Rigid partitions: from high connectivity to random graphs

Michael Krivelevich[*]   Alan Lew[†]   Peleg Michaeli[‡]

Abstract

A graph is called d-rigid if there exists a generic embedding of its vertex set into $\mathbb{R}^d$ such that every continuous motion of the vertices that preserves the lengths of all edges actually preserves the distances between all pairs of vertices. The rigidity of a graph is the maximal d such that the graph is d-rigid. We present new sufficient conditions for the d-rigidity of a graph in terms of the existence of "rigid partitions"—partitions of the graph that satisfy certain connectivity properties. This extends previous results by Crapo, Lindemann, and Lew, Nevo, Peled and Raz.

As an application, we present new results on the rigidity of highly-connected graphs, random graphs, random bipartite graphs, pseudorandom graphs, and dense graphs. In particular, we prove that random $Cd \log d$-regular graphs are typically d-rigid, demonstrate the existence of a giant d-rigid component in sparse random binomial graphs, and show that the rigidity of relatively sparse random binomial bipartite graphs is roughly the same as that of the complete bipartite graph, which we consider an interesting phenomenon. Furthermore, we show that a graph admitting $\binom{d+1}{2}$ disjoint connected dominating sets is d-rigid. This implies a weak version of the Lovász–Yemini conjecture on the rigidity of highly-connected graphs. We also present an alternative short proof for a recent result by Lew, Nevo, Peled, and Raz, which asserts that the hitting time for d-rigidity in the random graph process typically coincides with the hitting time for minimum degree d.

1 Introduction

A d-**dimensional framework** is a pair $(G, \mathbf{p})$ where $G = (V, E)$ is a finite simple graph and $\mathbf{p} : V \to \mathbb{R}^d$ is an embedding of its vertices. A framework is called **rigid** if every continuous motion of the vertices that preserves the lengths of all the edges does not change the distance between any two vertices. An embedding of $G = (V, E)$ is **generic** if its $d|V|$ coordinates are algebraically independent over the rationals. A framework with a generic embedding will be called a **generic** framework.

We define the **rigidity matrix** $\mathsf{R}(G, \mathbf{p}) \in \mathbb{R}^{|E| \times d|V|}$ of a framework $(G, \mathbf{p})$ as follows: the rows of $\mathsf{R}(G, \mathbf{p})$ are indexed by the edge set E and the columns are indexed by allocating d coordinates to each vertex of V. The row vector $\mathbf{r}_e$ indexed by $e = \{u, v\}$ is supported on the coordinates of u, v where it is equal to the d-dimensional row vectors $\mathbf{p}(u) - \mathbf{p}(v)$ and $\mathbf{p}(v) - \mathbf{p}(u)$, respectively. For simplicity, we assume that G has at least $d + 1$ vertices. It is known (see [6]) that the rank of $\mathsf{R}(G, \mathbf{p})$ is always at most $d|V| - \binom{d+1}{2}$. The framework $(G, \mathbf{p})$ is called **infinitesimally rigid** if the rank of $\mathsf{R}(G, \mathbf{p})$ is exactly $d|V| - \binom{d+1}{2}$. Infinitesimal rigidity is in general a stronger condition than rigidity; however, it was shown by Asimow and Roth in [6] that both conditions are equivalent for generic embeddings.

We say that a graph $G = (V, E)$ is **rigid in** $\mathbb{R}^d$ (or, d-**rigid**, for short) if there exists an embedding $\mathbf{p} : V \to \mathbb{R}^d$ such that the framework $(G, \mathbf{p})$ is infinitesimally rigid (or equivalently, if $(G, \mathbf{p})$ is rigid for all

[*]School of Mathematical Sciences, Tel Aviv University, Tel Aviv 6997801, Israel. Email: krivelev@tauex.tau.ac.il.

[†]Department of Mathematical Sciences, Mellon College of Science, Carnegie Mellon University, Pittsburgh, PA 15213, USA. Email: alanlew@andrew.cmu.edu.

[‡]Mathematical Institute, University of Oxford, Oxford, UK. Email: peleg.michaeli@maths.ox.ac.uk. Research partially supported by ERC Advanced Grant 883810.

generic embeddings $\mathbf{p} : V \to \mathbb{R}^d$; see [6]). We define the **rigidity** of G to be the maximal d such that G is d-rigid. d-rigidity is thus a graph-theoretic property; however, a purely combinatorial characterization remains a major open problem in rigidity theory for $d > 2$ (see, e.g., [59]).

Rigidity theory has a long history, tracing its origins to early results such as Cauchy's theorem (see, e.g., [1]), and has found applications (especially in two and three dimensions) in various fields outside of mathematics (see, e.g., the survey [61]). In particular, significant efforts have been directed towards characterising d-rigid graphs, with seminal works [56], [46], and [6, 7]. For further reading, we refer the reader to the textbook [31] and the survey [18].

In this paper, we present a new sufficient condition for d-rigidity based on graph partitions. We apply this to show that high connectivity implies high rigidity, a first result of its kind for dimensions greater than 2. We also utilize it to establish lower bounds for the rigidity of (pseudo)random and dense graphs, derive a simplified proof for the lower bound of rigidity in complete bipartite graphs, and examine a notable phenomenon in random bipartite graphs.

1.1 Rigid partitions

Let $G = (V, E)$ be a graph. For pairwise disjoint vertex sets $V_1, \ldots, V_k$, we say that $V_1, \ldots, V_k$ is a **partial colouring** of V if $V_1 \cup \cdots \cup V_k \subseteq V$, or a **colouring** of V if $V_1 \cup \cdots \cup V_k = V$. A **partition** of V is a colouring $V_1, \ldots, V_k$ of V in which $V_i \neq \varnothing$ for all $1 \leq i \leq k$. For two (not necessarily disjoint) vertex sets $A, B \subseteq V$, we let $E(A, B)$ be the set of edges having one endpoint in A and the other in B. Let $E_1, \ldots, E_k$ be a partial colouring of the edge set of G, and let $\hat{E} = E_1 \cup \cdots \cup E_k$. We say that $U \subseteq V$ has a **monochromatic cut** (with respect to the partial colouring) if there exists a partition $U = U' \cup U''$ such that $\hat{E} \cap E(U', U'') \subseteq E_i$ for some $i \in [k]$.

Now, fix $d \geq 1$, let $V_1 \cup \cdots \cup V_{d+1}$ be a colouring of V, and let $\{E_{ij}\}_{1 \leq i < j \leq d+1}$ be a partial colouring of E, for which $e \subseteq V_i \cup V_j$ for all $1 \leq i < j \leq d+1$ and $e \in E_{ij}$. Suppose the subgraphs $G_{ij} = (V_i \cup V_j, E_{ij})$ are connected for all $1 \leq i < j \leq d+1$ (where we consider a graph without vertices to be disconnected), and for every $1 \leq i \leq d+1$ and $U \subseteq V_i$ with $|U| \geq 2$, U has a monochromatic cut. Then, we call the pair $\left(\{V_i\}_{i=1}^{d+1}, \{E_{ij}\}_{1 \leq i < j \leq d+1}\right)$ a d-**rigid partition** of G. For notational convenience, a vertex partition $\{V_i\}_{i=1}^{d+1}$ is also called a d-rigid partition if there exists an associated partial colouring $\{E_{ij}\}_{1 \leq i < j \leq d+1}$ for which the aforementioned pair is a d-rigid partition. Note that in a d-rigid partition of G one of the vertex sets V_i may be empty.

Our main theorem states that any graph that admits a d-rigid partition is also d-rigid.

Theorem 1.1. *A graph that admits a d-rigid partition is d-rigid.*

The case $d = 2$ of Theorem 1.1 was addressed by Crapo [19], who proved that a graph is 2-rigid if and only if it admits a 2-rigid partition (in fact, Crapo used a different but equivalent notion of "proper $3T2$ decompositions"). For $d = 3$, Lindemann provided a sufficient condition for 3-rigidity in his PhD thesis [51]. His condition is described in different terms, using the notion of "proper d-framework decompositions" (which extends Crapo's notion of $3T2$ decompositions in the $d = 2$ case), but it can be shown to be equivalent to the condition stated in Theorem 1.1. Lindemann conjectured (using a different terminology; see [51, Conjecture 5.6]) that the existence of a 3-rigid partition characterizes 3-rigid graphs, and asked whether the same holds for general d. We show here that, for some values of d (and assuming the validity of certain conjectures on the rigidity of specific families of graphs, for all large enough d), the existence of a d-rigid partition is *not* a necessary condition for d-rigidity (see discussion in Section 2.7).

Recently, Jordán and Tanigawa [39] introduced the d-**dimensional algebraic connectivity** $\mathsf{a}_d(G)$ of a graph G, which is a high dimensional generalisation of the algebraic connectivity of G introduced by Fiedler in [25] (corresponding to the $d = 1$ case), and can be seen as a quantitative measure of

the d-rigidity of G (see Section 2.1 for a precise definition). Using this notion, we establish here a "quantitative" version of Theorem 1.1 (see Theorem 2.1), which directly implies Theorem 1.1.

In practice, it is often difficult to apply Theorem 1.1 directly. Therefore, we introduce a few concrete graph partitions that are more applicable. Let $G = (V, E)$ be a graph. A **dominating set** for G is a subset S of its vertices, such that any vertex of G is either in S, or has a neighbour in S. We say that a vertex set S is **connected** if the induced graph $G[S]$ is connected. A **connected dominating set** (CDS) is a dominating set that is also connected. A **CDS partition of size** k is a partition $V_1, \ldots, V_k$ of V such that each V_i is a CDS. Note that, since a set containing a CDS is also a CDS, the existence of k vertex disjoint connected dominating sets implies the existence of a CDS partition of size k. Our first application of Theorem 1.1 is the following.

Corollary 1.2. *A graph that admits a CDS partition of size $\binom{d+1}{2}$ is d-rigid.*

A proof of Corollary 1.2 appears in Section 2.4. We use Corollary 1.2 to prove that highly-connected graphs are highly-rigid. Let us make this statement precise. It is well known (see e.g. [33]) that d-rigid graphs are d-connected. Lovász and Yemini [52] conjectured in 1982 that there exists a (smallest) integer c_d, possibly $c_d = d(d+1)$, such that every c_d–connected graph is d-rigid. The $d = 2$ case was solved by Lovász and Yemini in [52]. Some progress has been made in recent years in the $d > 2$ case (see e.g. [36,38] for related results on powers of graphs, and [33] for a partial result in the case $d = 3$). In particular, Clinch, Jackson and Tanigawa proved in [16] an analogous version of the conjecture for the "C_2^1-cofactor matroid", which is conjectured to be equal to the generic 3-rigidity matroid ([68, Conjecture 10.3.2]). As an application of Corollary 1.2, we prove the following.

Theorem 1.3 (Highly connected graphs)**.** *There exists $C > 0$ such that for all $d = d(n) \geq 1$, every n-vertex $Cd^2\log^2 n$-connected graph is d-rigid.*

The proof of Theorem 1.3 follows by combining Corollary 1.2 with a result of Censor-Hillel, Ghaffari, Giakkoupis, Haeupler and Kuhn [12] that states that every k-connected graph has a CDS partition of size $\Omega(k/\log^2 n)$; see Section 2.5. We wish to highlight that shortly after making our manuscript available online, the full resolution of the Lovász–Yemini conjecture was announced by Villányi [66]. Since, as observed in [70], it is clear that every graph admitting a CDS partition of size k is either complete or k-connected, Villányi's results imply a weaker version of Corollary 1.2, requiring a partition of size $d(d+1)$ rather than $\binom{d+1}{2}$.

We proceed by presenting a few additional useful rigid partitions. For two (not necessarily disjoint) vertex sets $A, B \subseteq V$, write $G[A, B]$ for the subgraph of G on the vertex set $A \cup B$ with the edge set $E(A, B)$. We say that a partition $V_1, \ldots, V_d$ of V is a **(type I) strong d-rigid partition** of G, and that G *admits* a (type I) strong d-rigid partition, if $G[V_i, V_j]$ is connected for all $1 \leq i \leq j \leq d$ (see Fig. 1a). Throughout this paper we will mostly work with this definition, and omit the "type I" prefix. We say that a partition $V_1, \ldots, V_{d+1}$ of V is a **type II strong d-rigid partition** of G, and that G *admits* a (type II) strong d-rigid partition, if $G[V_i, V_j]$ is connected for all $1 \leq i < j \leq d + 1$ (see Fig. 1b). For strong d-rigid partitions of type II, unlike type I, there is no connectivity requirement for the induced subgraphs $G[V_i, V_i] = G[V_i]$, but a partition into $d + 1$ parts is necessary, as opposed to the d parts required in type I. It is important to note that neither definition implies the other (see Propositions 2.12 and 2.13); however, both types qualify as rigid partitions, as detailed in Section 2.4.

