

# Long cycles in subgraphs of (pseudo)random directed graphs

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## Abstract

We study the resilience of random and pseudorandom directed graphs with respect to the property of having long directed cycles. For every  $0 < \gamma < 1/2$  we find a constant  $c = c(\gamma)$  such that the following holds. Let  $G = (V, E)$  be a (pseudo)random directed graph on  $n$  vertices and with at least a linear number of edges, and let  $G'$  be a subgraph of  $G$  with  $(1/2 + \gamma)|E|$  edges. Then  $G'$  contains a directed cycle of length at least  $(c - o(1))n$ . Moreover, there is a subgraph  $G''$  of  $G$  with  $(1/2 + \gamma - o(1))|E|$  edges that does not contain a cycle of length at least  $cn$ .

## 1 Introduction

Given a property  $\mathcal{P}$ , a typical problem in extremal graph theory can be stated as follows. Given a number of vertices  $n$ , what is the minimal (or maximal) number  $f_{\mathcal{P}}(n)$  such that any graph on  $n$  vertices with  $f(n)$  edges possesses  $\mathcal{P}$ ? Many examples of such problems and results can be found, e.g., in [8].

Usually, the property  $\mathcal{P}$  we consider in extremal problems is either *monotone increasing* or *monotone decreasing*. A property  $\mathcal{P}$  is monotone increasing (respectively, decreasing) if it is preserved under edge addition (respectively, deletion).

The *resilience* of a graph  $G$  with respect to a property  $\mathcal{P}$  measures how *far* the graph is from any graph  $H$  that does not have  $\mathcal{P}$ . In particular, the study of resilience usually focuses on monotone properties, and the following two types of problems are studied.

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**Global Resilience.** Given a monotone increasing property  $\mathcal{P}$ , the global resilience of  $G$  with respect to  $\mathcal{P}$  is the maximal integer  $R$  such that for every subset  $E_0 \subseteq E(G)$  of  $|E_0| = R$  edges, the graph  $G - E_0$  still possesses  $\mathcal{P}$ . For the case of a monotone decreasing property  $\mathcal{P}$ , the global resilience of  $G$  with respect to  $\mathcal{P}$  is defined as the maximum number  $R$  such that the addition of any subset of  $R$  edges to  $G$  still results in a graph  $G' \in \mathcal{P}$ .

One can also define the notion of **local resilience** of a graph with respect to, say, a monotone increasing property  $\mathcal{P}$  as the maximum number  $r$  such that for any subgraph  $H \subseteq G$  of maximum degree  $r$ , the graph  $G - H$  is still in  $\mathcal{P}$ . Since in this paper we will be concerned with properties related to global resilience, we will not dwell on the notion of local resilience anymore.

To the best of our knowledge, Sudakov and Vu were the first [18] to define the notion of global resilience explicitly and quantitatively and to put it forward as a subject of independent study (it is closely related though to the well studied notion of fault tolerance, see, e.g., [1]). However, in a sense many well known theorems in extremal graph theory can be stated using this terminology. For example, given a fixed graph  $H$ , the Turán number of  $H$ , denoted by  $ex(H, n)$ , is the minimum number  $m$  such that any graph on  $n$  vertices with  $m$  edges contains a copy of  $H$ . Clearly, the study of Turán numbers is equivalent to the study of the global resilience of the complete graph  $K_n$  with respect to the property of having a copy of  $H$ .

Woodall [20] gave tight bounds for the number of edges in an undirected graph that guarantees the existence of a cycle of length at least  $\ell$ . In our terminology, he gave tight bounds on the global resilience of  $K_n$  with respect to the property of having a cycle of length at least  $\ell$ . We will discuss Woodall's result later and will also use his result in our work. Lewin [17] studied the analogous problem for directed graphs, and he gave tight bounds on the number of edges required for having a directed cycle of length at least  $\ell$ . Many extremal results regarding the existence of cycles in directed graphs can be found, e.g., in [7].

Recently, there has been a series of works studying the resilience of graphs with respect to different properties. Dellamonica et al. [11] studied the local and global resilience of long cycles in pseudorandom undirected graphs. Krivelevich et al. [16] studied the resilience with respect to pancyclicity (having a cycle of every possible length). Ben-Shimon et al. [6] studied the resilience of several graph properties in random regular graphs. Alon and Sudakov [2] studied the resilience of the chromatic number in random graphs. Böttcher et al. [9] studied the local resilience of  $G(n, p)$  with respect to the property of having an almost spanning bounded degree bipartite graph with sublinear bandwidth. Later, answering a question from [9], Huang et al. [14] addressed the resilience with respect to having a spanning subgraph  $H$ . Balogh et al. [4] studied the resilience of random and pseudorandom graphs with respect to containing a copy of a given nearly spanning tree of bounded maximum degree.

