Colouring powers of cycles from random lists

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Abstract

Let C_n^k be the k-th power of a cycle on n vertices (i.e. the vertices of C_n^k are those of the n-cycle, and two vertices are connected by an edge if their distance along the cycle is at most k). For each vertex draw uniformly at random a list of size c from a base set $\mathcal{S}$ of size $s = s(n)$. In this paper we solve the problem of determining the asymptotic probability of the existence of a proper colouring from the random lists for all fixed values of c, k, and growing n.

1 Introduction

Let G be a simple and undirected graph. Assign to each vertex x of G a set $L(x)$ of colours (positive integers). Such an assignment L of sets to vertices in G is referred to as a *colour scheme* for G. An *L-colouring* of G is a mapping f of $V(G)$ into the set of colours such that $f(x) \in L(x)$ for all $x \in V(G)$ and $f(x) \neq f(y)$ for each $(x, y) \in E(G)$. If G admits an L-colouring, then G is said to be *L-colourable* . In case of $L(x) = \{1, \ldots, k\}$ for all $x \in V(G)$, we also use the terms *k-colouring* and *k-colourable* respectively. A graph G is called *k-choosable* if G is L-colourable for every colour scheme L of G satisfying $|L(x)| = k$ for all $x \in V(G)$. The *chromatic number* $\chi(G)$ (*choice number* $ch(G)$) of G is the least integer k such that G is k-colourable (k-choosable).

The choosability concept was introduced, independently by Vizing [4] and by Erdős, Rubin and Taylor [2].

Denote by C_n^k the k-th power of a cycle on n vertices, i.e. $V(C_n^k) = \{0, \ldots, n-1\}$ and $(i, j) \in E(G)$ if $(i - j) \bmod n \in \{-k, \ldots, -1, 1, \ldots, k\}$. Equivalently, the vertices of C_n^k are those of the n-cycle, and two vertices are connected by an edge if their distance along the cycle is at most k. Let $\mathcal{S}$ be a set of colours of size $s(n)$. For each vertex of C_n^k draw uniformly at random c colours from $\mathcal{S}$ to form a colour scheme $L(c, k, s)$. Denote by $p(n, c, k, s(n))$ the probability that C_n^k is $L(c, k, s)$-colourable. The problem of determining what is the minimum $s(n)$ such that $p(n, c, k, s(n)) = 1 - o(1)$ was posed to us by Maurice Cochand. The problem originated in chemical industry and it is related to scheduling problems occurring in the production of colorants. The goal is to get an estimate of the resources required for the production process. Basically, the vertices represent jobs, the colours correspond to processors that perform the jobs, and the edges represent technological restrictions on the processors' assignments.

Of course, the same problem of colourability from randomly chosen lists can be posed for other asymptotic families such as complete graphs K_n and complete bipartite graphs $K_{m,n}$ as well. We plan to return to this question later.

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In this paper we investigate the asymptotic behaviour of $p(n) = p(n, c, k, s)$ for every fixed c, k and n tending to infinity. We show that if $c \le k$,

$$p(n) = \begin{cases} o(1), & s(n) = o(n^{1/c^2}), \\ e^{-\binom{k}{c}t^{-c^2}(c!)^c} + o(1), & s(n) \sim tn^{1/c^2}, \\ 1 - o(1), & s(n) = \omega(n^{1/c^2}), \end{cases}$$

On the other hand, if $c = k + 1$,

$$p(n) = \begin{cases} o(1), & s(n) < c, \\ 1 - o(1), & s(n) > c, \end{cases}$$

and if $c > k + 1$

$$p(n) = \begin{cases} o(1), & s(n) < c, \\ 1 - o(1), & s(n) \ge c. \end{cases}$$

(The limit is easily seen not to exist for the case $s = c = k + 1$.)