Corollary 1.4. *A graph that admits a (type I or type II) strong d-rigid partition is d-rigid.*

A quantitative version of the "type I" case in Corollary 1.4 has recently been proved by the second author with Nevo, Peled and Raz [50, Theorem 1.3]. Indeed, our proof of Theorem 2.1 relies on an extension of their arguments.
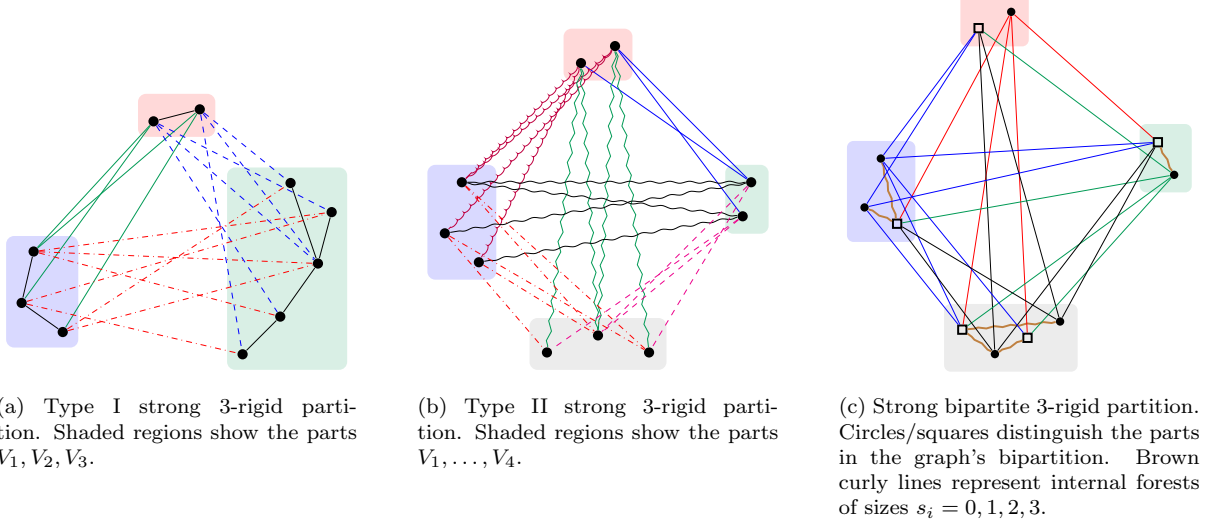
(a) Type I strong 3-rigid partition. Shaded regions show the parts V_1, V_2, V_3.

(b) Type II strong 3-rigid partition. Shaded regions show the parts $V_1, \ldots, V_4$.

(c) Strong bipartite 3-rigid partition. Circles/squares distinguish the parts in the graph's bipartition. Brown curly lines represent internal forests of sizes $s_i = 0, 1, 2, 3$.

Figure 1: Illustrations of strong 3-rigid partitions of *minimally 3-rigid graphs* (graphs which are 3-rigid with the minimum possible number of edges).

We would like to highlight a few limitations of proving d-rigidity of a graph by showing that it admits a strong d-rigid partition. The **domination number** of G, denoted by $\gamma(G)$, is the number of vertices in a smallest dominating set for G. Observe that if an n-vertex graph G admits a strong d-rigid partition then $\gamma(G) \leq n/d$. Thus, this method can only be used to prove that G is d-rigid for $d \leq n/\gamma(G)$. A second limitation is that bipartite graphs do not admit strong d-rigid partitions for $d \geq 3$. Indeed, let $G = (A \cup B, E)$ be a bipartite graph with bipartition (A, B), and let $V_1, \ldots, V_d$ be a partition of V for $d \geq 3$. If there exists $1 \leq i < j \leq d$ such that V_i, V_j are both in A or in B then $G[V_i, V_j]$ is not connected, hence this is not a strong d-rigid partition. Otherwise, there exists $1 \leq i \leq d$ such that V_i intersects both A and B. Thus, for $j \neq i$, $G[V_i, V_j]$ is not connected, and again, $V_1, \ldots, V_d$ is not a strong d-rigid partition. Nevertheless, bipartite graphs may be d-rigid for arbitrarily large d. This motivates the following bipartite variant of the definition above.

Let $G = (A \cup B, E)$ be a bipartite graph with bipartition (A, B). Let $V_1, \ldots, V_{d+1}$ be a partition of $V = A \cup B$. Write $A_i = V_i \cap A$ and $B_i = V_i \cap B$ for $1 \leq i \leq d+1$. We say that the partition is a **strong bipartite d-rigid partition** of G, and that G *admits* a strong bipartite d-rigid partition, if the following conditions hold: $G[A_i, B_j]$ is connected for all $i \neq j$, and there exists a sequence $0 \leq s_1 \leq s_2 \leq \cdots \leq s_{d+1}$ satisfying $\sum_{i=1}^{k} s_i \geq \binom{k}{2}$ for all $1 \leq k \leq d+1$, such that for every $1 \leq i \leq d+1$, $G[A_i, B_i]$ contains a forest with s_i edges. See Fig. 1c.

Corollary 1.5. *A bipartite graph that admits a strong bipartite d-rigid partition is d-rigid.*

Whiteley [67] (see also [37, Theorem 1.2] and [54, Corollary 4]), and independently Raymond [57], both relying on previous work by Bolker and Roth [8], showed that, for $m, n \geq 2$ and $d \geq 1$, the complete bipartite graph $K_{m,n}$ is d-rigid if and only if $m, n \geq d+1$ and $m + n \geq \binom{d+2}{2}$. We show here that, in fact, any complete bipartite graph $K_{m,n}$ satisfying these conditions admits a strong bipartite d-rigid partition.

Proposition 1.6. *Let $d \geq 1$. If $m, n \geq d+1$ and $m + n \geq \binom{d+2}{2}$, then $K_{m,n}$ admits a strong bipartite d-rigid partition.*

Combining Corollary 1.5 and Proposition 1.6, we obtain a new simple proof of the sufficient condition for the d-rigidity of complete bipartite graphs. The proofs appear in Sections 2.4 and 2.6, respectively.

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1.2 Applications to random and pseudorandom graphs

We present several applications of Corollaries 1.4 and 1.5 to random and pseudorandom graphs. For all instances where we apply Corollary 1.4, we demonstrate that a particular graph (typically) admits a (type I) strong d-rigid partition. In these scenarios, similar reasoning, which we forego detailing, can be used to show that the graph also admits a type II strong d-rigid partition.

Our first application is for pseudorandom graphs, for which we are going to use the following linear-algebraic definition. Let $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ be the eigenvalues of the adjacency matrix of a graph G. An n-vertex k-regular graph G is an (n, k, λ)-**graph** if $\max\{|\lambda_2(G)|, |\lambda_n(G)|\} \leq \lambda$. For a comprehensive review on pseudorandom graphs, see [44].

Theorem 1.7 (Pseudorandom graphs). *There exists $C > 1$ such that if G is an (n, k, λ)-graph with $k \geq \max\{9d\lambda, Cd \log d\}$ then G admits a strong d-rigid partition.*

We remark that here and later we prefer clarity of presentation over optimisation of the constants. Theorem 1.7 will follow immediately from a more general statement (Proposition 4.4) about "jumbled" graphs; see Section 4.1. Note that in light of Villányi's recent findings [66], one can establish a stronger rigidity bound for graphs with mild expansion. Given that every k-regular graph is $(k-\lambda_2(G))$-connected (see [25]), it follows that such a graph is $(1 - o(1))\sqrt{k - \lambda_2(G)}$-rigid. See also [15, 23] for some related results in the $d = 2$ case.

Since random regular graphs are, **whp**[1] (n, k, λ)-graphs for $\lambda = \Theta(\sqrt{k})$ (see, e.g., [27, 26, 58]), it follows immediately from Theorem 1.7 that a random $\Omega(d^2)$-regular graph is **whp** d-rigid. Using a more direct argument, we are able to prove the following stronger statement. Let $\mathcal{G}_{n,k}$ denote the distribution of random k-regular graphs. We denote a graph sampled from this distribution as $G \sim \mathcal{G}_{n,k}$, or occasionally, slightly abusing notation, simply as $\mathcal{G}_{n,k}$.

Theorem 1.8 (Random regular graphs). *There exists $C > 0$ such that for every fixed $d \geq 2$ and $k \geq Cd \log d$, $G \sim \mathcal{G}_{n,k}$ **whp** admits a strong d-rigid partition.*

Previously, the sole result regarding the rigidity of random regular graphs came from [34], demonstrating that $\mathcal{G}_{n,k}$ is **whp** 2-rigid when $k \geq 4$. It is plausible, however, that $\mathcal{G}_{n,2d}$ is **whp** d-rigid for every $d \geq 2$. Our result can be viewed as a first step in that direction.

Consider the **binomial random graph** $G(n, p)$ which is the distribution over graphs on the vertex set $[n]$ in which each pair of vertices is connected independently with probability p. It is well known [10] that every (nontrivial) monotone graph property has a threshold in $G(n, p)$. Since rigidity is evidently a monotone property, it is natural to try and identify its threshold. The initial efforts in this direction were made by Jackson, Servatius, and Servatius [34] who proved that $p \geq (\log n + \log \log n + \omega(1))/n$ suffices to guarantee 2-rigidity **whp**. As for $d > 2$, further advancements were achieved by Király and Theran [41], as well as by Jordán and Tanigawa [39]. Eventually, Lew, Nevo, Peled, and Raz [49] established that $p_c(d) = (\log n + (d - 1)\log \log n)/n$ is the threshold for d-rigidity for any fixed value of d. In fact, for fixed d, they also proved a more robust *hitting-time result*, which we discuss next.

The **random graph process** on the vertex set $[n]$ is a stochastic process that starts with the vertex set $[n]$ and no edges, and at each step, a new edge is added by selecting uniformly from the set of missing edges. We denote by $G(n, m)$ the random graph process on n vertices at time m; note that this is a uniformly chosen random graph on n vertices with m edges. The **hitting time** of a monotone property refers to the first time at which the random graph process possesses that particular property. An advantage of studying random graph processes is that it allows for a higher resolution analysis of the typical emergence of (monotone) graph properties. The main result of [49] is that for every fixed $d \geq 1$,

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[1]With high probability, i.e., with probability tending to 1 as n tends to ∞.

the hitting time for minimum degree d coincides, **whp**, with the hitting time for d-rigidity[2]. Here, we extend this result for sufficiently slowly growing d, and strengthen it by showing that the hitting time for minimum degree d coincides, **whp**, with the hitting time for the admittance of strong d-rigid partitions. Let τ_d denote the hitting time for minimum degree d in a random graph process.

Theorem 1.9 (Hitting time for d-rigidity). *There exists $c > 0$ for which the following holds. For every $1 \leq d = d(n) \leq c \log n / \log \log n$, a random graph $G \sim G(n, \tau_d)$ **whp** admits a strong d-rigid partition.*

The above theorem is sharp in the sense that d-rigidity implies d-connectivity, and hence also minimum degree d. For $d \gg \log n / \log \log n$, however, the threshold for d-rigidity remains unknown. In this work, we establish that $G(n, p)$ is **whp** d-rigid for $d = \Omega(np / \log(np))$, provided that $p \geq (1 + \varepsilon) \log n / n$, which is best possible up to logarithmic factors. Moreover, we show the existence of a strong d-rigid partition, an interesting result in its own right.

Theorem 1.10 (Binomial random graphs). *For every $\varepsilon > 0$ there exists $c > 0$ such that if $p \geq (1 + \varepsilon) \log n / n$, then $G \sim G(n, p)$ **whp** admits a strong d-rigid partition for $d = cnp / \log(np)$.*

We emphasize that the above statement is sharp, up to constants, in the following sense: a graph that admits a strong d-rigid partition has, in particular, a dominating set of size n/d. The smallest dominating set in $G(n, p)$ is **whp** of size $\Omega(\log(np)/p)$ (see, e.g., [29]), implying that the largest d for which $G(n, p)$ admits a strong d-rigid partition is **whp** $O(np / \log(np))$. Nevertheless, we conjecture that when $np \gg \log n$, the "bottleneck" for the rigidity of $G(n, p)$ lies in its number of edges: note that a d-rigid n-vertex graph must have at least $dn - \binom{d+1}{2}$ edges (otherwise, the rigidity matrix associated to any embedding of G into $\mathbb{R}^d$ has rank smaller than $dn - \binom{d+1}{2}$, so G is not d-rigid). Since the expected number of edges in $G(n, p)$ is $p\binom{n}{2}$, if $G(n, p)$ is d-rigid **whp** then it must satisfy the inequality $p\binom{n}{2} \geq dn - d(d+1)/2$, which holds for $d \leq (1 - o(1))(1 - \sqrt{1-p})n$. Accordingly, we conjecture:

Conjecture 1.11. *For every $p = \omega(\log n / n)$, $G \sim G(n, p)$ is **whp** d-rigid for $d = (1 - o(1))(1 - \sqrt{1-p})n$.*

Remark (Added in proof). A recent result of Peled and Peleg [55] establishes Conjecture 1.11 under the additional assumption that $p \ll n^{-1/2}$.

Next, we establish the following equivalent of Theorem 1.10 for random bipartite graphs. Let $G(n, n, p)$ be the **binomial bipartite random graph**, which is the distribution over bipartite graphs with two parts A and B, each of size n, in which each vertex in A is adjacent to each vertex in B independently with probability $p = p(n)$.

Theorem 1.12 (Binomial random bipartite graphs). *For every $\varepsilon > 0$ there exists $c > 0$ such that if $p \geq (1 + \varepsilon) \log n / n$, then $G \sim G(n, n, p)$ **whp** admits a strong bipartite d-rigid partition for $d = \min\{cnp / \log(np), 0.8\sqrt{n}\}$.*

The upper bound $O(\sqrt{n})$[3] on d in Theorem 1.12 should not come as a surprise, given that the complete bipartite graph $K_{n,n}$ is d-rigid for $d \sim 2\sqrt{n}$; see Proposition 1.6 and the preceding discussion. Nonetheless, it is a noteworthy observation that relatively sparse binomial random bipartite graphs (for $p = \Omega(\log n / \sqrt{n})$) attain this maximal possible rigidity, at least up to constant factors. This stands in contrast to what is seen in non-bipartite graphs, as indicated by Theorem 1.10 and Conjecture 1.11.

Since 1-rigidity is equivalent to connectivity, the notion of d-rigidity naturally arises as a generalisation for connectivity. A fundamental phenomenon in random graphs is the "phase transition", where a "giant" (linear-sized) connected component emerges. Extending this concept, it is natural to investigate the

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[2]In fact, the results from [49] hold also for more general notions of rigidity, defined in terms of "abstract rigidity matroids" (see e.g. [30]).

[3]Note that we have not made any particular effort to improve the value 0.8 in the theorem.