Here we study the resilience of pseudorandom (and hence, of random) directed graphs with respect to the property of having a long directed cycle (namely, a simple directed cycle that covers a constant fraction of the vertices). We prove asymptotically tight bounds, and thus provide the asymptotic value of the resilience of every graph with respect to this property, assuming it has some predefined pseudorandomness property. Our proof uses

a variant of the celebrated Szemerédi’s regularity lemma for sparse directed graphs, and a short and simple technique for finding a long directed path in pseudorandom directed graphs. Using these techniques we can reduce our problem to the case of undirected graphs, where by applying techniques of [11] we can give tight bounds.

## 1.1 The models

We consider here directed graphs on  $n$  vertices, where antiparallel edges are allowed. We say that a graph  $D = (V, E)$  has density  $p$  if  $|E| = pn^2$ .

Let  $D(n, p)$  be the following probability distribution on the set of  $n$ -vertex directed graphs. Every graph in the support of  $D(n, p)$  contains  $n$  vertices, and for every two distinct vertices  $x, y$ , there is an edge from  $x$  to  $y$  with probability  $p$ , and independently there is an edge from  $y$  to  $x$  with probability  $p$ . Clearly, the expected number of edges is  $2p\binom{n}{2}$ .

Once we define our random digraph model, it is usually desirable to define a pseudorandom analog. That is, we would like to define a property such that graphs with this property have many of the ‘nice’ properties of random graphs. Roughly speaking, we say that a directed graph is pseudorandom if the number of edges between every two large enough sets is close to the expected number of edges in a random directed graph with the same density. More formally, we say that a directed graph  $G$  is  $(p, r)$ -pseudorandom if it has edge density  $p$  and for every two disjoint sets  $A, B \subseteq V(G)$ ,  $|A| = |B|$ , the number of edges from  $A$  to  $B$ , denoted by  $e_G(A, B)$ , satisfies

$$|e_G(A, B) - p|A||B|| \leq r|A|\sqrt{pn}.$$

This is (up to normalization) a directed variant of the well known notion of *jumbled graphs*, that was introduced by Thomason [19]. In his celebrated work, Thomason essentially proved that a graph distributed as  $G(n, p)$  is  $(p, O(1))$ -pseudorandom with high probability.<sup>1</sup> On the other hand, there is no infinite sequence of  $(p, o(1))$ -pseudorandom graphs.

The following lemma can be easily verified by combining a Chernoff type bound with the union bound.

**Lemma 1.1.** *For every constant  $c > 0$  there is a constant  $C > 0$  such that for  $p \geq \frac{C}{n}$ , a random directed graph  $G \in D(n, p)$  is  $(p, c)$ -pseudorandom with high probability.*

Our results in this work will hold for every  $(p, r)$ -pseudorandom graph with  $p \geq \frac{C}{n}$  for some sufficiently large constant  $C$ , and every  $r \leq \mu\sqrt{pn}$  for some small constant  $\mu > 0$  that does not depend on  $C$ . By Lemma 1.1, a random directed graph distributed according to  $D(n, p)$  with  $p \geq \frac{C}{n}$  has this property with high probability.

We show here that the directed case is both similar and different from the undirected case. In fact, since we reduce here the directed case to the global resilience problem of the undirected case, we can use ideas from Dellamonica et al. [11] in order to get our bounds on the resilience for directed graphs. On the other hand, many of the techniques that were used

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<sup>1</sup>Here a sequence of events  $A_n$ ,  $n \geq 1$  is said to occur with high probability if  $\lim_{n \rightarrow \infty} \mathbb{P}[A_n] = 1$ .

for the undirected case cannot be applied in the directed case. Also, the range of parameters relevant to us is rather different, since in particular the result of Dellamonica et al. [11] shows that the removal of any 0.99-fraction of the edges of a (pseudo)random undirected graph still leaves a cycle of linear size. For the directed case it is easy to see that one can always remove half of the edges of any directed graph and get an acyclic directed graph, and hence a graph with no cycles at all.

## 1.2 Our results

Woodall [20] studied the minimal number of edges that guarantees the existence of a long cycle. In our terminology, he studied the global resilience of the complete graph  $K_n$  with respect to the property of having a cycle of length at least  $\ell$ . He proved the following.

**Theorem 1** (Woodall [20]). *Let  $3 \leq \ell \leq n$ . Every graph  $G$  on  $n$  vertices satisfying*

$$e(G) \geq \left\lceil \frac{n-1}{\ell-2} \right\rceil \cdot \binom{\ell-1}{2} + \binom{r+1}{2} + 1,$$

where  $r = (n-1) \bmod (\ell-2)$ , has a cycle of length at least  $\ell$ .

It is easy to verify that Woodall's bound is best possible. Indeed, take a graph formed by  $\lceil \frac{n-1}{\ell-2} \rceil$  disjoint cliques of size  $\ell-2$ , a single smaller clique of size  $r$  and a vertex that is connected to every other vertex in the graph. Clearly, the length of a longest cycle in this graph is at most  $\ell-1$ .

