right) = o(1).$$

This completes the proof.

3 Proof of Theorem 1

In this section we prove Theorem 1.

Proof. Let $G \sim \mathscr{G}_c(n,p)$, and let $\delta > 0$ be a sufficiently small constant (to be specified later). Throughout the proof we assume that G satisfies all the properties of the lemmas from the previous section.

Our proof strategy goes as follows. For each vertex $v \in SMALL$ let us arbitrarily choose a set $A(v) = \{x,y\}$ of exactly two distinct neighbors of v and set $V_0 = SMALL \cup (\bigcup_{v\in SMALL} A(v))$, and $E_0 = \{vz : v \in SMALL \text{ and } z \in A(v)\}$. By $(P4)$ and $(P5)$ of Lemma 4 we have that all the $A(v)$'s are disjoint and that E_0 is rainbow. Let $\mathscr{C}_{small} := \{c(e) : e \in E_0\}$ denote the set of colors used in E_0, and let $\mathscr{C}^* := \mathscr{C} \setminus \mathscr{C}_{small}$ be its complement. Observe that by $(P2)$ of Lemma 4 for a small enough δ we have $|\mathscr{C}^*| \geq (1+\varepsilon/2)n$.

Now, note that by $(P5)$ of Lemma 4 it easily follows that $\delta(G[V \setminus V_0]) \geq \delta \log n$. Therefore, letting $\alpha = \min\{\varepsilon/5, \delta e^{-10}\}$, by applying Lemma 9 to $G[V \setminus V_0]$ we find subsets $W \subseteq [n] \setminus V_0$ and $\mathscr{C}_0, \mathscr{C}_1 \subset \mathscr{C}^*$ for which

(i) $|W| = (1+o(1))\frac{n}{\log\log n}$, and
(ii) $\mathscr{C}_0 \cap \mathscr{C}_1 = \emptyset$, and

(*iii*) $|\mathscr{C}_0|, |\mathscr{C}_1| = (1+o(1))\alpha n$, and

(*iv*) for every $v \in [n] \setminus SMALL$ we have $d_{\mathscr{C}_1}(v,W) \in \left(\frac{\alpha\delta \log n}{2\log\log n}, \frac{20\alpha \log n}{\log\log n} \right)$, and

(*v*) the subgraph $H := G[W;\mathscr{C}_0]$ satisfies all the properties of Lemma 10.

In order to find the desired rainbow Hamilton cycle we proceed as follows. First, find a rainbow path P of length $n - n/\log^{0.4} n$ in $[n] \setminus (V_0 \cup W)$ whose edges receive colors from $\mathscr{C}^* \setminus (\mathscr{C}_0 \cup \mathscr{C}_1)$. The existence of such a path is ensured by Lemmas 12 and 13. Second, let x,y denote P's endpoints, define $S = ([n] \setminus (SMALL \cup V(P) \cup W)) \cup \{x,y\}$ be the set of "unused" vertices, and consider the bipartite graph $B := G[S,W;\mathscr{C}_1]$. Lemma 5 ensures that B contains (say) a rainbow 9-matching M from S to W. Let x' and y' be two neighbors (in M) of x and y, respectively, and define $M' := M \setminus \{e \in M : e \cap \{x,y\} \neq \emptyset\}$.

Next, by applying Lemma 11 to $G[W;\mathscr{C}_0]$, we find a subgraph $R \subseteq G[W;\mathscr{C}_0]$ which satisfies the following:

(*a*) R is rainbow, and

(*b*) R is an $(\alpha\delta|W|/100, 100)$-expander, and

(*c*) $|E(R)| = \Theta(n/\log\log n)$.

Now, let us define G_1 to be the subgraph of G on vertex set $V_1 := [n] \setminus V(P)$, with edge set $M' \cup E_0 \cup E(R)$. Note that since R is an $(\alpha\delta|W|/100, 100)$-expander, and since for $S' := S \setminus \{x,y\}$, there exists a 9-matching from S' to W, it follows by Lemma 14 that adding S' and M' to R yields an $(\alpha\delta|W|/100, 4)$-expander. Now, since by ($P5$) we have that vertices in $SMALL$ are far apart, by Lemma 15 it follows that G_1 is an $(\alpha\delta|V_1|/200, 2)$-expander with $\Theta(n/\log\log n)$ edges, and is clearly rainbow. Finally, we wish to turn G_1 (in G) into a graph which contains a rainbow Hamilton path with x' and y' as its endpoints. Note that both x' and y' are not neighbors of vertices of degree 2 in G_1. Now, one can repeatedly apply Lemma 17 to G_1 with respect to the set of available colors to obtain a rainbow Hamilton path of G_1 connecting x' to y' which uses only colors not appearing on P.(Each time we add a booster e whose color $c(e) \in \mathscr{C}^* \setminus (\mathscr{C}_0 \cup \mathscr{C}_1)$ has not been used before we update the set of available colors by excluding $c(e)$. Since $|W| = o(n)$, along the process we still have a linear number of colors available, and thus Lemma 17 applies..) A moment's thought now reveals that such a path, together with P and the edges xx' and yy', yields a rainbow Hamilton cycle in G. This completes the proof.