(sudden?) emergence of a giant d-rigid component. A d-**rigid component** of a graph $G = (V, E)$ is a maximal d-rigid induced subgraph of G. That is, for $U \subseteq V$, $G[U]$ is a d-rigid component of G if it is d-rigid but $G[U']$ is not d-rigid for any $U \subsetneq U'$. It is conjectured by Lew, Nevo, Peled and Raz [49] (and proved for $d = 2$ by Kasiviswanathan, Moore and Theran in [40]), that the random graph $G(n, p)$ undergoes a phase transition for the containment of a giant d-rigid component exactly at the "time" that the average degree of the $(d + 1)$-core of the graph exceeds $2d$. This coincides with the threshold for d-orientability of $G(n, p)$ (see [24]), which happens at $p \sim d/n$ (see [20, Theorem 1]).

In this work, we establish that for sufficiently large $c = c(d)$, the random graph $G(n, c/n)$ contains, **whp**, a giant—and in fact almost spanning—d-rigid component.

Theorem 1.13 (Binomial random bipartite graphs). *For every $\varepsilon \in (0, 1)$ there exists $C > 0$ such that for every fixed $d \geq 2$, $G \sim G(n, Cd \log d/n)$ contains **whp** a d-rigid component with at least $(1 - \varepsilon)n$ vertices.*

1.3 Applications to dense graphs

Next, we consider the rigidity of dense graphs. More specifically, we look at "Dirac graphs", that is, n-vertex graphs with minimum degree at least $n/2$.

Theorem 1.14. *Let $\ell = \ell(n)$ satisfy $3 \log n/2 \leq \ell < n/2$. Let $G = (V, E)$ be an n-vertex graph with $\delta(G) \geq \frac{n}{2} + \ell$. Then, G admits a strong d-rigid partition for $d = 2\ell/(3 \log n)$.*

Theorem 1.14 is an immediate consequence of a more general result about graphs in which every pair of vertices has "many" common neighbours; see Theorem 5.1. For $\ell = \Theta(n)$, Theorem 1.14 is sharp up to constants. Consider, for instance, $G \sim G(n, p)$ for $p = \frac{1}{2} + 2\varepsilon$; here, $\delta(G) \geq \left(\frac{1}{2} + \varepsilon\right)n$ **whp**. However, the largest d for which G admits a strong d-rigid partition is **whp** $O(n/\log n)$. For smaller values of ℓ, the following theorem offers a significantly improved bound on the rigidity of the graph.

Theorem 1.15. *There exists $c > 0$ such that for every $0 \leq d < c\sqrt{n}/\log n$, if G is an n-vertex graph with $\delta(G) \geq \frac{n}{2} + d$ then G is $(d + 1)$-rigid.*

Theorem 1.15 is sharp up to a multiplicative constant. For example, consider a graph composed of two cliques, each of size $n/2 + k$, intersecting in $2k$ vertices. This graph, an n-vertex graph with a minimum degree of $n/2 + k - 1$, has vertex-connectivity of $2k$ and, hence, is at most $2k$-rigid (and in fact, it is easy to check that its rigidity is exactly $2k$). It is also sharp in terms of the minimum degree, as there are disconnected n-vertex graphs with $\delta(G) = \lfloor n/2 \rfloor - 1$. We conjecture that for all $1 \leq \ell \leq n/2 - 1$, every graph with a minimum degree of at least $n/2 + \ell$ is $\Omega(\ell)$-rigid:

Conjecture 1.16. *Let $1 \leq \ell(n) < n/2$. Let G be an n-vertex graph with minimum degree $\delta(G) \geq \frac{n}{2} + \ell(n)$. Then, G is d-rigid for $d = \Omega(\ell(n))$.*

Remark (Added in proof). In recent work [43], we study the rigidity of graphs with large minimum degree in more detail. In particular, we present an improved bound on the rigidity of graphs with minimum degree $n/2 + \ell$, for $\ell = O(n/\log^2 n)$ (which is sharp for $\ell = O(\sqrt{n})$), and propose a refined version of Conjecture 1.16.

Notation and terminology Let $G = (V, E)$ be a graph. For two (not necessarily disjoint) vertex sets A, B, we let $E(A, B)$ be the set of edges having one endpoint in A and the other in B, and write $E(A) = E(A, A)$. We denote further $e(A, B) = |\{(u, v) \in A \times B : \{u, v\} \in E\}|$. Note that if $A \cap B \neq \varnothing$ then it might be the case that $e(A, B) > |E(A, B)|$. We denote by $N(A)$ the *external* neighbourhood of

A, that is, the set of all vertices in $V \smallsetminus A$ that have a neighbour in A. In the above notation we often replace $\{v\}$ with v for abbreviation. The degree of a vertex $v \in V$, denoted by $\deg(v)$, is its number of incident edges. We let $\delta(G)$ and $\Delta(G)$ be the minimum and maximum degrees of G, respectively.

Throughout the paper, all logarithms are in the natural basis. If f, g are functions of n, we write $f \ll g$ if $f = o(g)$, $f \gg g$ if $f = \omega(g)$, and $f \sim g$ if $f = (1 + o(1))g$, where $o(\cdot)$ and $\omega(\cdot)$ denote the standard Little-o and Little-omega asymptotic notations, respectively. For the sake of clarity of presentation, we will systematically omit floor and ceiling signs.

Paper organisation The structure of this paper is as follows. In Section 2 we present and prove the quantitative counterpart of Theorem 1.1 (Theorem 2.1), and subsequently detail the proofs of Corollaries 1.2, 1.4 and 1.5 in Section 2.4, of Theorem 1.3 in Section 2.5, and of Proposition 1.6 in Section 2.6. In Section 2.7 we discuss some of the limitations of d-rigid partitions, and present examples of d-rigid graphs that do not admit d-rigid partitions (for certain values of d). Section 3, which is self-contained, introduces general easily verifiable sufficient conditions for the existence of rigid partitions. In Section 4, we delve into the applications of this method to various (pseudo)random graphs, presenting the proofs of Theorems 1.7 to 1.10, 1.12 and 1.13. We conclude in Section 5 with the proofs of Theorems 1.14 and 1.15.

2 Rigidity via rigid partitions

2.1 Stiffness matrices and d-dimensional algebraic connectivity

Let $G = (V, E)$ be a graph and $\mathbf{p} : V \to \mathbb{R}^d$ be an embedding. For $u, v \in V$, let

$$d_{uv}(\mathbf{p}) = \frac{\mathbf{p}(u) - \mathbf{p}(v)}{\|\mathbf{p}(u) - \mathbf{p}(v)\|} \in \mathbb{R}^d.$$

We define the **normalized rigidity matrix** $\hat{\mathsf{R}}(G, \mathbf{p}) \in \mathbb{R}^{d|V| \times |E|}$ as follows: the columns of $\hat{\mathsf{R}}(G, \mathbf{p})$ are indexed by the edges of G, and the rows are indexed by assigning d consecutive coordinates to each vertex of V. For an edge $\{u, v\} \in E$, the column of $\hat{\mathsf{R}}(G, p)$ indexed by $\{u, v\}$ is the vector $\mathbf{v}_{u,v} \in \mathbb{R}^{d|V|}$ defined by

$$\mathbf{v}_{u,v}^{\mathsf{T}} = \left[\begin{array}{ccccccccccc} 0 & \ldots & 0 & \overset{u}{d_{uv}(\mathbf{p})^{\mathsf{T}}} & 0 & \ldots & 0 & \overset{v}{d_{vu}(\mathbf{p})^{\mathsf{T}}} & 0 & \ldots & 0 \end{array} \right].$$

That is, the vector $\mathbf{v}_{u,v}$ is supported on the coordinates associated to the vertices u and v, where it is equal to the d-dimensional vectors d_{uv} and d_{vu} respectively.

Note that $\hat{\mathsf{R}}(G, \mathbf{p})$ can be obtained from the unnormalized rigidity matrix $\mathsf{R}(G, \mathbf{p})$ by first dividing each row of $\mathsf{R}(G, \mathbf{p})$ by a non-zero constant $\|\mathbf{p}(u) - \mathbf{p}(v)\|$, and then by taking the transpose of the matrix. In particular, $\hat{\mathsf{R}}(G, \mathbf{p})$ and $\mathsf{R}(G, \mathbf{p})$ have the same rank.

The **stiffness matrix** of the framework $(G, \mathbf{p})$ is the matrix

$$\mathsf{L}(G, \mathbf{p}) = \hat{\mathsf{R}}(G, \mathbf{p})\hat{\mathsf{R}}(G, \mathbf{p})^{\mathsf{T}} \in \mathbb{R}^{d|V| \times d|V|}.$$

For a symmetric matrix $M \in \mathbb{R}^{m \times m}$ and $1 \le i \le m$, let $\lambda_i(M)$ be the i-th smallest eigenvalue of M. Note that $\mathsf{L}(G, \mathbf{p})$ is a positive semi-definite matrix satisfying $\mathrm{rank}(\mathsf{L}(G, \mathbf{p})) = \mathrm{rank}(\hat{\mathsf{R}}(G, \mathbf{p})) = \mathrm{rank}(\mathsf{R}(G, \mathbf{p})) \le d|V| - \binom{d+1}{2}$. Therefore, we have $\lambda_i(\mathsf{L}(G, \mathbf{p})) = 0$ for $1 \le i \le \binom{d+1}{2}$, and $(G, \mathbf{p})$ is infinitesimally rigid if and only if $\lambda_{\binom{d+1}{2}+1}(\mathsf{L}(G, \mathbf{p})) > 0$.

In [39], Jordán and Tanigawa introduced the **d-dimensional algebraic connectivity of G**, defined

by

$$\mathsf{a}_d(G) = \sup\left\{\lambda_{\binom{d+1}{2}+1}(\mathsf{L}(G,\mathbf{p}))\right\},$$

where the supremum is taken over all embeddings $\mathbf{p}: V \to \mathbb{R}^d$.[4]

Notice that $\mathsf{a}_d(G) > 0$ if and only if G is d-rigid. We can think of $\mathsf{a}_d(G)$ as a "quantitative measure" of the d-rigidity of G—we expect graphs with large $\mathsf{a}_d(G)$ to be more "strongly rigid", in certain senses (see e.g. [39, Corollary 8.2] for a result quantifying such a phenomenon).

For $d = 1$, it is easy to check that, for any embedding $\mathbf{p}: V \to \mathbb{R}$, the normalized rigidity matrix $\hat{\mathsf{R}}(G,\mathbf{p})$ is just the incidence matrix of the orientation of G induced by the embedding, and the stiffness matrix $\mathsf{L}(G,\mathbf{p})$ is exactly the graph Laplacian $\mathsf{L}(G)$. Therefore, the one-dimensional algebraic connectivity of G coincides with the classical notion of algebraic connectivity (a.k.a. Laplacian spectral gap) of a graph, introduced by Fiedler in [25]. We denote in this case $\mathsf{a}_1(G) = \mathsf{a}(G) = \lambda_2(\mathsf{L}(G))$. For convenience, we define $\mathsf{a}(G) = \infty$ for the graph $G = (\{v\}, \varnothing)$.

We can now state the following quantitative version of Theorem 1.1.

Theorem 2.1. *Let $G = (V, E)$ be a graph, and let $\left(\{V_i\}_{i=1}^{d+1}, \{E_{ij}\}_{1 \le i < j \le d+1}\right)$ be a d-rigid partition of G. For $1 \le i < j \le d+1$, let $G_{ij} = (V_i \cup V_j, E_{ij})$. Then,*

$$\mathsf{a}_d(G) \ge \min\left\{\frac{\mathsf{a}(G_{ij})}{2} : 1 \le i < j \le d+1\right\}.$$

In particular, since G_{ij} is connected for all $1 \le i < j \le d+1$, G is d-rigid.

2.2 Limit frameworks

For the proof of Theorem 2.1, we will need to consider the limit of a sequence of frameworks. In order to do that, we introduce the notion of *generalized frameworks*, first studied by Tay in [62] (although we use here slightly different definitions and notation; see also [60]).

Let $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ denote the $(d-1)$-dimensional unit sphere. Let $G = (V, E)$ be a graph, and let $\mathbf{q}: \{(u,e) : e \in E, u \in e\} \to \mathbb{S}^{d-1}$. The pair $(G, \mathbf{q})$ is called a **generalized d-dimensional framework**. For $e = \{u, v\} \in E$, we define $\mathbf{v}_{u,v} \in \mathbb{R}^{d|V|}$ by

$$\mathbf{v}_{u,v}^{\mathsf{T}} = \begin{bmatrix} 0 & \dots & 0 & \overset{u}{\mathbf{q}(u,e)^{\mathsf{T}}} & 0 & \dots & 0 & \overset{v}{\mathbf{q}(v,e)^{\mathsf{T}}} & 0 & \dots & 0 \end{bmatrix}.$$

We define the **rigidity matrix** $\hat{\mathsf{R}}(G,\mathbf{q}) \in \mathbb{R}^{d|V| \times |E|}$ as the matrix whose columns are the vectors $\mathbf{v}_{u,v}$ for all $\{u,v\} \in E$. We define the **stiffness matrix** of $(G, \mathbf{q})$ as

$$\mathsf{L}(G,\mathbf{q}) = \hat{\mathsf{R}}(G,\mathbf{q})\hat{\mathsf{R}}(G,\mathbf{q})^{\mathsf{T}} \in \mathbb{R}^{d|V| \times d|V|}.$$

Similarly, we define the **lower stiffness matrix** of $(G, \mathbf{q})$ as

$$\mathsf{L}^-(G,\mathbf{q}) = \hat{\mathsf{R}}(G,\mathbf{q})^{\mathsf{T}}\hat{\mathsf{R}}(G,\mathbf{q}) \in \mathbb{R}^{|E| \times |E|}. \tag{2.1}$$

We have the following explicit description of the lower stiffness matrix, which is an immediate consequence of (2.1).

---

[4]The original definition in [39] allows $\mathbf{p}$ to be non-injective, however both definitions are equivalent (see [48, Lemma 2.4]).