The work of Dellamonica et al. [11] can be viewed as a generalization of Woodall's work from the case of  $K_n$  to the case of general pseudorandom graphs. In order to cite their result and also for future reference in our paper the following function is defined.

**Definition 2.** For a given  $0 \leq \alpha < 1$ , define

$$w(\alpha) = 1 - (1 - \alpha) \lfloor (1 - \alpha)^{-1} \rfloor.$$

It is easy to verify that we have  $w(0) = 0$  and  $\lim_{x \nearrow 1} w(x) = 0$ . The following asymptotic version of Woodall's result is proved in [11].

**Theorem 3** ([11]). *Let  $\alpha > 0$ . For every  $\beta > 0$  there is  $n_0$  such that for every graph  $G$  on  $n > n_0$  vertices satisfying*

$$|E(G)| \geq \binom{n}{2} \cdot \left( 1 - (1 - w(\alpha))(\alpha + w(\alpha)) + \beta \right)$$

has a cycle of length at least  $(1 - \alpha) \cdot n$ .

Observe that we can partition a vertex set of size  $n$  into  $k = \lceil \frac{1}{1-\alpha} \rceil$  sets, each of size  $(1-\alpha)n$ , and a remaining set of size  $w(\alpha)n$ , as  $1 - w(\alpha) = k(1 - \alpha)$ .

Dellamonica et al. proved in [11] that Theorem 3 can be extended to (sparse) pseudorandom graphs; more specifically, they proved that any subgraph  $G' = (V, E')$  of a  $(p, r)$ -pseudorandom graph  $G = (V, E)$  (where  $p \gg n^{-1}$ , and  $r$  is fixed) with  $|E'| \geq (1 - (1 - w(\alpha))(\alpha + w(\alpha)) + o(1))|E|$  edges has a cycle of length at least  $(1 - \alpha) \cdot |V|$ . Here we provide tight bounds on the resilience of pseudorandom directed graphs with respect to the property of having a long directed cycle.

Our main theorem is a directed version of their result.

**Theorem 4.** *Fix  $0 < \gamma < \frac{1}{2}$  and let  $G = (V, E)$  be a  $(p, r)$ -pseudorandom directed graph on  $n$  vertices, where  $r \leq \mu\sqrt{np}$  and  $\mu(\gamma) > 0$  is a sufficiently small constant that depends only on  $\gamma$  and  $n$  is sufficiently large. Let  $G'$  be a subgraph of  $G$  with at least  $(\frac{1}{2} + \gamma)|E|$  edges. Then  $G'$  contains a directed cycle of length at least  $(1 - \alpha - o(1)) \cdot n$ , where  $\alpha$  satisfies*

$$2\gamma = 1 - (1 - w(\alpha))(\alpha + w(\alpha)).$$

Observe crucially that every directed graph  $G = (V, E)$  contains an acyclic subgraph  $G'$  with at least  $|E|/2$  edges. Indeed, fix a permutation  $\sigma : V \rightarrow V$ , and let  $G_1$  be the subgraph with all edges  $xy$  such that  $\sigma(x) > \sigma(y)$ , and  $G_2$  be the subgraph with all edges  $xy$  such that  $\sigma(x) < \sigma(y)$ . Then both  $G_1$  and  $G_2$  are acyclic, and at least one of them contains at least half of the edges of  $G$ .

Theorem 4 yields the following two immediate corollaries.

**Corollary 5.** *For every  $\gamma > 0$  there is a constant  $c_1(\gamma) > 0$  such that the following holds. Let  $G$  be a  $(p, r)$ -pseudorandom graph on  $n$  vertices,  $r \leq \mu\sqrt{np}$  where  $\mu(\gamma) > 0$  is some sufficiently small constant that depends only on  $\gamma$  and  $n$  is sufficiently large. Let  $G'$  be a subgraph of  $G$  with at least  $(1/2 + \gamma)|E(G)|$  edges. Then  $G'$  contains a directed cycle of length at least  $c_1 n$ .*

In other words, the above corollary guarantees that the deletion of less than half of the edges of a pseudorandom digraph leaves a cycle of linear length.

**Corollary 6.** *There exists a function  $c_2(\epsilon)$  with  $\lim_{\epsilon \rightarrow 0} c_2(\epsilon) = 0$  such that the following holds. Let  $G$  be a  $(p, r)$ -pseudorandom graph on  $n$  vertices,  $r \leq \mu\sqrt{np}$  where  $\mu(\gamma) > 0$  is some sufficiently small constant that depends only on  $\gamma$  and  $n$  is sufficiently large. Let  $G'$  be a subgraph of  $G$  with at least  $(1 - \epsilon)|E(G)|$  edges. Then  $G'$  contains a directed cycle of length at least  $(1 - c_2) \cdot n$ .*

Here, we prove that deleting a sufficiently small fraction of the edges of a pseudorandom digraph leaves a cycle of length close to  $n$ .