We shall use the standard asymptotic notation and assumption. In particular we assume that the parameter n is large enough whenever necessary. For two functions $f(n)$ and $g(n)$, we write $f = o(g)$ if $\lim_{n\to\infty} f/g = 0$, and $f = \omega(g)$ if $g = o(f)$. Also, $f = O(g)$ if there exists an absolute constant $c > 0$ such that $f(n) < cg(n)$ for all large enough n; $f = \Theta(g)$ if both $f = O(g)$ and $g = O(f)$ hold; and $f \sim g$ if $\lim_{n\to\infty} f/g = 1$.

2 Case $c \le k$

For all results in this section we assume $c \le k$. We shall see that with probability tending to 1 the graph becomes L-colourable at the same time cliques of size $c+1$ with the same list drawn for every vertex, vanish.

Proposition 2.1 *If $s(n) = o(n^{1/c^2})$, then $p(n) = o(1)$.*

Proof: Every set $\{v_1, ..., v_{c+1}\}$ of $c + 1$ vertices of C_n^k satisfies:

$$Pr[v_1, ..., v_{c+1} \text{ have the same list }] = \binom{s}{c}^{-c};$$

Partition the first $(c+1)\lfloor \frac{n}{c+1} \rfloor$ vertices of $V(C_n^k)$ into $\lfloor \frac{n}{c+1} \rfloor$ disjoint sets of $c+1$ consecutive (in the sense of the cycle) vertices. Clearly, the events "The set gets the same list drawn for all its vertices" are mutually independent. Therefore, the probability that none of those sets has the same list is:

$$\left[1 - \frac{1}{\binom{s}{c}^c}\right]^{\lfloor n/(c+1) \rfloor} = o(1) \ .$$

Hence with probability tending to 1, there exists a set of $c + 1$ consecutive vertices, which forms a clique in C_n^k, with the same list drawn for every vertex. Therefore, C_n^k cannot be coloured.

$\square$

Theorem 2.2 *If $s(n) = \omega(n^{1/c^2})$, then $p(n) = 1 - o(1)$.*

Proof: Fix with foresight a constant d satisfying $d > \frac{c^2(c^2+c-1)}{c-1}$. Order the vertices $v_1, ..., v_n$ according to the cycle, with an arbitrary starting point. Call a colour scheme L of C_n^k *good* if it satisfies the following conditions:

1. There are no $c+1$-cliques with the same list drawn for every vertex of the clique.
2. There exists a family of sets of $k+1$ consecutive vertices (in what follows: $k+1$-sets) such that:
   (a) The lists drawn for each of the sets do not intersect with any of the $2k$ lists of its neighbours.
   (b) The number of vertices between two such consecutive $k+1$-sets is no more than $n^{1/d}$.
3. Every set of at most $n^{1/d}$ consecutive vertices, U, has the following property: every subset $X \subset U$, $|X| \geq c+2$, satisfies
$$|X| \leq \left| \bigcup_{x \in X} L(x) \right| .$$

We prove the theorem in two steps. First we show that if L is good, then C_n^k is L-colourable. Secondly, we show that with probability tending to 1, $L(c, k, s(n))$ is good.

Assume L is good, then every set of at most $n^{1/d}$ consecutive vertices, W, can be coloured from L. To see this construct a bipartite graph, $H_{L,W}$ with sides W, A where $A = \mathcal{S}$ and $(v, a) \in E(H)$ if $a \in L(v)$. By Condition 3, $|X| \leq |N(X)|$ for all X of size $|X| \geq c+2$. Since the degree of every $v \in W$ is c, it follows that $|X| - |N(X)| \leq 1$ for every $X \subset W$. By the defect version of Hall's theorem (see [5]) there exists a matching in H which saturates all but at most one vertex of W. This matching ensures us a legal choice of colours for the saturated vertices. Furthermore, it is clear that if $x \in W$ is the unsaturated vertex, then x is in some $X \subset W$ where $|X| - |N(X)| = 1$. This can only happen if $|X| = c+1$, meaning also that all the lists of X's vertices are identical. By Condition 1, X is not a clique. Colour two non-adjacent vertices of X with the same colour, if x is still not coloured, colour it with the available colour. This clearly completes the matching and thus completes an L-colouring of W.