Lemma 2.2. *Let $(G, \mathbf{q})$ be a generalized d-dimensional framework. Let $e_1, e_2 \in E$. Then,*

$$\mathsf{L}^-(G, \mathbf{q})(e_1, e_2) = \begin{cases} 2 & \text{if } e_1 = e_2, \\ \mathbf{q}(u, e_1) \cdot \mathbf{q}(u, e_2) & \text{if } e_1 = \{u, v\} \text{ and } e_2 = \{u, w\}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the non-zero eigenvalues of $\mathsf{L}(G, \mathbf{q})$ coincide with those of $\mathsf{L}^-(G, \mathbf{q})$. In particular, assuming $|E| \geq d|V| - \binom{d+1}{2}$, we have

$$\lambda_k(\mathsf{L}(G, \mathbf{q})) = \lambda_{|E|-d|V|+k}(\mathsf{L}^-(G, \mathbf{q})), \tag{2.2}$$

for all $k \geq \binom{d+1}{2} + 1$.

Given $G = (V, E)$ and an embedding $\mathbf{p} : V \to \mathbb{R}^d$, we can define a generalized framework $(G, \mathbf{q})$ by

$$\mathbf{q}(u, \{u, v\}) = d_{uv}(\mathbf{p}) = \frac{\mathbf{p}(u) - \mathbf{p}(v)}{\|\mathbf{p}(u) - \mathbf{p}(v)\|} \tag{2.3}$$

for all $\{u, v\} \in E$. The generalized framework $(G, \mathbf{q})$ is equivalent to the framework $(G, \mathbf{p})$, in the sense that $\hat{\mathsf{R}}(G, \mathbf{q}) = \hat{\mathsf{R}}(G, \mathbf{p})$, and therefore $\mathsf{L}(G, \mathbf{q}) = \mathsf{L}(G, \mathbf{p})$. The main "advantage" of generalized frameworks is that they allow us to talk about limits of frameworks:

We say that a generalized framework $(G, \mathbf{q})$ is a **d-dimensional limit framework** if there exists a sequence of d-dimensional frameworks $\{(G, \mathbf{p}_n)\}_{n=1}^\infty$ such that

$$\mathbf{q}(u, \{u, v\}) = \lim_{n \to \infty} d_{uv}(\mathbf{p}_n)$$

for all $\{u, v\} \in E$. We denote in this case $(G, \mathbf{p}_n) \to (G, \mathbf{q})$. Note that if $(G, \mathbf{q})$ is a limit d-dimensional framework, it must satisfy $\mathbf{q}(u, \{u, v\}) = -\mathbf{q}(v, \{u, v\})$ for all $\{u, v\} \in E$.

Generalized 1-dimensional frameworks correspond to *signings* of the graph G (that is, to functions $\eta : \{(u, e) : e \in E, u \in e\} \to \{-1, 1\}$), and their stiffness matrix is the "signed Laplacian" corresponding to the signing η (for more on the theory of signed graphs and their associated matrices, see, e.g., [69]). The 1-dimensional limit frameworks are exactly those corresponding to *acyclic orientations* of G (that is, to signings satisfying $\eta(u, \{u, v\}) = -\eta(v, \{u, v\})$ for all $\{u, v\} \in E$, for which the directed graph $H = (V, \{(u, v) : \{u, v\} \in E, \eta(u, \{u, v\}) = 1\})$ has no directed cycles). Actually, any 1-dimensional limit framework can be obtained from an embedding $\mathbf{p} : V \to \mathbb{R}^1$ as in (2.3) (without the need of taking a limit of a sequence of embeddings). The stiffness matrix $\mathsf{L}(G, \eta)$ in this case is just the usual graph Laplacian $\mathsf{L}(G)$.

Lemma 2.3. *If $(G, \mathbf{q})$ is a d-dimensional limit framework, then*

$$\mathsf{a}_d(G) \geq \lambda_{\binom{d+1}{2}+1}(\mathsf{L}(G, \mathbf{q})).$$

Proof. Let $(G, \mathbf{p}_n) \to (G, \mathbf{q})$, where $\{\mathbf{p}_n\}_{n=1}^\infty$ is a sequence of embeddings of V into $\mathbb{R}^d$. Note that $\hat{\mathsf{R}}(G, \mathbf{q})$ is the entry-wise limit of the sequence of matrices $\hat{\mathsf{R}}(G, \mathbf{p}_n)$, and therefore $\mathsf{L}(G, \mathbf{q})$ is the limit of the matrices $\mathsf{L}(G, \mathbf{p}_n)$. Hence, by the continuity of eigenvalues,

$$\lambda_{\binom{d+1}{2}+1}(\mathsf{L}(G, \mathbf{q})) = \lim_{n \to \infty} \lambda_{\binom{d+1}{2}+1}(\mathsf{L}(G, \mathbf{p}_n)).$$

Thus, we obtain

$$\mathsf{a}_d(G) \geq \sup_n \lambda_{\binom{d+1}{2}+1}(\mathsf{L}(G, \mathbf{p}_n)) \geq \lambda_{\binom{d+1}{2}+1}(\mathsf{L}(G, \mathbf{q})). \qquad \square$$

Lemma 2.4. *Let $G = (V, E)$. Let $V = V_1 \cup \cdots \cup V_k$ be a colouring of V, and let $x_1, \ldots, x_k$ be k distinct points in $\mathbb{R}^d$. Assume we have for each $1 \leq i \leq k$ a d-dimensional limit framework $(G[V_i], \mathbf{q}_i)$. Define $\mathbf{q} : \{(u, e) : e \in E, u \in e\} \to \mathbb{S}^{d-1}$ by*

$$\mathbf{q}(u, \{u, v\}) = \begin{cases} (x_i - x_j)/\|x_i - x_j\| & \text{if } u \in V_i, v \in V_j \text{ for } i \neq j, \\ \mathbf{q}_i(u, \{u, v\}) & \text{if } u, v \in V_i \text{ for } 1 \leq i \leq k, \end{cases}$$

for all $\{u, v\} \in E$. Then, $(G, \mathbf{q})$ is a d-dimensional limit framework.

Proof. For each $1 \leq i \leq k$, let $\mathbf{p}_{i,n} : V_i \to \mathbb{R}^d$ be a sequence of embeddings such that $(G[V_i], \mathbf{p}_{i,n}) \to (G[V_i], \mathbf{q}_i)$. We may assume without loss of generality that $\|\mathbf{p}_{i,n}(u)\| \leq 1$ for all $1 \leq i \leq k$ and $u \in V_i$ (since re-scaling of a map $\mathbf{p}$ does not affect the values of $d_{uv}(\mathbf{p})$).

Define $\mathbf{p}_n : V \to \mathbb{R}^d$ by

$$\mathbf{p}_n(u) = nx_i + \mathbf{p}_{i,n}(u)$$

for all $1 \leq i \leq k$ and $u \in V_i$. We will show that $(G, \mathbf{p}_n) \to (G, \mathbf{q})$. Let $\{u, v\} \in E$. If $u, v \in V_i$ for some $1 \leq i \leq k$, then

$$d_{uv}(\mathbf{p}_n) = \frac{\mathbf{p}_n(u) - \mathbf{p}_n(v)}{\|\mathbf{p}_n(u) - \mathbf{p}_n(v)\|} = \frac{\mathbf{p}_{i,n}(u) - \mathbf{p}_{i,n}(v)}{\|\mathbf{p}_{i,n}(u) - \mathbf{p}_{i,n}(v)\|} \to \mathbf{q}_i(u, \{u, v\}) = \mathbf{q}(u, \{u, v\}),$$

as required. If $u \in V_i$ and $v \in V_j$ for some $i \neq j$, then, since $\|\mathbf{p}_{i,n}(u)\| \leq 1$ and $\|\mathbf{p}_{j,n}(v)\| \leq 1$, we have

$$\lim_{n \to \infty} \mathbf{p}_n(u)/n = x_i$$

and

$$\lim_{n \to \infty} \mathbf{p}_n(v)/n = x_j.$$

Therefore,

$$d_{uv}(\mathbf{p}_n) = \frac{\mathbf{p}_n(u) - \mathbf{p}_n(v)}{\|\mathbf{p}_n(u) - \mathbf{p}_n(v)\|} = \frac{\mathbf{p}_n(u)/n - \mathbf{p}_n(v)/n}{\|\mathbf{p}_n(u)/n - \mathbf{p}_n(v)/n\|} \to \frac{x_i - x_j}{\|x_i - x_j\|}. \qquad \square$$

The following lemma is a simple generalisation of an argument by Tay [62, Theorem 3.4] (see also [60, Theorem 8]).

Lemma 2.5. *Let $G = (V, E)$ be a graph, and let $E = E_1 \cup \cdots \cup E_k$ be a colouring of its edge set. Let $y_1, \ldots, y_k \in \mathbb{S}^{d-1}$. If every $U \subseteq V$ of size at least 2 has a monochromatic cut, then there exists a d-dimensional limit framework $(G, \mathbf{q})$ such that, for every $i \in [k]$, $\mathbf{q}(u, \{u, v\}) \in \{y_i, -y_i\}$ for all $\{u, v\} \in E_i$.*

Proof. We argue by induction on $|V|$. If $|V| = 1$, we are done. Otherwise, there exist some $i \in [k]$ and a partition $V = A \cup B$ such that all the edges in G between A and B belong to E_i. By the induction hypothesis, there exist limit frameworks $(G[A], \mathbf{q}_A)$ and $(G[B], \mathbf{q}_B)$ such that for every $j \in [k]$ and $\{u, v\} \in E_j$, $\mathbf{q}_A(u, \{u, v\}) \in \{y_j, -y_j\}$ if $u, v \in A$, and $\mathbf{q}_B(u, \{u, v\}) \in \{y_j, -y_j\}$ if $u, v \in B$. Let $x_A = y_i$ and $x_B = 0$. Note that $(x_A - x_B)/\|x_A - x_B\| = y_i$. Therefore, by Lemma 2.4, there is a d-dimensional limit framework $(G, \mathbf{q})$ satisfying

$$\mathbf{q}(u, \{u, v\}) = \begin{cases} y_i & \text{if } u \in A \text{ and } v \in B, \\ -y_i & \text{if } u \in B \text{ and } v \in A, \\ \mathbf{q}_A(u, \{u, v\}) & \text{if } u, v \in A, \\ \mathbf{q}_B(u, \{u, v\}) & \text{if } u, v \in B, \end{cases}$$

for all $\{u,v\} \in E$. Note that for every $j \in [k]$ and $\{u,v\} \in E_j$, we obtain $\mathbf{q}(u,\{u,v\}) \in \{y_j, -y_j\}$, as desired. $\qquad\square$

2.3 Proof of Theorem 2.1

For the proof of Theorem 2.1 we will need the following very simple lemma from [50].

Lemma 2.6 ([50, Lemma 3.1]). *Let $n \geq 1$ be an integer, and let $n_1, \ldots, n_k \geq 0$ such that $n = n_1 + \cdots + n_k$. Let $M \in \mathbb{R}^{n \times n}$ be a block diagonal symmetric matrix with blocks $M_1, \ldots, M_k$, where $M_i \in \mathbb{R}^{n_i \times n_i}$ for all $1 \leq i \leq k$. Let $1 \leq m \leq n$ and $r_1, \ldots, r_k$ such that $1 \leq r_i \leq \max\{1, n_i\}$ for all $1 \leq i \leq k$ and $\sum_{i=1}^{k} r_i = m + k - 1$. Then,*

$$\lambda_m(M) \geq \min\{\lambda_{r_i}(M_i) : 1 \leq i \leq k\}.$$

For convenience, given a "0×0" matrix M, we define $\lambda_1(M) = \infty$. Note then that Lemma 2.6 applies even if we allow values $n_i = 0$ and $r_i = 1$ for one or more $1 \leq i \leq k$.

Proof of Theorem 2.1. Without loss of generality, we assume that $\{E_{ij}\}_{1 \leq i < j \leq d+1}$ is a colouring of E. Otherwise, we can apply the arguments below to the subgraph $G' = (V, \cup_{1 \leq i < j \leq d+1} E_{ij})$ (and then use the fact that $a_d(G) \geq a_d(G')$, see e.g. [48, Theorem 2.3]).

Recall that, for $1 \leq i < j \leq d+1$, we define $G_{ij} = (V_i \cup V_j, E_{ij})$. For convenience, for $i > j$, we define $G_{ij} = G_{ji}$. Let $x_1, \ldots, x_{d+1} \in \mathbb{R}^d$ be the vertices of a regular simplex. For $i \neq j$, let $y_{ij} = (x_i - x_j)/\|x_i - x_j\|$.

Let $1 \leq i \leq d+1$. By Lemma 2.5 (using the fact that every $U \subseteq V_i$ of size at least 2 has a monochromatic cut), there exists a d-dimensional limit framework $(G[V_i], \mathbf{q}_i)$, satisfying $\mathbf{q}_i(u, \{u,v\}) \in \{y_{ij}, -y_{ij}\}$ for all $j \neq i$ and $\{u,v\} \in E_{ij}$. Therefore, by Lemma 2.4, there exists a d-dimensional limit framework $(G, \mathbf{q})$ satisfying, for all $i \neq j$ and $\{u,v\} \in E_{ij}$, $\mathbf{q}(u, \{u,v\}) \in \{y_{ij}, -y_{ij}\}$ if $u, v \in V_i$ or $u, v \in V_j$, and $\mathbf{q}(u, \{u,v\}) = y_{ij}$ if $u \in V_i$ and $v \in V_j$.

Let $\eta : \{(u,e) : e \in E, u \in e\} \to \{1, -1\}$ be defined by

$$\eta(u, \{u,v\}) = \begin{cases} 1 & \text{if } \mathbf{q}(u, \{u,v\}) = y_{ij}, \\ -1 & \text{if } \mathbf{q}(u, \{u,v\}) = -y_{ij} \end{cases}$$

for all $\{u,v\} \in E_{ij}$ such that $u \in V_i$. Note that, if $u \in V_i$ and $v \in V_j$ for $i \neq j$, we have $q(u, \{u,v\}) = y_{ij}$ and $q(v, \{u,v\}) = -y_{ij} = y_{ji}$. Therefore, by the definition of η, we have $\eta(u, \{u,v\}) = \eta(v, \{u,v\}) = 1$ in this case.