Finally, we prove the following matching lower bound.

**Proposition 7.** Fix  $0 < \gamma < \frac{1}{2}$  and let  $G$  be a  $(p, r)$ -pseudorandom directed graph on  $n$  vertices, where  $r = O(\sqrt{np})$  and  $pn \rightarrow \infty$ . There is a subgraph  $G'$  with  $(\frac{1}{2} + \gamma)|E|$  edges that does not contain any directed cycle of length at least  $(1 - \alpha + o(1)) \cdot n$ , where  $\alpha$  satisfies

$$2\gamma = 1 - (1 - w(\alpha))(\alpha + w(\alpha)).$$

**Our Tools.** One of the main tools we use in this work is a sparse directed variant of Szemerédi's regularity lemma (Lemma 2.1), that was stated in [12]. This allows us to partition our graph into a constant number of regular pairs, and essentially to reduce the problem to finding almost spanning paths in regular pairs.

To this end, we use a simple yet powerful lemma that finds almost spanning paths in expanding graphs (Lemma 2.3). In our case, a regular pair is a bipartite expander in both directions. The approach is based on ideas from [10, 3, 5].

The rest of the paper is organized as follows. In Section 2 we state the sparse directed regularity lemma, and prove that regular pairs have an almost spanning directed path. In Section 3, we reduce the resilience problem in directed graphs to undirected graphs, and then apply ideas from [11] and prove Theorem 4. In Section 4 we prove Proposition 7 and show that our results are essentially tight.

Throughout the paper we assume that the order of  $G$ , denoted by  $n$ , is large enough. We do not try to optimize constants and omit floor and ceiling signs whenever these are not crucial.

## 2 The regularity lemma for sparse directed graphs and long paths in regular pairs

### 2.1 The regularity lemma

In this section we follow [12] and state a regularity lemma for sparse directed graphs. We first provide some notation.

Given a directed graph  $G = (V, E)$ , for any pair of disjoint sets of vertices  $U, W$ , we let  $E_G(U, W)$  be the set of edges directed from  $U$  to  $W$ , and let  $e_G(U, W) = |E_G(U, W)|$ . We say that  $G$  is  $(\delta, D, p)$ -bounded if for any two disjoint sets  $U, W$  such that  $|U|, |W| \geq \delta|V|$  we have

$$e_G(U, W) \leq Dp|U||W|.$$

The *edge density* from a set  $U$  to a set  $W$  is defined by  $\frac{e_G(U, W)}{|U||W|}$ . We say that two sets  $U$  and  $W$  span a bipartite directed graph of *bi-density*  $p$  if it has edge density at least  $p$  in both directions. Also define the *directed  $p$ -density* from  $U$  to  $W$  by

$$d_{G,p}(U, W) = \frac{e_G(U, W)}{p|U||W|}.$$

We omit the index graph  $G$  and write  $d_p(U, W)$  whenever the base graph is clear from the context.

For  $0 < \delta \leq 1$ , a pair  $(U, W)$  is  $(\delta, p)$ -regular in a digraph  $G$  if for every  $U' \subseteq U$  and  $W' \subseteq W$  such that  $|U'| \geq \delta|U|$  and  $|W'| \geq \delta|W|$  we have both

$$|d_{G,p}(U, W) - d_{G,p}(U', W')| < \delta,$$

and

$$|d_{G,p}(W, U) - d_{G,p}(W', U')| < \delta.$$

A partition  $\mathcal{P} = \{V_0, V_1, \dots, V_k\}$  of  $V$  is  $(\delta, k, p)$ -regular if the following properties hold.

- (i).  $|V_0| \leq \delta|V|$ .
- (ii).  $|V_i| = |V_j|$  for all  $1 \leq i < j \leq k$ .
- (iii). At least  $(1 - \delta)\binom{k}{2}$  of the pairs  $(V_i, V_j), 1 \leq i < j \leq k$ , are  $(\delta, p)$ -regular.

We will use the following variant of Szemerédi's regularity lemma, that is stated in [12], and whose proof follows lines similar to the proof of the regularity lemma for sparse undirected graphs, proved independently by Kohayakawa and by Rödl (see, e.g., [15]). In [12] the lemma is stated for oriented graphs (where no anti parallel edges are allowed), yet the result can be easily adjusted to our case, where anti parallel edges are allowed.

**Lemma 2.1** (Lemma 3 in [12]). *For any real number  $\delta > 0$ , any integer  $k_0 \geq 1$  and any real number  $D > 1$ , there exist constants  $\eta = \eta(\delta, k_0, D)$  and  $K = K(\delta, k_0, D) \geq k_0$  such that for any  $0 < p(n) \leq 1$ , any  $(\eta, D, p)$ -bounded directed graph  $G$  admits a  $(\delta, k, p)$ -regular partition for some  $k_0 \leq k \leq K$ .*

## 2.2 Every regular pair contains a long path

We next prove that every regular pair of positive bi-density contains an almost spanning path. To this end, we first show a trivial expansion property of regular pairs, and then use this property to prove the desired result.