Take a family of $k+1$-sets as in Condition 2, find an L-colouring of every $k+1$-set and an L-colouring of every run of vertices between consecutive $k+1$-sets. This is clearly possible by the above argument and this completes an L-colouring of C_n^k.

We show now that with probability tending to 1, $L(c, k, s(n))$ is good.

Condition 1. Let X be the random variable counting the number of $c+1$-cliques with the same list drawn for every vertex. The number of cliques in C_n^k is $n\binom{k}{c}$ (choose the leftmost vertex of the clique in n possible manners, then choose k vertices from its c neighbours to the right in $\binom{k}{c}$ possible manners), so clearly:
$$E[X] = n\binom{k}{c}\binom{s}{c}^{-c} = o(1),$$
So $Pr[X \geq 1] = o(1)$.

Condition 2. For a fixed $k+1$-set, the probability that its lists do not intersect with its neighbours' lists is clearly more than:

$$\left[\frac{\binom{s-2kc}{c}}{\binom{s}{c}}\right]^{k+1} = 1-\Theta(\frac{1}{s});$$

Hence, the chance that the $k+1$-set's lists are not entirely disjoint from its neighbours' lists is at most $\Theta(1/s)$.
Partition the first $(3k+3)\lfloor\frac{n}{3k+3}\rfloor$ vertices of $V(C_n^k)$ into disjoint $k+1$-sets, with distance of $2k+1$ between them (this is a mere technicality to keep the events independent). So, by the union bound, the event that there exists a run of at least $n^{1/d}$ consecutive vertices in which every $k+1$-set (in the former division) has failed to satisfy the list disjointness condition occurs with probability at most:

$$n[\Theta(1/s)]^{\Theta(n^{1/d})} = o(1);$$

Condition 3. We show that for a given set of $\lfloor n^{1/d}\rfloor$ consecutive vertices, W, the probability that Condition 3 does not hold is $o(1/n)$, thus with probability tending to 1, every such W satisfies Condition 3. This obviously implies that, with probability tending to 1, every set of at most $n^{1/d}$ vertices satisfies Condition 3. For a fixed W, construct a bipartite graph $H_{L,W}$ as before. The probability that Condition 3 does not hold is clearly:

$$Pr[\exists X\subset W, |X|\geq c+2, |X|>|N(X)|] \leq \sum_{i=c+2}^{\lfloor n^{1/d}\rfloor}\binom{\lfloor n^{1/d}\rfloor}{i}\binom{s}{i-1}\left[\frac{\binom{i-1}{c}}{\binom{s}{c}}\right]^{i},$$

The first term in the summation implies the choice of such X, the second implies the choice of $N(X)$ and the third is the probability that the neighbours of X are in $N(X)$, i.e., that the lists of the vertices of X were chosen from the $i-1$ colours (at most) of $N(X)$. This is $o(\frac{1}{n})$:

$$\begin{aligned}
\sum_{i=c+2}^{\lfloor n^{1/d}\rfloor}\binom{\lfloor n^{1/d}\rfloor}{i}\binom{s}{i-1}\left[\frac{\binom{i-1}{c}}{\binom{s}{c}}\right]^{i} &\leq \sum_{i=c+2}^{n^{1/d}}\left(\frac{en^{1/d}}{i}\right)^{i}\left(\frac{es}{i-1}\right)^{i-1}\left[\frac{\left(\frac{e(i-1)}{c}\right)^{c}}{\left(\frac{s}{c}\right)^{c}}\right]^{i} \\
&\leq \sum_{i=c+2}^{n^{1/d}}\left(\frac{e^{c+2}n^{1/d}(i-1)^{c-2}}{s^{c-1}}\right)^{i}\frac{i-1}{es} \\
&\leq \frac{n^{1/d}}{es}\sum_{i=c+2}^{n^{1/d}}\left(\frac{e^{c+2}n^{\frac{c-1}{d}}}{s^{c-1}}\right)^{i} \leq \frac{n^{1/d}}{es}\frac{\left(\frac{e^{c+2}n^{\frac{c-1}{d}}}{s^{c-1}}\right)^{c+2}}{1-\left(\frac{e^{c+2}n^{\frac{c-1}{d}}}{s^{c-1}}\right)} \\
&= e^{(c+2)^2}(1+o(1))\frac{n^{\frac{c^2+c-1}{d}}}{s^{c^2}s^{c-1}} \leq e^{(c+2)^2}(1+o(1))\frac{n^{\frac{c^2+c-1}{d}-\frac{c-1}{c^2}}}{s^{c^2}} \\
&= o(\frac{1}{n})\ .
\end{aligned}$$