Moreover, since $\mathbf{q}$ is a d-dimensional limit framework, we have $\mathbf{q}(u, \{u,v\}) = -\mathbf{q}(v, \{u,v\})$ for all $\{u,v\} \in E$. Therefore, we have $\eta(u, \{u,v\}) = -\eta(v, \{u,v\})$ whenever $u, v \in V_i$ for some $1 \leq i \leq d+1$.

It is easy to check that if $e = \{u,v\} \in E_{ij}$ and $e' = \{u,w\} \in E_{ik}$ for $i \neq j, k$, then

$$\mathbf{q}(u, e) \cdot \mathbf{q}(u, e') = (y_{ij} \cdot y_{ik})\eta(u,e)\eta(u,e').$$

Note that, if $j \neq k$, $y_{ij} \cdot y_{ik} = 1/2$, and if $j = k$, $y_{ij} \cdot y_{ik} = 1$. Therefore, by Lemma 2.2, we obtain

$$\mathsf{L}^-(G,\mathbf{q})(e,e') = \begin{cases} 2 & \text{if } e = e', \\ \eta(u,e) \cdot \eta(u,e') & \text{if } e \cap e' = \{u\}, \ e, e' \in E_{ij} \text{ for } i \neq j, \\ \frac{1}{2}\eta(u,e) \cdot \eta(u,e') & \text{if } e \cap e' = \{u\}, \ e \in E_{ij}, \ e' \in E_{ik} \text{ for pairwise distinct } i, j, k, \\ 0 & \text{otherwise.} \end{cases}$$

We can write $\mathsf{L}^-(G, \mathbf{q})$ as a sum

$$\mathsf{L}^-(G, \mathbf{q}) = \frac{1}{2}M + \frac{1}{2}T,$$

where M is the matrix defined by

$$M(e, e') = \begin{cases} 2 & \text{if } e = e', \\ \eta(u, e) \cdot \eta(u, e') & \text{if } e \cap e' = \{u\}, \\ 0 & \text{otherwise,} \end{cases}$$

and T is a block diagonal matrix with blocks T_{ij} for $1 \leq i < j \leq d+1$, where each block T_{ij} is supported on the rows and columns corresponding to E_{ij}, and is defined by

$$T_{ij}(e, e') = \begin{cases} 2 & \text{if } e = e', \\ \eta(u, e) \cdot \eta(u, e') & \text{if } e \cap e' = \{u\}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $M = \mathsf{L}^-(G, \eta)$, and in particular it is a positive semi-definite matrix. Thus, by Weyl's inequality (see, e.g., [11, Theorem 2.8.1(iii)]),

$$\lambda_i(\mathsf{L}^-(G, \mathbf{q})) \geq \frac{1}{2}\lambda_i(T) \text{ for all } 1 \leq i \leq |E|. \tag{2.4}$$

Let $1 \leq i < j \leq d+1$. Define a signing $\eta_{ij} : \{(u, e) : e \in E_{ij}, u \in e\} \to \{1, -1\}$ by

$$\eta_{ij}(u, \{u, v\}) = \begin{cases} \eta(u, \{u, v\}) & \text{if } u, v \in V_i, \\ -\eta(u, \{u, v\}) & \text{if } u, v \in V_j, \\ 1 & \text{if } u \in V_i, v \in V_j, \\ -1 & \text{if } u \in V_j, v \in V_i. \end{cases}$$

Note that $\eta_{ij}(v, \{u, v\}) = -\eta_{ij}(u, \{u, v\})$, so η_{ij} is an orientation of G_{ij}, and therefore $\mathsf{L}(G_{ij}, \eta_{ij})$ is actually the Laplacian matrix $\mathsf{L}(G_{ij})$. Moreover, for any $\{u, v\}, \{u, w\} \in E_{ij}$,

$$\eta(u, \{u, v\}) \cdot \eta(u, \{u, w\}) = \eta_{ij}(u, \{u, v\}) \cdot \eta_{ij}(u, \{u, w\}),$$

and therefore $T_{ij} = \mathsf{L}^-(G_{ij}, \eta_{ij})$. Let $r_{ij} = |E_{ij}| - |V_i \cup V_j| + 2$. Since G_{ij} is connected, we have $|E_{ij}| \geq |V_i \cup V_j| - 1$. So, by (2.2), we obtain

$$\lambda_{r_{ij}}(T_{ij}) = \lambda_{r_{ij}}(\mathsf{L}^-(G_{ij}, \eta_{ij})) = \lambda_2(\mathsf{L}(G_{ij}, \eta_{ij})) = \lambda_2(\mathsf{L}(G_{ij})) = \mathsf{a}(G_{ij}).$$

Let $m = |E| - d|V| + \binom{d+1}{2} + 1$. Note that

$$\sum_{1 \leq i < j \leq d+1} r_{ij} = \sum_{1 \leq i < j \leq d+1} |E_{ij}| - d\sum_{i=1}^{d+1} |V_i| + 2\binom{d+1}{2}$$

$$= |E| - d|V| + 2\binom{d+1}{2} = m + \binom{d+1}{2} - 1.$$

So, by Lemma 2.6,

$$\lambda_m(T) \geq \min\left\{\lambda_{r_{ij}}(T_{ij}) : 1 \leq i < j \leq d+1\right\} = \min\left\{\mathsf{a}(G_{ij}) : 1 \leq i < j \leq d+1\right\}.$$

By Lemma 2.3, (2.2) and (2.4), we obtain

$$\mathsf{a}_d(G) \geq \lambda_{\binom{d+1}{2}+1}(\mathsf{L}(G,\mathbf{q})) = \lambda_m(\mathsf{L}^-(G,\mathbf{q})) \geq \frac{1}{2}\lambda_m(T) \geq \min\left\{\frac{\mathsf{a}(G_{ij})}{2} : 1 \leq i < j \leq d+1\right\},$$

as wanted. $\qquad\qquad\square$

2.4   Special rigid partitions

We proceed to prove Corollaries 1.2, 1.4 and 1.5. Corollary 1.2 follows from the next proposition together with Theorem 1.1.

Proposition 2.7. *A graph that admits a CDS partition of size $\binom{d+1}{2}$ admits a d-rigid partition.*

Proof. If $d = 1$ then the claim is trivial; we therefore assume that $d \geq 2$. Let $\{A_{ij} : 1 \leq i < j \leq d+1\}$ be a partition of V into $\binom{d+1}{2}$ connected dominating sets. Let $V_1 = A_{13}$, $V_2 = A_{12}$, $V_3 = A_{23}$, and $V_j = \bigcup_{i<j} A_{ij}$ for $4 \leq j \leq d+1$. For $1 \leq i < j \leq d+1$, let $E_{ij} = E(V_i, V_j) \cup E(A_{ij})$, and let $G_{ij} = (V_i \cup V_j, E_{ij})$. For convenience, we define $E_{ji} = E_{ij}$ and $A_{ji} = A_{ij}$.

First, we show that the graphs G_{ij} are connected. Let $1 \leq i < j \leq d+1$. Assume that $A_{ij} \subseteq V_j$ (this is the case unless $i = 1$ and $j = 3$, in which case the proof follows in essentially the same way). Since A_{ij} is a dominating set, every vertex in V_i is adjacent in G_{ij} to a vertex in A_{ij}. Similarly, since V_i is a dominating set (as it contains a dominating set), every vertex in V_j is adjacent in G_{ij} to a vertex in V_i. Therefore, every vertex in $V_i \cup V_j$ is connected in G_{ij} to a vertex in A_{ij} (by a path of length at most two). Since A_{ij} is connected in G_{ij}, this implies that G_{ij} is connected.

We proceed to show that for every $1 \leq i \leq d+1$, every $U \subseteq V_i$ of size at least 2 has a monochromatic cut (that is, a partition into U', U'' such that $\hat{E} \cap E(U', U'') \subseteq E_{ij}$ for some $j \neq i$, where $\hat{E} = \bigcup_{1 \leq i' < j' \leq d+1} E_{i'j'}$). But this is trivially true: Let $u \in U$. Then $u \in A_{ij}$ for some $j \neq i$. Let $U' = \{u\}$ and $U'' = U \setminus \{u\}$. Then, $\hat{E} \cap E(U', U'') \subseteq E(A_{ij}) \subseteq E_{ij}$. Therefore, $\left(\{V_i\}_{i=1}^{d+1}, \{E_{ij}\}_{1 \leq i < j \leq d+1}\right)$ is a d-rigid partition of G. $\qquad\square$

Corollary 1.4 follows from the next two propositions together with Theorem 1.1. Recall that, given a colouring $V_1, \ldots, V_{d+1}$ of V and a partial colouring $\{E_{ij}\}_{1 \leq i < j \leq d+1}$ of E, we denote by G_{ij} the subgraph $(V_i \cup V_j, E_{ij})$ of G and set $\hat{E} = \bigcup_{1 \leq i < j \leq d+1} E_{ij}$.

Proposition 2.8. *A graph that admits a type I strong d-rigid partition admits a d-rigid partition.*

Proof. Let $V_1, \ldots, V_d$ be a type I strong d-rigid partition of a graph $G = (V, E)$. Let $V_{d+1} = \varnothing$, so $V_1, \ldots, V_{d+1}$ is a colouring of V. For $1 \leq i < j \leq d$ let $E_{ij} = E(V_i, V_j)$, and for $1 \leq i \leq d$ and $j = d+1$ let $E_{ij} = E(V_i)$. Thus, $\{E_{ij}\}_{1 \leq i < j \leq d+1}$ is a partial colouring of E. Moreover, for $1 \leq i < j \leq d$, G_{ij} is connected since $G_{ij} = G[V_i, V_j]$, and for $1 \leq i \leq d$ and $j = d+1$, G_{ij} is connected since $G_{ij} = G[V_i]$. Finally, let $1 \leq i \leq d+1$ and let $U \subseteq V_i$ with $|U| \geq 2$ (so $i \leq d$). We have to show that U has a monochromatic cut. However, that follows directly from the fact that $E(U) \subseteq E_{i,d+1}$. $\qquad\square$

Proposition 2.9. *A graph that admits a type II strong d-rigid partition admits a d-rigid partition.*

Proof. Let $V_1, \ldots, V_{d+1}$ be a type II strong d-rigid partition of a graph $G = (V, E)$. So $V_1, \ldots, V_{d+1}$ is a colouring of V. For $1 \leq i < j \leq d+1$ let $E_{ij} = E(V_i, V_j)$. Thus, $\{E_{ij}\}_{1 \leq i < j \leq d+1}$ is a partial colouring of E. Moreover, for $1 \leq i < j \leq d+1$, G_{ij} is connected since $G_{ij} = G[V_i, V_j]$. Finally, let $1 \leq i \leq d+1$ and let $U \subseteq V_i$ with $|U| \geq 2$. We have to show that U has a monochromatic cut. However, that follows directly from the fact that $\hat{E} \cap E(V_i) = \varnothing$ for every $1 \leq i \leq d+1$. $\qquad\square$

Corollary 1.5 follows from the next proposition together with Theorem 1.1.

Proposition 2.10. *A graph that admits a strong bipartite d-rigid partition admits a d-rigid partition.*

Proof. Let $V_1, \ldots, V_{d+1}$ be a strong bipartite d-rigid partition of a bipartite graph $G = (A \cup B, E)$ with bipartition (A, B). So $V_1, \ldots, V_{d+1}$ is a colouring of $V = A \cup B$. For $1 \leq i \leq d+1$, write $A_i = V_i \cap A$ and $B_i = V_i \cap B$. There is a sequence $0 \leq s_1 \leq s_2 \leq \cdots \leq s_{d+1}$ satisfying $\sum_{i=1}^{k} s_i \geq \binom{k}{2}$ for all $1 \leq k \leq d+1$ such that $G[A_i, B_i]$ contains a forest F_i with s_i edges for all $1 \leq i \leq d+1$. Without loss of generality, we may assume that $\sum_{i=1}^{d+1} s_i = \binom{d+1}{2}$. By a theorem of Landau [47], there exists a tournament T on vertex set $[d+1]$ such that the out-degree of vertex i is exactly s_i for all $1 \leq i \leq d+1$. For each $1 \leq i \leq d+1$, denote the edges of F_i by $e_i^{j_1}, \ldots, e_i^{j_{s_i}}$, where $j_1, \ldots, j_{s_i}$ are the s_i out-neighbours of i in T.

Let $1 \leq i < j \leq d+1$. If (i, j) is a directed edge of T, set $e_{ij} = e_i^j$; otherwise set $e_{ij} = e_j^i$. In any case, define $E_{ij} = E(V_i, V_j) \cup \{e_{ij}\}$. Thus, $\{E_{ij}\}_{1 \leq i < j \leq d+1}$ is a partial colouring of E. Moreover, for $1 \leq i < j \leq d+1$, $G_{ij} = (V_i \cup V_j, E_{ij})$ is connected since $G[V_i, V_j]$ is a subgraph of G_{ij} with two connected components (as $G[A_i, B_j]$ and $G[A_j, B_i]$ are both connected) and e_{ij} is an edge connecting them.