4 Proof of Theorem 2

In this section we prove Theorem 2.

Proof. Let us define the following sequence $\Gamma_0, \Gamma_1, \ldots, \Gamma_N$ of random edge-colored k-uniform hypergraphs, where $N = \binom{n}{k}$, in the following way: Let $e_1, \ldots, e_N$ be an arbitrary enumeration of all the elements of $\binom{[n]}{k}$. Now, in Γ_i, for every $j > i$ we add the corresponding edge with probability p, independently at random and assign it

all the colors in $[c]$ (these edges can be seen as multiple edges with multiplicity c). For every $j \leq i$, we add e_j to Γ_i with probability q, independently at random and then assign it a unique color from $[c]$ uniformly, independently at random. Note that $\Gamma_0 \sim \mathscr{H}^k(n, p)$ while $\Gamma_N \sim \mathscr{H}_c^k(n, q)$. Therefore, in order to complete the proof it is enough to show that

$$\Pr[\Gamma_i \text{ contains some } C \in \mathscr{C}] \geq \Pr[\Gamma_{i-1} \text{ contains some } C \in \mathscr{C}].$$

To this end, expose all edges but e_i and its color(s) in both spaces. There are three possible scenarios:

(a) Γ_{i-1} contains some $C \in \mathscr{C}$ not using e_i, or
(b) Γ_{i-1} does not contain any member $C \in \mathscr{C}$ even if we add e_i with all the possible colors, or
(c) Γ_{i-1} contains a member of $\mathscr{C}$ if we add e_i with all the possible colors.

Note that in (a) and (b) there is nothing to prove. Therefore, it is enough to consider case (c). The crucial observation here is that if e_i is needed for finding a copy of some $C \in \mathscr{C}$, then since $\mathscr{C}$ is ℓ-rich, it follows that at least ℓ colors are valid for e_i in order to obtain such a copy. Now, the probability for Γ_{i-1} to contain a member of $\mathscr{C}$ is precisely p (recall that e_i is crucial for this aim and that we add e_i with all possible colors), where the probability for Γ_i to have such a copy is at least $q\ell/c = p$. This completes the proof.

5 Applications of Theorem 2

In this section we show how to use Theorem 2 in order to derive Theorems 3 and 4. For Theorem 3 we prove a stronger statement from which the proof immediately follows.

Theorem 6. *Let S be any k-uniform hypergraph on n vertices with m edges, and let p be such that*

$$\Pr\left[H \sim \mathscr{H}^k(n, p) \text{ contains a copy of } S\right] = 1 - o(1).$$

Then, for every $\varepsilon \geq 0$, letting $c = (1 + \varepsilon)m$ and $q = \frac{cp}{\varepsilon m + 1}$, if $q \leq 1$ then we have

$$\Pr\left[H \sim \mathscr{H}_c^k(n, q) \text{ contains a rainbow } S\right] = 1 - o(1).$$

Proof. Let $\mathscr{C}$ be the set of all possible rainbow copies of S on n vertices with colors from $[c]$, where $c = (1 + \varepsilon)m$. Note that for any $e \in E(C)$, $C - e$ has exactly $m - 1$ edges and since there are $(1 + \varepsilon)m$ colors, it follows that there are $\varepsilon m + 1$ ways to color e to obtain a rainbow copy of S. All in all, $\mathscr{C}$ is $(\varepsilon m + 1)$-rich, and therefore by applying Theorem 2 to $\mathscr{C}$ we obtain the desired claim.

Now we prove Theorem 4 which informally speaking states that for $c = \omega(n)$ and $p = \omega(\log n/n)$, in a typical $G \sim \mathscr{G}_c(n, p)$ one can find $(1 - o(1))np/2$ edge-disjoint Hamilton cycles, each of which is rainbow.