□

Theorem 2.3 *For any constant $t > 0$, if $s(n) \sim tn^{1/c^2}$, then $\lim_{n\to\infty} p(n) = e^{-\binom{k}{c}t^{-c^2}(c!)^c}$.*

Proof: The same calculations as before show that if $s(n) \sim tn^{1/c^2}$, Condition 2 and Condition 3 hold with probability tending to 1. Thus, the problem reduces to calculating the asymptotic probability of the appearance of a $c+1$-clique with identical lists drawn for each vertex. Using Brun's Sieve (see, e.g., [1], Chapter 8) we show that the number of such $c+1$-cliques has asymptotically Poisson distribution.

Let $m = n\binom{k}{c}$ be the number of $c+1$-cliques in C_n^k, as explained previously. Let B_i, $i = 1, \ldots, m = n\binom{k}{c}$, be the event that the i-th $c+1$-clique has the same list drawn for it. For disjoint $c+1$-cliques, it is clear that the corresponding B_i's are independent. Let X_i, $i = 1, \ldots, m$, be B_i's indicator random variables, and let $X = \sum_{i=1,\ldots,m} X_i$ denote the random variable counting the number of $c+1$-cliques having the same list drawn for its vertices. We wish to estimate the probability that $X = 0$. Define:

$$\mu = \binom{k}{c} t^{-c^2}(c!)^c,$$

$$S^{(r)} = \sum_{i_1,\ldots,i_r} Pr[B_{i_1} \cap \ldots \cap B_{i_r}],$$

where the sum is over all sets $\{i_1, \ldots, i_r\} \subset \{1, \ldots, m\}$. Now:

$$E[X] = S^{(1)} = \sum_{i=1}^{m} Pr[B_i] = n\binom{k}{c}\binom{s}{c}^{-c} \to \mu,$$

For $r \geq 2$, divide the sum $S^{(r)}$ into two parts:

$$S^{(r)} = \sum_{i_1,\ldots,i_r}^{*} Pr[B_{i_1} \cap \ldots \cap B_{i_r}] + \sum_{i_1,\ldots,i_r}^{**} Pr[B_{i_1} \cap \ldots \cap B_{i_r}],$$

where the first summation goes over all r-tuples of pairwise disjoint $c+1$-cliques and second goes over all r-tuples of $c+1$-cliques which are not pairwise disjoint. Note that every $c+1$-clique intersects with a constant number of other $c+1$-cliques, therefore, the number of terms in the first summation is $\binom{m}{r} - O(n^{r-1})$ and:

$$\sum_{i_1,\ldots,i_r}^{*} Pr[B_{i_1} \cap \ldots \cap B_{i_r}] = \left[\binom{m}{r} - O(n^{r-1})\right]\left[\binom{s}{c}^{-cr}\right] \to \frac{\mu^r}{r!};$$

To deal with the second summation, note that for every $1 \leq i \leq r-1$ there are $\Theta(n^i)$ manners to choose r $c+1$-cliques such that the size of the maximal disjoint family is i. The probability that every $c+1$-clique in this maximal disjoint family has the same list drawn is clearly $\binom{s}{c}^{-ci}$. If $i < r$, then there exists at least one vertex of a clique not in the maximal disjoint family whose list is identical to one of the lists of the cliques in the maximal disjoint family. Therefore:

$$\sum_{i_1,\ldots,i_r}^{**} Pr[B_{i_1} \cap \ldots \cap B_{i_r}] \leq \sum_{i=1}^{r-1} \Theta(n^i)\frac{1}{\binom{s}{c}^{ci}}\frac{1}{\binom{s}{c}} = o(1),$$

Thus $S^{(r)} \to \mu^r/r!$, and by Brun's Sieve, $Pr[X = 0] \to e^{-\mu}$. It follows by the above discussion that $p(n)$ is also asymptotic to $e^{-\mu}$.