Finally, let $1 \leq i \leq d+1$ and let $U \subseteq V_i$ with $|U| \geq 2$. We have to show that U has a monochromatic cut. Let $\hat{E} = \cup_{1 \leq i' < j' \leq d+1} E_{i'j'}$. If U does not span any of the edges e_i^j for $j \neq i$ then $\hat{E} \cap E(U) = \varnothing$, hence U has a monochromatic cut trivially. Otherwise, suppose $e_i^j \subseteq U$ for some $j \neq i$, and write $e_i^j = \{u, v\}$. Let $H_i = F_i \setminus e_i^j$. Let U' be the connected component of u in $H_i[U]$, and $U'' = U \setminus U'$. Since F_i is a forest, we have $v \in U''$, and therefore $U', U'' \neq \varnothing$. Then, $\hat{E} \cap E(U', U'') = \{e_i^j\}$, which is a subset of either E_{ij} (if $i < j$) or E_{ji} (if $j < i$). Thus, U', U'' is a monochromatic cut for U, as wanted. $\square$

For our arguments on random bipartite graphs, we will use the following special class of strong bipartite d-rigid partitions:

Lemma 2.11. *Let $G = (A \cup B, E)$ be a bipartite graph with bipartition (A, B). Let $V_1, \ldots, V_{d+1}$ be a partition of $V = A \cup B$. Write $A_i = V_i \cap A$ and $B_i = V_i \cap B$ for $1 \leq i \leq d+1$. Assume that $G[A_i, B_j]$ is connected for all $i \neq j$, and that $G[A_i, B_i]$ contains a matching of size d for all $1 \leq i \leq d+1$. Then, $V_1, \ldots, V_{d+1}$ is a strong bipartite d-rigid partition of G.*

Proof. The claim follows immediately from the definition of a strong bipartite d-rigid partition, using the fact that a matching is a forest, and that for all $1 \leq k \leq d+1$ we have $kd \geq \binom{k}{2}$. $\square$

Before concluding this section, we demonstrate that type I and type II strong rigid partitions are genuinely distinct notions.

Proposition 2.12. *For every $d \geq 2$ there exists a graph that admits a type I strong d-rigid partition but not a type II strong d-rigid partition.*

Proof. Fix $d \geq 2$. Let $G = G_d^{\mathrm{I}} = (V, E)$ be the graph obtained from the complete graph on the vertex set $[d+3]$ by removing the edges $\{1, 2\}$, $\{2, 3\}$, and $\{4, 5\}$.

We first claim that G admits a type I strong d-rigid partition. Indeed, consider the partition $V_1 = \{2, 3, 4\}$, $V_2 = \{1, 5\}$, and $V_j = \{j + 3\}$ for $j = 3, \ldots, d$. Each $G[V_j]$ is obviously connected. Moreover, $G[V_1, V_2]$ is connected, and, since $\deg_G(j + 3) = d + 2$ for all $3 \leq j \leq d$, $G[V_i, V_j]$ is connected for all $6 \leq i \leq d+3$ and $j \neq i$.

Now suppose, for contradiction, that G admits a type II strong d-rigid partition $V_1, \ldots, V_{d+1}$. Since we have $d+3$ vertices and $d+1$ nonempty parts, at least $d-1$ of the parts are singletons. Let $V_i = \{v\}$ be one of these singletons. Note that v must be adjacent to every other vertex in the graph—otherwise there exists $j \neq i$ for which the graph $G[V_i, V_j]$ is disconnected. Therefore, v has degree $d + 2$. However, G has only $d - 2$ vertices of degree $d + 2$ (the vertices in $[d+3] \setminus [5]$), a contradiction. $\square$

Proposition 2.13. *For every $d \geq 2$ there exists a graph that admits a type II strong d-rigid partition but not a type I strong d-rigid partition.*

Proof. Fix $d \geq 2$, and define the graph $G = G_d^{\mathrm{II}} = (V, E)$ as follows: let

$$V = \{(x, i) \mid x \in \{0, 1\}, \ i \in [d+1]\},$$

and

$$E = \{\{(x, i), (x', j)\} : i \neq j, \ x + x' \in \{0, 1\}\}.$$

In other words, the neighbours of a vertex of the form $(0, i)$ are the vertices (x, j) for $x \in \{0, 1\}$ and $j \neq i$, and the neighbours of a vertex of the form $(1, i)$ are the vertices $(0, j)$ for $j \neq i$. Observe that $\Delta(G) = 2d$.

We first claim that G admits a type II strong d-rigid partition. Indeed, consider the partition $V_1, \ldots, V_{d+1}$, where $V_j = \{(0, j), (1, j)\}$ for all $1 \leq j \leq d + 1$. It is easy to check that the bipartite subgraph $G[V_i, V_j]$ is connected for all $i \neq j$.

Now suppose, for contradiction, that G admits a type I strong d-rigid partition $V_1, \ldots, V_d$. Define $A_0 = \{(0, i) \mid i \in [d+1]\}$ and $A_1 = \{(1, i) \mid i \in [d+1]\}$. First, note that $|V_j| \geq 2$ for all $j \in [d]$, since any singleton part would contain a vertex of degree $2d + 1$, in contradiction to $\Delta(G) = 2d$. Next, we show that each V_j must in fact contain at least two vertices from A_0. Notice that, for all $j \in [d]$, $|V_j \cap A_0| \geq 1$, since otherwise V_j contains only vertices of A_1, and therefore forms an independent set of size at least 2, in contradiction to $G[V_j]$ being connected. Now, assume for contradiction that $|V_j \cap A_0| = 1$ for some $j \in [d]$. That is, $V_j \cap A_0 = \{(0, i)\}$, for some $i \in [d+1]$. If $(1, i) \in V_j$, then in order for $G[V_j]$ to be connected, V_j must also contain a neighbour of $(1, i)$, say $(0, k)$ with $k \neq i$. But this contradicts the assumption that $V_j \cap A_0 = \{(0, i)\}$. Alternatively, if $(1, i) \in V_{j'}$ for some $j' \neq j$, then for $G[V_j, V_{j'}]$ to be connected, V_j must again contain a neighbour $(0, k)$ of $(1, i)$ with $k \neq i$, yielding the same contradiction. Hence, each V_j must contain at least two elements of A_0, implying that $|A_0| \geq 2d$. But $|A_0| = d + 1$, so this is a contradiction for $d \geq 2$. $\qquad\square$

2.5 Rigidity of highly-connected graphs

In this section we prove Theorem 1.3. Let us restate it here for convenience:

Theorem 1.3 (Highly connected graphs)**.** *There exists $C > 0$ such that for all $d = d(n) \geq 1$, every n-vertex $Cd^2\log^2 n$-connected graph is d-rigid.*

For the proof, we will need the following result by Censor-Hillel et al. [12], showing that highly-connected graphs admit a large CDS partition.

Theorem 2.14 ([12, Corollary 1.6])**.** *There exists $c > 0$ such that every k-connected n-vertex graph admits a CDS partition of size $ck/\log^2 n$.*

Proof of Theorem 1.3. Let $c > 0$ be the constant from Theorem 2.14, and let $C = 1/c$. Let $d = d(n) \geq 1$, and let $k = Cd^2\log^2 n$. Let $G = (V, E)$ be a k-connected n-vertex graph. By Theorem 2.14, G admits a CDS partition of size $ck/\log^2 n = d^2 \geq \binom{d+1}{2}$. Hence, by Corollary 1.2, G is d-rigid. $\qquad\square$

Note that improving the bound in Theorem 2.14 would immediately imply an improved bound in Theorem 1.3. However, Censor-Hillel, Ghaffari and Kuhn showed in [13] that there exist k-connected graphs that can be partitioned into at most $O(k/\log n)$ connected dominating sets. Therefore, the best that we can hope for, using our current arguments, is to reduce the requirement on the connectivity of G in Theorem 1.3 from $\Omega(d^2\log^2 n)$ to $\Omega(d^2 \log n)$. It is also worth noting that for every *fixed* k, there exists a k-connected graph with no CDS partition of size 2—this follows from, e.g., [45, Proposition 6].

2.6 Rigidity of complete bipartite graphs

We conclude this section with a proof of Proposition 1.6.

Proof of Proposition 1.6. We define for $1 \leq i \leq d+1$,

$$s_i = \begin{cases} 1 & \text{if } i \leq 3, \\ i-1 & \text{if } i > 3. \end{cases}$$

Note that $0 \leq s_i \leq s_{i+1}$ for all $1 \leq i \leq d$, $\sum_{i=1}^{k} s_i \geq \binom{k}{2}$ for all $1 \leq k \leq d+1$, and $\sum_{i=1}^{d+1} s_i = \binom{d+1}{2}$. Let $V = A \cup B$ be the vertex set of $G = K_{m,n}$, and let A and B be its two parts, with $|A| = m$ and $|B| = n$. Since $m, n \geq d+1$ and $|V| = m+n \geq \binom{d+2}{2} = \sum_{i=1}^{d+1}(s_i+1)$, there is a partition $V_1, \ldots, V_{d+1}$ of V such that $V_i \cap A \neq \varnothing$, $V_i \cap B \neq \varnothing$ and $|V_i| \geq s_i + 1$ for all $1 \leq i \leq d+1$.

For $1 \leq i \leq d+1$, let $A_i = V_i \cap A$ and $B_i = V_i \cap B$. Note that $G[V_i] = G[A_i, B_i]$ is connected for all $1 \leq i \leq d+1$ (since it is a complete bipartite graph itself, with parts $A_i, B_i \neq \varnothing$). Therefore, it contains a spanning tree with $|V_i| - 1 \geq s_i$ edges. Moreover, for any $i \neq j$, $G[A_i, B_j]$ is connected (again, since it is a complete bipartite graph with parts $A_i, B_j \neq \varnothing$). Therefore, $V_1, \ldots, V_{d+1}$ is a strong bipartite d-rigid partition of $G = K_{m,n}$. $\qquad\square$

2.7 Limitations of rigid partitions

We do not expect the existence of a d-rigid partition to be a necessary condition for d-rigidity, except for $d = 2$. Indeed, we have the following restriction on the maximum d for which a graph G admits a d-rigid partition:

Proposition 2.15. *Let $G = (V, E)$ be a graph with clique number $\omega(G)$. Assume that G admits a d-rigid partition. Then*

$$d \leq \frac{|V| + \omega(G)}{2}.$$

Proof. Let $\left(\{V_i\}_{i=1}^{d+1}, \{E_{ij}\}_{1 \leq i < j \leq d+1}\right)$ be a d-rigid partition of G. For $1 \leq i < j \leq d+1$, let $G_{ij} = (V_i \cup V_j, E_{ij})$. First, note that at most one of the sets V_i may be empty. Let $I = \{1 \leq i \leq d+1 : |V_i| = 1\}$. Note that, for $i, j \in I$, with $i < j$, the unique vertex in V_i must be adjacent in G to the unique vertex in V_j, otherwise the graph G_{ij} would be disconnected. Therefore, the vertices in $\bigcup_{i \in I} V_i$ form a clique in G. In particular, we must have $|I| \leq \omega(G)$. On the other hand, we have

$$0 \cdot 1 + 1 \cdot |I| + 2 \cdot (d+1-|I|-1) \leq \sum_{i=1}^{d+1} |V_i| = |V|,$$

and therefore $|I| \geq 2d - |V|$. We obtain $d \leq (|V| + \omega(G))/2$, as wanted. $\qquad\square$

Now, let n be an even number, and let G_n be the n-**hyperoctahedral graph**—that is, the graph obtained by removing a perfect matching from the complete graph on n vertices. It is conjectured that the rigidity of G_n is $n - 1 - \lfloor \sqrt{n} + 1/2 \rfloor$ (see [14, Conjecture 3.4]). We verified this conjecture by computer calculation for $n \leq 150$. On the other hand, since a maximal clique in G_n is of size $n/2$, then by Proposition 2.15 G_n does not admit a d-rigid partition for $d > \frac{3n}{4}$.

Let us consider another example that, assuming the validity of Conjecture 1.11, highlights an even greater disparity between graph rigidity and the existence of rigid partitions. Consider the binomial random graph $G \sim G(n, 1/2)$ (any constant edge probability would serve similarly). If G has a d-rigid partition $\left(\{V_i\}_{i=1}^{d+1}, \{E_{ij}\}_{1 \leq i < j \leq d+1}\right)$, then there must be an edge in G between each pair (V_i, V_j) for $G_{ij} = (V_i \cup V_j, E_{ij})$ to be connected. However, standard calculations show (see [9]) that for $d \geq Cn/\sqrt{\log n}$

(for some $C > 0$), such a family of d disjoint vertex sets does not exist **whp**. Therefore, if we accept the validity of Conjecture 1.11, it follows that G is **whp** d-rigid for $d = \Theta(n)$, yet it does not admit a d-rigid partition for any $d \geq Cn/\sqrt{\log n}$.

The following construction, showing the existence, for all $d \geq 4$, of d-rigid graphs that do not admit d-rigid partitions, was suggested to us by an anonymous referee.

Fix $d \geq 4$. Let $G = (V, E)$ be a triangle-free d-rigid graph on at least $d + 1$ vertices (for example, we may take $G = K_{n,m}$, for $n, m \geq d + 1$ and $n + m \geq \binom{d+2}{2}$; see Proposition 1.6). Let $U = \{u_A : A \in \binom{V}{d}\}$ and $V' = V \cup U$. Write $E' = E \cup \{\{u_A, v\} : A \in \binom{V}{d}, v \in A\}$, and let $G' = (V', E')$. That is, G' is the graph obtained from G by adding, for every set A of d vertices in V, a new vertex u_A adjacent to exactly the d vertices in A. Since G' is obtained from G by sequentially adding vertices, each connected to the existing graph by d edges, it follows (see, e.g., [63]) that G' is d-rigid. However, we will show that G' does not admit a d-rigid partition.

Proposition 2.16. *The graph $G' = (V', E')$ does not admit a d-rigid partition.*

For the proof of Proposition 2.16, we will need the following simple lemma about the distribution of the edges incident to a vertex of degree d in a d-rigid partition.