Proof. First, observe that for example by the main results of [14],[16], it follows in particular that for $p = \omega(\log n/n)$ we have

$$\Pr[G \sim \mathscr{G}(n, p) \text{ contains } (1 - o(1))np/2 \text{ edge-disjoint Hamilton cycles}] = 1 - o(1).$$

Now, let C be such that $\frac{Cn}{(C-1)n+1} \leq 1 + \varepsilon/2$ and let $c = Cn$. Let us define $\mathscr{C}$ to be the family of all collections C of $(1 - \varepsilon/2)np/2$ edge-disjoint Hamilton cycles, each of which is rainbow. Note that for every $C \in \mathscr{C}$ and every $e \in E(C)$, since e belongs to a given rainbow Hamilton cycle, there are at most $n - 1$ colors which are forbidden for it. Therefore, there are $Cn - (n - 1) = (C - 1)n + 1$ ways to color e in order to obtain an element of $\mathscr{C}$ and we conclude that $\mathscr{C}$ is $((C - 1)n + 1)$-rich. Now, by applying Theorem 2 for $q = \frac{Cnp}{(C-1)n+1}$ we obtain

$$\Pr[G \sim \mathscr{G}_c(n, q) \text{ contains } (1 - o(1))np/2 \text{ edge-disjoint rainbow Hamilton cycles}] = 1 - o(1).$$

All in all, since $q \leq (1 + \varepsilon/2)p$ we obtain that $(1 - o(1))np \geq \frac{(1-o(1))nq}{1+\varepsilon/2} \geq (1 - \varepsilon)nq$ as desired.

References

1. R. Aharoni and P. Haxell, Hall's theorem for hypergraphs, Journal of Graph Theory 35 (2000), 83–88.
2. N. Alon and J. H. Spencer, **The Probabilistic Method**, 3rd Ed., John Wiley & Sons, 2008.
3. D. Bal and A. Frieze, Rainbow matchings and hamilton cycles in random graphs, arXiv preprint arXiv:1311.6423, 2013.
4. I. Ben-Eliezer, M. Krivelevich and B. Sudakov, The size Ramsey number of a directed path, Journal of Combinatorial Theory Series B 102 (2012), 743–755.
5. S. Ben-Shimon, A. Ferber, D. Hefetz and M. Krivelevich, Hitting time results for Maker-Breaker games, Proceedings of the 22nd Symposium on Discrete Algorithms (SODA'11), 900–912. Journal version: Random Structures and Algorithms 41 (2012), 23–46.
6. B. Bollobás, **Random graphs**, 2nd Ed., Cambridge University Press, 2001.
7. C. Cooper and A. Frieze, Multi-coloured hamilton cycles in random edge-coloured graphs, Combinatorics, Probability and Computing 11 (2002), 129–133.
8. A. Dudek and A. Frieze, Tight Hamilton cycles in random uniform hypergraphs, Random Structures and Algorithms 42 (2013), 374–385.
9. A. Ferber, M. Krivelevich and H. Naves, Generating random graphs in biased Maker-Breaker games, Random Structures and Algorithms, to appear.
10. A. Ferber, R. Nenadov and U. Peter, Universality of random graphs and rainbow embedding, Random Structures and Algorithms, to appear.
11. A. Frieze and P.-S. Loh, Rainbow hamilton cycles in random graphs, Random Structures and Algorithms 44 (2014), 328–354.
12. S. Janson, T. Łuczak and A. Rucinski, **Random graphs**, John Wiley & Sons, 2000.
13. A. Johansson, J. Kahn and V. Vu, Factors in random graphs, Random Structures and Algorithms, 33 (2008), 1–28.

14. F. Knox, D. Kühn, and D. Osthus, Edge-disjoint Hamilton cycles in random graphs, Random Structures and Algorithms 46 (2015), 397–445.
15. M. Krivelevich, C. Lee and B. Sudakov, Compatible Hamilton cycles in random graphs, arXiv preprint arXiv:1410.1438 (2014).
16. M. Krivelevich and W. Samotij, Optimal packings of Hamilton cycles in sparse random graphs, SIAM Journal on Discrete Mathematics 26 (2012), 964–982.
17. D. Kühn and D. Osthus, Hamilton decompositions of regular expanders: applications, Journal of Combinatorial Theory Series B 104 (2014), 1–27.
18. C. McDiarmid, Clutter percolation and random graphs, Math. Programming Stud. 13 (1980), 17–25.
19. L. Pósa, Hamiltonian circuits in random graphs, Discrete Math. 14 (1976), 359–364.
20. D. B. West, **Introduction to Graph Theory**, Prentice Hall, 2001.