**Claim 2.2.** *Let  $(U, W)$  be a  $(\delta, p)$ -regular pair for  $|U| = |W|$  with bi-density at least  $2\delta p$ , where  $p > 0$ . Then for every two sets  $U' \subseteq U$  and  $W' \subseteq W$  such that  $|U'| \geq \delta|U|$  and  $|W'| \geq \delta|W|$  there is a directed edge from  $U'$  to  $W'$ .*

*Proof.* By regularity we have

$$e_G(U', W') \geq (d_p(U, W) - \delta)p|U'||W'| \geq (2\delta - \delta)p|U'||W'| = \delta p|U'||W'| > 0.$$

The claim follows. □

We next show that a bipartite directed graph with a simple expansion property contains a long directed path. The proof follows ideas from [10, 3, 5].

**Lemma 2.3.** *Let  $H = (V_1, V_2, E)$ , where  $|V_1| = |V_2| = t$ , be a directed bipartite graph that satisfies the following property: for every two sets  $A \subseteq V_1, B \subseteq V_2$  of size  $k$ , there is at least one edge from  $A$  to  $B$  and there is at least one edge from  $B$  to  $A$ . Then  $H$  contains a directed path of length  $2t - 4k + 3$ .*

*Proof.* Recall that DFS (Depth First Search) is a graph search algorithm that visits all the vertices of a (directed or undirected) graph  $G$  as follows. It maintains three sets of vertices, letting  $S$  be the set of vertices which we have completed exploring,  $T$  be the set of unvisited vertices, and  $U = V(G) \setminus (S \cup T)$ , where the vertices of  $U$  are kept in a *stack* (a *last in, first out* data structure). It is also assumed that some order  $\sigma$  on the vertices of  $G$  is fixed, and the algorithm prioritizes vertices according to  $\sigma$ . The algorithm starts with  $S = \emptyset, U = \emptyset$  and  $T = V(G)$ .

While there is a vertex in  $V(G) \setminus S$ , if  $U$  is non-empty, let  $v$  be the last vertex that was added to  $U$ . If  $v$  has an out-neighbor  $u \in T$ , the algorithm inserts  $u$  to  $U$ . If  $v$  does not have an out-neighbor in  $T$  then  $v$  is popped out from  $U$  and is moved to  $S$ . If  $U$  is empty, the algorithm chooses an arbitrary vertex from  $T$  and pushes it to  $U$ .

We now proceed to the proof of the lemma. We execute the DFS algorithm for an arbitrary chosen order  $\sigma$  on the vertices of the graph  $H$ . We let again  $S, T, U$  be three sets of vertices as defined above. At the beginning of the algorithm, all the vertices are in  $T$ , and at each step a single vertex either moves from  $T$  to  $U$  or from  $U$  to  $S$ . At the end of the algorithm, all the vertices are in  $S$ .

Consider the point during the execution of the algorithm when  $|S| = |T|$ . Observe crucially that all the vertices in  $U$  form a directed path, and we have  $||U \cap V_1| - |U \cap V_2|| \leq 1$ . Since  $|U| = 2t - |S| - |T| = 2t - 2|S|$  is even, we have in fact

$$|U \cap V_1| = |U \cap V_2|.$$

We get that

$$|S| = |S \cap V_1| + |S \cap V_2| = |T \cap V_1| + |T \cap V_2| = |T|,$$

and

$$|V_1 \setminus U| = |S \cap V_1| + |T \cap V_1| = |S \cap V_2| + |T \cap V_2| = |V_2 \setminus U|.$$

Hence, we get both

$$|S \cap V_2| = |T \cap V_1|,$$

and

$$|S \cap V_1| = |T \cap V_2|.$$

Assume without loss of generality that

$$|S \cap V_1| \geq |S|/2 \geq |S \cap V_2|.$$

Then  $|S \cap V_1| \geq t/2 - |U|/4$  and therefore  $|T \cap V_2| \geq t/2 - |U|/4$ . Observe crucially that there are no edges from  $S$  to  $T$ . By the assumption of the lemma we conclude that  $|S \cap V_1| = |T \cap V_2| \leq k - 1$  and therefore we get  $t/2 - |U|/4 \leq k - 1$  and hence  $|U| \geq 2t - 4k + 4$ . Thus  $H$  contains a directed path  $|U|$  of length  $2t - 4k + 3$ , as desired.  $\square$



We therefore have the following corollary.