Lemma 2.17. *Let $G = (V, E)$ be a graph, and let $(\{V_i\}_{i=1}^{d+1}, \{E_{ij}\}_{1 \leq i < j \leq d+1})$ be a d-rigid partition of G. For $1 \leq i < j \leq d+1$, write $G_{ji} = G_{ij} = (V_i \cup V_j, E_{ij})$. Let $i \in [d+1]$, and let $v \in V_i$ be a vertex with $\deg_G(v) = d$. Then, if $|V_i| + |V_j| \geq 2$ for all $j \neq i$, then $\deg_{G_{ij}}(v) = 1$ for all $j \neq i$. Otherwise, $|V_i| = 1$ and $V_j = \varnothing$ for some $j \neq i$. In this case, there exists $k \neq i, j$ such that $\deg_{G_{ik}}(v) = 2$, and $\deg_{G_{ir}}(v) = 1$ for all $r \neq i, j, k$.*

Proof. Let $\hat{E} = \cup_{1 \leq i < j \leq d+1} E_{ij}$, and let $\hat{G} = (V, \hat{E})$. Note that $(\{V_i\}_{i=1}^{d+1}, \{E_{ij}\}_{1 \leq i < j \leq d+1})$ is a d-rigid partition of $\hat{G}$, and therefore $\hat{G}$ is d-rigid. Thus, since $\deg_G(v) = d$ and $v \in V_i$, we have $\sum_{j \neq i} \deg_{G_{ij}}(v) = d$ (otherwise, v is a vertex of degree smaller than d in $\hat{G}$, in contradiction to the fact that $\hat{G}$ is d-rigid). Consider first the case that $|V_i| + |V_j| \geq 2$ for all $j \neq i$. Then, for each $j \neq i$, the graph G_{ij} has at least 2 vertices, and since it is connected, $\deg_{G_{ij}}(v) \geq 1$. But $\deg_G(v) = d$, hence $\deg_{G_{ij}}(v) = 1$ for all $j \neq i$.

Now suppose instead that $|V_i| = 1$ and $V_j = \varnothing$ for some $j \neq i$. Observe that for $r \neq j$, $V_r \neq \varnothing$, as otherwise G_{jr} is disconnected (by our convention that a vertex-less graph is disconnected). Then, for each $r \neq i, j$, the graph G_{ir} has at least 2 vertices, and since it is connected, $\deg_{G_{ir}}(v) \geq 1$. But $\deg_G(v) = d$, hence $\deg_{G_{ik}}(v) = 2$ for some $k \neq i, j$ and $\deg_{G_{ir}}(v) = 1$ for all $r \neq i, j, k$. $\square$

Lemma 2.18. *Let $G = (V, E)$ be a graph, and let $(\{V_i\}_{i=1}^{d+1}, \{E_{ij}\}_{1 \leq i < j \leq d+1})$ be a d-rigid partition of G. For $1 \leq i < j \leq d+1$, write $G_{ji} = G_{ij} = (V_i \cup V_j, E_{ij})$. Suppose that for some $1 \leq i < j \leq d+1$, W, W' is a partition of $V_i \cup V_j$, such that $E_{i'j'} \cap E_G(W, W') = \varnothing$ unless $(i', j') = (i, j)$. Then, for every $v \in V$ with $\deg_G(v) = d$, either $|W \cap N(v)| \leq 1$ or $|W' \cap N(v)| \leq 1$.*

Proof. Assume for contradiction that there exist $1 \leq i < j \leq d+1$, a partition W, W' of $V_i \cup V_j$ satisfying the required property, and $v \in V$ with $\deg_G(v) = d$, such that $|W \cap N(v)| \geq 2$ and $|W' \cap N(v)| \geq 2$. Assume $v \in V_k$ for $k \in [d+1]$. Suppose first that $k \neq i, j$. In that case, $\deg_{G_{ik}}(v) + \deg_{G_{jk}}(v) \geq 4$, contradicting Lemma 2.17. We can thus assume that $v \in V_i \cup V_j$. Assume, without loss of generality, that $v \in W \cap V_i$. If $V_i = \{v\}$, then $\deg_{G_{ij}}(v) \geq 4$, contradicting Lemma 2.17. Otherwise, $\deg_{G_{ij}}(v) \geq 2$, again contradicting Lemma 2.17. $\square$

We are ready to prove Proposition 2.16.

Proof of Proposition 2.16. Assume, for the sake of contradiction, that G' admits a d-rigid partition $(\{V_i\}_{i=1}^{d+1}, \{E_{ij}\}_{1 \leq i < j \leq d+1})$. For $1 \leq i < j \leq d+1$, let $G_{ji} = G_{ij} = (V_i \cup V_j, E_{ij})$. We consider two cases.

Case I. Assume that there exists $A \in \binom{V}{d}$ such that (without loss of generality) $V_{d+1} = \varnothing$ and $V_d = \{u_A\}$. By Lemma 2.17, we may assume, without loss of generality, that $|N(u_A) \cap V_i| = 1$ for all $i \in [d-2]$. Assume, for contradiction, that for some $i \in [d-2]$ it holds that $|V_i| > 1$. Then, by the monochromatic cut property, there exists a partition $V_i = W \cup W'$ such that $E_G(W, W') \cap E_{ij} \neq \varnothing$ for at most one $j \neq i$. Since V_{d+1} is empty and $G_{i,d+1}$ is connected, that j must be $d+1$. Assume, without loss of generality, that the unique neighbour of u_A in V_i is in W. Then, $E_{G_{di}}(W \cup V_d, W') = \varnothing$, contradicting the fact that G_{di} is connected. Hence, $|V_i| = 1$ for all $i \in [d-2]$.

For $i \in [d-2]$, write $V_i = \{v_i\} \subseteq V$. In particular, since G_{ij} is connected for all $1 \leq i < j \leq d-2$, $\{v_1, \ldots, v_{d-2}\}$ is a clique in G (this is already a contradiction to the fact that G is triangle-free if $d \geq 5$, but we continue to also address the case $d = 4$). Note that $|V_{d-1}| \geq 2$. By the monochromatic cut property, there is a partition W, W' of V_{d-1} such that $E_G(W, W') \cap E_{j,d-1} \neq \varnothing$ for all but at most one $j \neq d-1$. By the same argument as before, that j is $d+1$. We may assume without loss of generality that $|W \cap V| \leq 1$ — otherwise, $|W \cap V|, |W' \cap V| \geq 2$, implying that, since $d \geq 4$, there exists $B \in \binom{V}{d}$ such that $|W \cap N(u_B)|, |W' \cap N(u_B)| \geq 2$, in contradiction to Lemma 2.18 (applied to the partition of $V_{d-1} \cup V_{d+1} = V_{d-1}$ into W and W'). Since $G_{d-1,d}$ is connected, u_A has a neighbour in W. Hence, $W \cap V \neq \varnothing$. Write $W \cap V = \{w\}$.

We now show that $\{v_i, w\} \in E$ for all $i \in [d-2]$. Indeed, fix $i \in [d-2]$. Since $G_{i,d-1}$ is connected, there is a path in $G_{i,d-1}$ between v_i and w. Note that this path does not involve any vertices in U, since by Lemma 2.17, every vertex in $U \cap V_{d-1}$ has degree 1 in $G_{i,d-1}$. On the other hand, note that, since $W \cap V = \{w\}$ and $E_{G_{i,d-1}}(W, W') = \varnothing$, w has no neighbours in $V_{d-1} \cap V$ in the graph $G_{i,d-1}$. Therefore, the path from w to v_i must actually consist of the single edge $\{v_i, w\}$.

We conclude that $\{v_1, \ldots, v_{d-2}, w\}$ is a clique of size $d - 1 \geq 3$ in G, a contradiction.

Case II. Assume that the previous assumption does not hold. Then, by Lemma 2.17, for all $u_A \in U$, if $u_A \in U \cap V_k$, then $\deg_{G_{ik}}(u_A) = 1$ for all $i \neq k$.

For every $1 \leq i < j \leq d+1$, by Lemma 2.18, applied to the partition of $V_i \cup V_j$ into V_i and V_j, either $|V_i \cap V| \leq 1$ or $|V_j \cap V| \leq 1$ (otherwise, since $d \geq 4$, there exists $B \in \binom{V}{d}$ such that $|V_i \cap N(u_B)|, |V_j \cap N(u_B)| \geq 2$, in contradiction to Lemma 2.18). Therefore, without loss of generality, we may assume that $|V_i \cap V| \leq 1$ for all $1 \leq i \leq d$. We will show that $|V_i \cap V| = 1$ for all but at most one $1 \leq i \leq d$. Assume for contradiction that there exist $1 \leq i < j \leq d$ such that $V \cap V_i = V \cap V_j = \varnothing$. Then, G_{ij} has at least 2 vertices (since having 0 vertices would contradict connectivity, and having a single vertex would contradict the assumption of Case II), but no edges (since U is independent in G'), contradicting the connectivity of G_{ij}. Therefore, without loss of generality, we may assume that $|V_i \cap V| = 1$ for all $1 \leq i \leq d-1$. For each $1 \leq i \leq d-1$, let w_i be the unique vertex in $V_i \cap V$.

Now, fix $1 \leq i < j \leq d-1$. Since every $u_A \in (V_i \cup V_j) \cap U$ has $\deg_{G_{ij}}(u_A) = 1$, it follows that the only possible path between w_i, w_j in G_{ij} is the edge $\{w_i, w_j\}$. This implies that $\{w_1, \ldots, w_{d-1}\}$ is a clique, contradicting the fact that G is triangle-free. $\square$

Proposition 2.16 shows that the existence of a d-rigid partition does not characterise d-rigid graphs for $d \geq 4$. It remains open whether this characterisation holds for $d = 3$, and it would be interesting to try and find examples of 3-rigid graphs that do not admit 3-rigid partitions.

3 Conditions for rigid partitions

In this section, we develop general and easily verifiable sufficient conditions for the existence of rigid partitions. Central to our findings is the notion that "sparse" graphs that exhibit certain weak expansion

properties admit strong rigid partitions. In Section 4, this observation will be applied to various scenarios.

We will repeatedly make use of the following version of Chernoff bounds (see, e.g., in [35, Chapter 2]). Let $\varphi(x) = (1+x)\log(1+x) - x$ for $x > -1$.

Theorem 3.1 (Chernoff bounds)**.** *Let $n \geq 1$ be an integer and let $p \in [0,1]$, let $X \sim \mathsf{Bin}(n,p)$, and let $\mu = \mathbb{E}X = np$. Then, for every $\nu > 0$,*

$$\mathbb{P}(X \leq \mu - \nu) \leq \exp\left(-\mu\varphi\left(-\frac{\nu}{\mu}\right)\right) \leq \exp\left(-\frac{\nu^2}{2\mu}\right),$$

$$\mathbb{P}(X \geq \mu + \nu) \leq \exp\left(-\mu\varphi\left(\frac{\nu}{\mu}\right)\right) \leq \exp\left(-\frac{\nu^2}{2(\mu + \nu/3)}\right),$$

from which it follows, in particular, that for $\nu < \mu$,

$$\mathbb{P}(|X - \mu| \geq \nu) \leq 2\exp\left(-\frac{\nu^2}{3\mu}\right).$$

We will also need the following version of the Lovász Local Lemma; see, e.g., [4, Section 5.1].

Theorem 3.2 (Lovász Local Lemma)**.** *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be events in some probability space. Suppose that for every $i \in [n]$, each $\mathcal{A}_i$ is mutually independent of a set of all other events A_j but at most D, and that $\mathbb{P}(\mathcal{A}_i) \leq 1/(e(D+1))$. Then:*

$$\mathbb{P}\left(\bigcap_{i=1}^n \overline{\mathcal{A}_i}\right) \geq (1 - 1/(D+1))^n.$$

Our next lemma is the key lemma of this section and plays a crucial role throughout the rest of the paper. In essence, it asserts that for any graph with a sufficiently large minimum degree and a bounded min-to-max degree ratio, there exists a partition of its vertex set into d parts in which every vertex has an ample number of neighbours in each part.

Lemma 3.3 (Partition lemma)**.** *For every $\alpha \in (0,1)$ and $\Gamma \geq 1$ there exists $C = C(\alpha, \Gamma) > 1$ such that the following holds. For every $d \geq 2$, any n-vertex graph $G = (V, E)$ with minimum degree $\delta \geq Cd\log d$ and maximum degree $\Delta \leq \Gamma\delta$ has a partition $V = V_1 \cup \cdots \cup V_d$ such that every vertex in V has at least $(1-\alpha)\delta/d$ neighbours in V_i for all $1 \leq i \leq d$.*

Moreover, the probability of obtaining such a partition, if we partition the vertex set at random by independently assigning each vertex $v \in V$ to the set V_i with probability $1/d$, is at least $(1 - 1/\Delta^2)^n$.

Proof. Let $\{m(v) : v \in V\}$ be a family of independent random variables, each uniformly distributed on $[d]$. For all $1 \leq i \leq d$, let $V_i = \{v \in V : m(v) = i\}$. For $v \in V$ and $i \in [d]$ let $\mathcal{A}_{v,i}$ be the event that $|N(v) \cap V_i| \leq (1-\alpha)\delta/d$, and let $\mathcal{A}_v = \bigcup_{i \in [d]} \mathcal{A}_{v,i}$. Note that $|N(v) \cap V_i| \sim \mathsf{Bin}(\deg(v), 1/d)$, so by Chernoff's bounds (Theorem 3.1) we obtain

$$\mathbb{P}(\mathcal{A}_{v,i}) = \mathbb{P}(|N(v) \cap V_i| \leq (1-\alpha)\delta/d) \leq \mathbb{P}(|N(v) \cap V_i| \leq (1-\alpha)\deg(v)/d) \leq e^{\frac{-\alpha^2 \deg(v)}{2d}} \leq e^{\frac{-\alpha^2 \delta}{2d}}.$$

By the union bound we have

$$p := \mathbb{P}(\mathcal{A}_v) \leq \sum_{i=1}^d \mathbb{P}(\mathcal{A}_{v,i}) \leq d e^{\frac{-\alpha^2 \delta}{2d}}.$$

Note that for every $u \in V$, there are at most $\Delta(\Delta - 1) \leq \Delta^2 - 1$ vertices v for which $N(u) \cap N(v) \neq \varnothing$. Thus, the event $\mathcal{A}_u$ is mutually independent of the set of all other events $\mathcal{A}_v$ but at most $\Delta^2 - 1$. To use the Lovász Local Lemma (Theorem 3.2), it is enough to show that $ep\Delta^2 \leq 1$. Indeed, letting $\delta = Cd\log d$,

$$ep\Delta^2 \leq e \cdot d e^{\frac{-\alpha^2 \delta}{2d}}(\Gamma\delta)^2 = e\Gamma^2\log^2 d \cdot C^2 d^{3 - \frac{\alpha^2 C}{2}}.$$

Write $\psi(d) = \log^2 d/d$. The function $\psi(d)$ attains its maximum at $d = e^2$, where it equals $4/e^2$. Thus, for $C \geq 8/\alpha^2$, we obtain

$$e\Gamma^2\log^2 dC^2d^{3-\frac{\alpha^2 C}{2}} = e\Gamma^2\psi(d)C^2d^{-(\alpha^2 C/2-4)} \leq \frac{4\Gamma^2}{e}C^2d^{-(\alpha^2 C/2-4)} \leq \frac{4\Gamma^2}{e}C^22^{-(\alpha^2 C/2-4)}.$$

Thus, there exists $C_1 = C_1(\alpha, \Gamma)$ for which for every $C \geq C_1$ the above expression is at most 1.