□

3 Case $c > k$

For all results in this section we assume $c > k$.

A recent result of Prowse and Woodall [3] states that for all values of n and k, $ch(C_n^k) = \chi(C_n^k)$. It is easy to see that when $n \geq k(k+1)$, if $k+1$ divides n then $\chi(C_n^k) = k+1$, otherwise $\chi(C_n^k) = k+2$. Their result makes the following proposition true:

Proposition 3.1 *If $s = s(n)$ satisfies $s \geq c \geq k+2$, then $p(n) = 1 - o(1)$.*

Furthermore, it is trivial that if $s = c = k+1$, no limit exists for the probability of a legal colour assignment. We present an argument settling the case $c = k+1$ and $s > c$, which also implies Proposition 3.1 without the use of the result of Prowse and Woodall.

Theorem 3.2

1. *If $c = k+1$ and $s > c$, then $p(n) = 1 - o(1)$.*
2. *If $c > k+1$ and $s \geq c$, then $p(n) = 1 - o(1)$.*

Proof: If $s = \omega(1)$, then as shown before, for an arbitrary set of k+1 consecutive vertices, the probability that its lists intersect with its $2k$ neighbours lists is $\Theta(1/s)$. So, with probability tending to 1, a fixed set of k+1 consecutive vertices will have lists disjoint from its neighbours' lists, which allows us to first colour the k+1-set and then complete the colouring greedily: colour by the order of the cycle, for each vertex choose a colour not chosen by its preceding neighbours.

Consider now the case $s = O(1)$. We prove the theorem for $c = k+1$, $s = k+2$. This implies Theorem 3.2, since increasing c increases the probability of a colouring, and in this case, the proof works with higher values of s as long as it stays bounded. Assume then $\mathcal{S} = \{1, ..., k+2\}$.

Given a set of $2[(k+1)^2 + (k+1)]$ consecutive vertices, divide the set into the first and last k+1-sets, and into $k+1$ pairs of disjoint k+1-sets. Such a set will be called *good* if it satisfies the following conditions:

1. Each vertex of the first and the last k+1-set has the list $\{1, ..., k+1\}$.
2. For every $1 \leq i \leq k+1$, each vertex of the second member of the ith pair has the list $\{1, ..., k+1\}$ and each vertex of the first member has the list $\{1, ..., k+2\} - \{i\}$.

The probability that a set of consecutive $2[(k+1)^2 + (k+1)]$ vertices satisfies these conditions is:

$$\left(\frac{1}{k+2}\right)^{2[(k+1)^2+(k+1)]}.$$

After choosing $\lfloor \frac{n}{2[(k+1)^2+(k+1)]} \rfloor$ disjoint consecutive sets in $V(C_n^k)$ in the usual manner, the probability that none of them has the above properties tends to 0 and therefore a good set exists with probability tending to 1.

Assume we have a good set, colour C_n^k in the following manner:
Colour the last k+1-set with $(1, ..., k+1)$ by that order. Continue colouring greedily the vertices after the last k+1-set, by the order of the cycle (again, for each vertex choose a colour not chosen by its preceding neighbours) until we complete colouring the first k+1-set (this can be done since every vertex after the first k+1-set has exactly k coloured neighbours). For each $1 \leq i \leq k+1$ colour as

follows:
To the first $i-1$ vertices of the first member of the ith pair, give colours $(1, ..., i-1)$ by that order. Give the ith vertex of that member the colour $k+2$, and colour the rest of the first member greedily. To the first i vertices of the second member of the ith pair, give colours $(1, ..., i)$ by that order. Give the rest of the vertices the same colours as in the matching vertices of the first member of the ith pair. It is easy to check that this colouring is possible, and hence the theorem is proven.

□

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