**Corollary 2.4.** *Let  $(U, W)$  be a  $(\delta, p)$ -regular pair with bi-density at least  $2\delta p$  and  $|U| = |W| = t$ ,  $p > 0$ . Then the bipartite directed graph between  $U$  and  $W$  contains a directed path of length  $(1 - 2\delta) \cdot 2t + 2$  that starts at  $U$ .*

*Proof.* By Claim 2.2, there is an edge in each direction between every two sets of size  $\delta t$  in  $U$  and  $W$ . Therefore Lemma 2.3 implies the existence of a directed path of length  $(1 - 2\delta)2t + 3$ . Note that if the first vertex in the path is from  $W$  we may remove it, thus getting a directed path of length at least  $(1 - 2\delta)2t + 2$  that starts at  $U$ .  $\square$

### 3 Proof of Theorem 4

In this section we prove our main result. Given a constant  $\gamma > 0$ , we essentially want to prove that every subgraph with a  $(1/2 + \gamma)$ -fraction of the edges of a pseudorandom directed graph contains a long directed cycle. Let  $\delta = \delta(\gamma)$  be a constant that we will fix later,  $K = K(\delta, 1/\delta, 1 + \delta)$  and  $\eta = \eta(\delta, 1/\delta, 1 + \delta)$  be the constants defined by the regularity lemma (Lemma 2.1).

Let  $G = (V, E)$  be a  $(p, r)$ -pseudorandom directed graph with  $r \leq \mu\sqrt{np}$ . For  $\mu \leq \delta \cdot \min\{\eta, 1/K\}$  we have

$$r \leq \delta\sqrt{np} \cdot \min\{\eta, 1/K\}.$$

Let  $A, B$  be two sets of vertices of size  $\eta n$  in  $G$ . Observe that

$$r\sqrt{pn|A||B|} \leq \delta\sqrt{np}\eta\sqrt{p\eta^2n^3} = \delta\eta^2n^2p \leq \delta\eta^2n^2 = \delta|A||B|.$$

Therefore, we get that  $G$  is  $(\eta, 1 + \delta, p)$ -bounded.

Given a subgraph  $G' = (V, E')$  of  $G$  that contains  $(1/2 + \gamma)|E|$  edges, our goal is to show that  $G'$  contains a long directed cycle.

Clearly,  $G'$  is  $(\eta, 1 + \delta, p)$ -bounded as well, and hence we can apply the sparse directed regularity lemma (Lemma 2.1) to  $G'$  and get a partition of  $V$  to clusters  $V_0, V_1, \dots, V_m$ , where  $1/\delta \leq m \leq K$ ,  $|V_0| \leq \delta n$ ,  $|V_1| = |V_2| = \dots = |V_m| = t$  and all but at most a  $\delta$ -fraction of the pairs  $(V_i, V_j)$  are  $(\delta, p)$ -regular. Note that  $\frac{n(1-\delta)}{m} \leq t \leq \frac{n}{m}$ .

We next define an **undirected** auxiliary graph  $H$  on the clusters  $V_1, \dots, V_m$ . With a slight abuse of notation, we denote the vertices of  $H$  by  $V_1, V_2, \dots, V_m$ . Two vertices  $V_i$  and  $V_j$  are connected if the pair  $(V_i, V_j)$  is  $(\delta, p)$ -regular and has bi-density at least  $2\delta p$ .

Since  $G$  is  $(p, r)$ -pseudorandom and  $r \leq \frac{\delta\sqrt{np}}{K} \leq \frac{\delta\sqrt{np}}{m}$ , we get that the  $p$ -density of every pair  $(V_i, V_j)$  in  $G$  is at least  $1 - \delta$  and at most  $1 + \delta$ .

Observe that if  $V_i$  and  $V_j$  are not connected by an edge in  $H$ , one of the following must happen.

- The pair  $(V_i, V_j)$  is not regular.
- Either  $|E_{G'}(V_i, V_j)| < 2\delta p|V_i||V_j|$  or  $|E_{G'}(V_j, V_i)| < 2\delta p|V_i||V_j|$ . In other words, at least  $(1 - 3\delta)p|V_i||V_j|$  of the edges from  $V_i$  to  $V_j$  or from  $V_j$  to  $V_i$  in  $G$  are not in  $G'$ .

The number of non-regular pairs is at most  $\delta \binom{m}{2}$ . Since in every pair with edge bi-density less than  $2\delta p$  at least  $(1 - 3\delta)pt^2$  edges were lost when moving from  $G$  to  $G'$ , we get that the number of pairs in  $G'$  with edge bi-density less than  $2\delta p$  is bounded by

$$\frac{(1/2 - \gamma)pn^2}{(1 - 3\delta)pt^2} \leq \frac{(1/2 - \gamma)pn^2}{(1 - 3\delta)p\left(\frac{n(1-\delta)}{m}\right)^2} = \frac{1/2 - \gamma}{(1 - 3\delta)(1 - \delta)^2} \cdot m^2.$$

Observe that by our choice of parameters for the regularity lemma we have  $m \geq \frac{1}{\delta}$ , and hence we have

$$m^2 \leq 2 \binom{m}{2} (1 + 2/m) \leq 2 \binom{m}{2} (1 + 2\delta).$$

We conclude that the fraction of non-edges in  $H$  is bounded by

$$\delta + \frac{(1 - 2\gamma)(1 + 2\delta)}{(1 - 3\delta)(1 - \delta)^2}.$$

By taking  $\delta \ll \gamma$ , we get that the fraction of edges in  $H$  is at least  $2\gamma(1 - o(1))$ , where the  $o(1)$  term depends only on  $\delta$  and can be made arbitrary small.

Let  $\alpha$  be the minimal solution of the equation

$$2\gamma = 1 - (1 - w(\alpha))(\alpha + w(\alpha)),$$

where  $w$  is the function defined in Subsection 1.2. By Theorem 3, we get that  $H$  contains an undirected cycle of length at least  $(1 - \alpha - o(1))m$ .

We complete the proof by showing how to construct a long cycle in  $G'$  given a long cycle in  $H$ . We start with the case that  $H$  contains a long cycle of even length, and later show how to modify the proof in the case of odd length.

Let  $V_{i_1}, V_{i_2}, \dots, V_{i_{2b}}$  be a cycle of length  $(1 - \alpha + o(1))m$  in  $H$ . Note that for every  $1 \leq q \leq 2b$ , the pair  $(V_{i_q}, V_{i_{q+1}})$  (where we identify  $V_{i_{2b+1}}$  with  $V_{i_1}$ ) is  $(\delta, p)$ -regular and has edge bi-density at least  $2\delta p$ . Therefore, by Corollary 2.4, for every  $1 \leq q \leq b$ , there is a directed path  $P_q$  of length at least  $(1 - 2\delta) \cdot 2t$  that alternates between  $V_{i_{2q-1}}$  and  $V_{i_{2q}}$ . For every  $1 \leq q \leq b$ , let  $P_q^R = P_q \cap V_{i_{2q}}$  and let  $P_q^L = P_q \cap V_{i_{2q-1}}$ .

Observe that for every  $1 \leq q \leq b$ , by Claim 2.2 there is an edge from the last  $\delta t$  vertices of  $P_q^R$  to the first  $\delta t$  vertices of  $P_{q+1}^L$  (here we identify  $P_{b+1}^L$  with  $P_1^L$ ). Thus we can paste every two consecutive paths together, losing at most  $2\delta t$  vertices from each path. We conclude that we can paste all the paths together to get a directed cycle of length at least

$$b \cdot (1 - 2\delta - \delta) \cdot 2t = (1 - \alpha - o(1))mt = (1 - \alpha - o(1))|V(G')|,$$

where the  $o(1)$  term depends only on  $\delta(\gamma)$  and can be made arbitrary small (as  $\alpha$  depends on  $\gamma$  only).

Finally, suppose there is an odd cycle of length  $(1 - \alpha + o(1))m$  in  $H$ , whose vertices are by  $V_{i_1}, V_{i_2}, \dots, V_{i_{2b+1}}$ . As in the previous case, for every  $1 \leq q \leq b$ , let  $P_q$  be a path of length

at least  $(1 - 2\delta) \cdot 2t$  that alternates between  $V_{i_{2q-1}}$  and  $V_{i_{2q}}$ . Let  $Z'$  be the first  $\delta t$  vertices in  $P_1^L$  and  $Z''$  be the last  $\delta t$  vertices in  $P_b^R$ . Moreover, let

$$V'_{i_{2b+1}} = \{v \in V_{i_{2b+1}} : \forall u \in Z', (u, v) \notin E\},$$

and

$$V''_{i_{2b+1}} = \{v \in V_{i_{2b+1}} : \forall u \in Z'', (v, u) \notin E\}.$$

By Claim 2.2, we have  $|V'_{i_{2b+1}}|, |V''_{i_{2b+1}}| \leq \delta t$ , and therefore for all but  $2\delta t$  of the vertices in  $V_{i_{2b+1}}$  we have an edge from the last  $2\delta t$  vertices in  $P_b$  and an edge to the first  $2\delta t$  first vertices in  $P_1$ . Since  $t > 2\delta t$  we conclude that we can connect  $P_b$  and  $P_1$  through a vertex in  $V_{i_{2b+1}}$ , thus getting a path of length at least

$$(b - 1) \cdot (1 - 2\delta - \delta)2t = (1 - \alpha - o(1))|V(G')|.$$

Theorem 4 follows. □

## 4 Lower bounds

Let  $G = (V, E)$  be a directed graph. Recall that by fixing a permutation  $\sigma$  on the vertices, we can partition the edges of  $G$  into two acyclic sets as follows. The first set contains all directed edges  $xy$  where  $\sigma(x) > \sigma(y)$ , and the second set contains all directed edges  $xy$  where  $\sigma(y) > \sigma(x)$ . Therefore, the global resilience of every directed graph with respect to the property of having directed cycles is at most  $1/2$ . Here we extend this idea and show that our main result is asymptotically tight.

**Proof of Proposition 7.** We show that there is a subgraph  $G'$  with a  $(1/2 + \gamma)$ -fraction of the edges, whose longest directed cycle is of length at most  $(1 - \alpha + o(1))n$ . Our approach follows [11].

Recall that  $G$  is  $(p, r)$ -pseudorandom with  $r \leq \mu\sqrt{np}$  and  $pn \rightarrow \infty$ . We first claim that for every two disjoint sets  $A, B$  of size  $\Omega(n)$ , the number of edges from  $A$  to  $B$  is  $p|A||B|(1 + o(1))$ . Indeed, let  $|A| = an$  and  $|B| = bn$ , and suppose that  $a < b$ . If  $B'$  is a random subset of  $B$  of size  $an$ , then by linearity of expectation the number of edges between  $A$  and  $B'$  is  $E(A, B) \cdot \frac{|B'|}{|B|}$ . Therefore, if the number of edges between  $A$  and  $B$  is smaller than (respectively, larger than)  $p|A||B|(1 + o(1))$  then there is a choice of a set  $B'$  such that  $A$  and  $B'$  contradicts the  $(p, r)$ -pseudorandomness of  $G$ .

We partition the vertices of  $G$  into  $k$  classes  $V_1, V_2, \dots, V_k$  of size  $(1 - \alpha)n$  each, and one additional class  $V_{k+1}$  of size  $w(\alpha)n \leq (1 - \alpha)n$ . Let  $G'$  be the subgraph with all edges from  $V_i$  to  $V_j$ , for  $1 \leq i < j \leq k + 1$ , and all the edges within each class  $V_i$ . Clearly,  $G'$  does not contain a cycle longer than  $(1 - \alpha)n$ , since a directed path leaving a certain  $V_i$  cannot return there. Therefore, we can conclude the proof by showing that the number of edges in  $G'$  is as claimed.

Indeed, since  $G$  is  $(p, r)$ -pseudorandom and each  $V_i$  is of size  $\Omega(n)$ , we get that for every  $1 \leq i < j \leq k$ , the number of edges from  $V_j$  to  $V_i$  is  $(1 - \alpha)^2 \cdot pn^2(1 + o(1))$ . The number of edges from  $V_{k+1}$  to  $\cup_{i \leq k} V_i$  in  $G$  is  $k(1 - \alpha)w(\alpha) \cdot pn^2(1 + o(1))$ . Recalling that by definition of  $w(x)$  we have  $(1 - \alpha)k = 1 - w(\alpha)$ , we get that the number of edges we deleted from  $G$  to get  $G'$  is

$$\begin{aligned} & \left( \binom{k}{2} (1 - \alpha)^2 + k(1 - \alpha)w(\alpha) \right) \cdot pn^2(1 + o(1)) \\ &= (1 - \alpha)k((1 - \alpha)(k - 1) + 2w(\alpha))(1 + o(1)) \frac{pn^2}{2} \\ &= (1 - w(\alpha))(\alpha - w(\alpha) + 2w(\alpha)) \cdot \frac{pn^2(1 + o(1))}{2} \\ &= (1 - w(\alpha))(\alpha + w(\alpha)) + o(1) \frac{pn^2}{2}. \end{aligned}$$

Note that the number of edges in  $G$  is  $pn^2(1 + o(1))$ . Let the number of edges in  $G'$  be  $(1/2 + \gamma) \cdot pn^2$ . Then the number of edges we deleted satisfies

$$(1 + o(1))(1/2 - \gamma)pn^2 = (1 - w(\alpha))(\alpha + w(\alpha)) + o(1) \frac{pn^2}{2},$$

and therefore  $\gamma$  satisfies

$$2\gamma = 1 - (1 - w(\alpha))(\alpha + w(\alpha)) + o(1),$$

as claimed. The statement follows. □

## 5 Concluding remarks

We studied the global resilience of pseudorandom directed graphs with respect to the property of having a long directed cycle. We gave matching lower and upper bounds, and our proof essentially reduced our problem to the case of undirected graphs.

A variety of questions regarding the resilience of directed graphs can be asked. A few, somewhat arbitrary examples are the problem of local resilience with respect to having a long directed cycle, the resilience with respect to the property of having some fixed directed graph. Another interesting problem is the resilience with respect to Hamiltonicity, which in the dense case is settled in [13].

In this work we considered subgraphs with a  $(1/2 + \gamma)$ -fraction of the edges, and observed that every directed graph contains an acyclic subgraph with a  $1/2$ -fraction of the edges. In [5], the authors proved that every two-coloring of the edges of a pseudorandom digraph contains a relatively long monochromatic path. That is, instead of proving that a large subgraph has a certain property, it is proved that every partition of the edges of the graph has a certain property. It will be interesting to give such results for other properties of directed graphs.

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