Therefore, by the Lovász Local Lemma (Theorem 3.2), for $C \geq C_1$, the probability none of the $\mathcal{A}_v$ occur is at least $(1 - 1/\Delta^2)^n$. $\qquad\qquad\square$

In certain applications, we need the partition to be "balanced" with respect to some underlying colouring of the vertex set. That is, given a colouring of the vertex set, we need our partition to split each of the colour classes into roughly equally sized parts. In our specific setting, we will exclusively apply this lemma to the case of two colour classes (see Proposition 3.10).

Lemma 3.4 (Balanced partition lemma). *For every $\alpha, \zeta \in (0,1)$ and $\Gamma, t \geq 1$ there exist $C = C(\alpha, \zeta, \Gamma, t) > 1$ such that the following holds. Let $d \geq 2$, and let $G = (V, E)$ be an n-vertex graph with minimum degree $\delta \geq Cd\log d$ and maximum degree $\Delta \leq \Gamma\delta$. Assume that V is partitioned into t sets $A_1, \ldots, A_t$, such that $|A_i| = n/t$ for all $1 \leq i \leq t$. Then, there exists a partition $V = V_1 \cup \cdots \cup V_d$ such that every vertex in V has at least $(1-\alpha)\delta/d$ neighbours in V_i for all $1 \leq i \leq d$, and $(1-\zeta)n/(td) \leq |V_i \cap A_j| \leq (1+\zeta)n/(td)$ for all $1 \leq i \leq d$ and $1 \leq j \leq t$.*

Proof. Let $\{m(v) : v \in V\}$ be a family of independent random variables, each uniformly distributed on $[d]$. For all $1 \leq i \leq d$, let

$$V_i = \{v \in V : m(v) = i\}.$$

Let $\mathcal{A}$ be the event that all vertices in V have at least $(1-\alpha)\delta/d$ neighbours in each V_i, and let $\mathcal{B}$ be the event that for all $1 \leq i \leq d$ and $1 \leq j \leq t$, $||V_i \cap A_j| - n/(td)| \leq \zeta n/(td)$. Let $p_1 = \mathbb{P}(\mathcal{A})$ and $p_2 = \mathbb{P}(\mathcal{B})$. In order to show that $\mathbb{P}(\mathcal{A} \cap \mathcal{B}) > 0$, it is enough to show that $p_1 + p_2 > 1$. By the partition lemma (Lemma 3.3), there exists $C_1 = C_1(\alpha, \Gamma)$ such that, for $C \geq \max\{C_1, 2\}$, one has

$$p_1 \geq \left(1 - \frac{1}{\Delta^2}\right)^n \geq e^{-n/(\Delta^2-1)} \geq e^{-n/(\delta^2-1)} \geq e^{-2n/(C^2d)}, \qquad (3.1)$$

where we used the inequality $1 - x > e^{-x/(1-x)}$ which holds for $x \in (0,1)$, and the fact that $\delta^2 - 1 \geq C^2d^2\log^2 d - 1 \geq C^2d/2$ (since $d \geq 2$ and $C \geq 2$). On the other hand, for $1 \leq i \leq d$ and $1 \leq j \leq t$, we have by Chernoff's bounds (Theorem 3.1),

$$\mathbb{P}\left(||V_i \cap A_j| - n/(td)| \geq \zeta n/(td)\right) \leq 2e^{-\zeta^2 n/(3td)},$$

so by the union bound,

$$p_2 \geq 1 - 2tde^{-\zeta^2 n/(3td)}. \qquad (3.2)$$

Define

$$f(C) = (\zeta^2/(3t))C^2 - (1 + \log(2t)/\log 2)C - 2.$$

Let C_2 be the largest root of $f(C)$. Note that $f(C) > 0$ for $C > C_2$, and that, by the quadratic formula,

$$C_2 = \frac{3t}{2\zeta^2}\left(1 + \frac{\log(2t)}{\log 2} + \sqrt{\left(1 + \frac{\log(2t)}{\log 2}\right)^2 + \frac{8\zeta^2}{3t}}\right) \geq \frac{3t}{2\zeta^2}\left(1 + 1 + \sqrt{(1+1)^2 + 0}\right) = \frac{6t}{\zeta^2}.$$

Let $C > \max\{C_1, C_2, 2\}$. By (3.1) and (3.2), we have

$$\log(p_1/(1 - p_2)) = \log p_1 - \log(1 - p_2) \geq -\frac{2n}{C^2 d} - \left(-\frac{\zeta^2 n}{3td} + \log(2td)\right) = \frac{n}{d}\left(\frac{\zeta^2}{3t} - \frac{2}{C^2}\right) - \log(2td).$$

Using the inequality $n \geq \delta \geq Cd \log d$ (and the fact that $\zeta^2/(3t) - 2/C^2 > 0$, since $C > C_2 \geq 6t/\zeta^2$), we obtain

$$\log(p_1/(1 - p_2)) \geq C \log d \left(\frac{\zeta^2}{3t} - \frac{2}{C^2}\right) - \log(2td) = \frac{\log d}{C}\left(\frac{\zeta^2}{3t}C^2 - \frac{\log(2td)}{\log d}C - 2\right).$$

Note that

$$\frac{\log(2td)}{\log d} = 1 + \frac{\log(2t)}{\log d} \leq 1 + \frac{\log(2t)}{\log 2},$$

and therefore,

$$\log(p_1/(1 - p_2)) \geq \frac{\log d}{C}\left(\frac{\zeta^2}{3t}C^2 - \left(1 + \frac{\log(2t)}{\log 2}\right)C - 2\right) = \frac{\log d}{C} \cdot f(C) > 0.$$

Thus, for $C > \max\{C_1, C_2, 2\}$, we obtain $p_1/(1 - p_2) > 1$, or equivalently $p_1 + p_2 > 1$, as wanted. $\square$

We now embark on the task of identifying basic graph properties that will guarantee the existence of rigid partitions. Say that a graph $G = (V, E)$ is an R-**expander** if every set $A \subseteq V$ with $|A| \leq R$ has $|N(A)| \geq 2|A|$. We say it is (x, y)-**thin** if every set $A \subseteq V$ with $|A| = a \leq x$ spans at most ay edges. Note that if G is (x, y)-thin then it is (x', y')-thin for every $x' \leq x$ and $y' \geq y$. Moreover, if G is (x, y)-thin then every subgraph G' of G is (x, y)-thin. The next lemma roughly shows that thin graphs with sufficiently high minimum degree are expanders.

Lemma 3.5. *Let $G = (V, E)$ be an (x, y)-thin graph with minimum degree $\delta \geq 6y$. Then, G is an $(x/3)$-expander.*

Proof. Let $A \subseteq V$ with $|A| \leq x/3$, and let $B = A \cup N(A)$. Assume for contradiction that $|N(A)| < 2|A|$. Then $|B| < 3|A| \leq x$, and therefore, since G is (x, y)-thin, we have

$$|E(B)| \leq |B|y < 3|A|y.$$

On the other hand, since $\delta(G) = \delta \geq 6y$, we have

$$|E(B)| \geq \frac{1}{2}\sum_{v \in A}\deg(v) \geq \delta|A|/2 \geq 3|A|y,$$

a contradiction. $\square$

Say that a graph $G = (V, E)$ is a K-**connector** if every two disjoint vertex sets of size at least K are connected by an edge. Similarly, if $G = (V_1 \cup V_2, E)$ is a bipartite graph, we say that G is a K-**bi-connector** if every $A_1 \subseteq V_1$ and $A_2 \subseteq V_2$, both of size at least K, are connected by an edge. Note that if G is a K-connector, then it is also a K'-connector for all $K' > K$. Moreover, for a real number K, G is a K-connector if and only if it is a $\lceil K \rceil$-connector. If G is a K-connector, then, for every $U \subseteq V$, $G[U]$ is also a K-connector, and for every two disjoint sets $U_1, U_2 \subseteq V$, $G[U_1, U_2]$ is a K-bi-connector. Finally, note that the property of being a K-connector is monotone increasing.

The next lemma roughly shows that graphs that are both expanders and connectors are connected.

Lemma 3.6. *Let G be an R-expander. Then, if G is a $3R$-connector, G is connected. Similarly, if G is bipartite and a $2R$-bi-connector, then G is connected.*

Proof. Note that for any R-expander G, every connected component of G is of size at least $3R$. On the other hand, if G is a K-connector, any connected component except possibly the largest one must have less than K vertices. Therefore, any R-expander that is also a $3R$-connector must be connected.

Now, assume that G is bipartite with parts V_1 and V_2. Note that if G is an R-expander, then every connected component of G has at least $2R$ elements in each V_i. On the other hand, if G is a K-bi-connector, any connected component except possibly the largest one has less than K elements in one of V_1, V_2. Thus, a bipartite R-expander that is also a $(2R)$-bi-connector is connected. $\square$

Using the last two lemmas, we now show that in thin connectors, induced subgraphs with sufficiently large minimum degree are connected.

Lemma 3.7. *Let $G = (V, E)$ be an (x, y)-thin graph that is also a $(2x/3)$-connector. Let $U_1, U_2 \subseteq V$ such that either $U_1 = U_2$ or $U_1 \cap U_2 = \varnothing$, and assume that $\delta(G[U_1, U_2]) \geq 6y$. Then, $G[U_1, U_2]$ is connected.*

Proof. Let $G' = G[U_1, U_2]$. Note that any subgraph of an (x, y)-thin graph is also (x, y)-thin. In particular, G' is (x, y)-thin. Since the minimum degree of G' is at least $6y$, then by Lemma 3.5 G' is an $(x/3)$-expander.

We now divide into two cases: if $U_1 = U_2$ then, since G is a $(2x/3)$-connector, so is $G' = G[U_1]$. In particular, G' is an x-connector, so by Lemma 3.6 it is connected. If $U_1 \cap U_2 = \varnothing$, then, since G is a $(2x/3)$-connector, $G' = G[U_1, U_2]$ is a $(2x/3)$-bi-connector. Thus, by Lemma 3.6, it is connected. $\square$

We now deduce from the last lemma and the partition lemma (Lemma 3.3) a useful sufficient condition for the admittance of *strong* rigid partitions.

Proposition 3.8. *For every $\Gamma \geq 1$ there exists $C = C(\Gamma) > 1$ such that the following holds. Let $d \geq 2$ and let $G = (V, E)$ be a graph with minimum degree $\delta \geq Cd \log d$ and maximum degree $\Delta \leq \Gamma \delta$. Assume that G is $(x, \delta/(7d))$-thin and a $(2x/3)$-connector, for some x. Then G admits a strong d-rigid partition.*

Proof. Let $\alpha = 1/7$. By the partition lemma (Lemma 3.3), there is $C = C(\alpha, \Gamma) = C(\Gamma)$ such that if $\delta \geq Cd \log d$ and $\Delta \leq \Gamma \delta$, there exists a partition $V = V_1 \cup \cdots \cup V_d$ such that the minimum degree of $G[V_i, V_j]$ is at least $(1 - \alpha)\delta/d = 6\delta/(7d)$ for all $1 \leq i, j \leq d$. By Lemma 3.7, $G[V_i, V_j]$ is connected for all $1 \leq i, j \leq d$, as wanted. $\square$

In order to address the bipartite scenario, we will utilize the following simple lemma.

Lemma 3.9. *Let $m, K \geq 1$ be integers, and let G be a bipartite graph with sides A, B of size at least m each. Suppose further that G is K-bi-connector. Then, G admits a matching of size $m - K + 1$.*

Proof. Let M be a maximal matching in G. Assume for contradiction that $|M| \leq m - K$. Let $A' \subseteq A$ and $B' \subseteq B$ be the vertices not incident to any edge of M. Then, we have $|A'| \geq K$ and $|B'| \geq K$. Since G is a K-bi-connector, there exists an edge in G connecting a vertex $u \in A'$ and a vertex $v \in B'$. But then $M \cup \{\{u, v\}\}$ is a matching containing M, in contradiction to the maximality of M. $\square$

Proposition 3.10. *For every $\Gamma \geq 1$ there exists $C = C(\Gamma) > 1$ such that the following holds. Let $d \geq 2$, write $d' = d + 1$, and let $G = (V, E)$ be a bipartite graph with bipartition (A, B) with $|A| = |B|$, minimum degree $\delta \geq Cd' \log d'$ and maximum degree $\Delta \leq \Gamma \delta$. Assume that $d' \leq \sqrt{3|A|/4}$, and that G is $(x, \delta/(7d'))$-thin and a $(2x/3)$-bi-connector for $x \leq |A|/(7d')$. Then G admits a strong bipartite d-rigid partition.*

Proof. Set $\alpha = \zeta = 1/7$ and $t = 2$. By the balanced partition lemma (Lemma 3.4), there is $C = C(\alpha, \zeta, \Gamma, t)$ such that if $\delta \geq Cd' \log d'$ and $\Delta \leq \Gamma\delta$, there exists a partition $V = V_1 \cup \cdots \cup V_d \cup V_{d+1}$ with the following properties. Write $A_i = V_i \cap A$ and $B_i = V_i \cap B$. Then, $\delta(G[V_i, V_j]) \geq \frac{6\delta}{7d'}$ for all $1 \leq i < j \leq d+1$, and hence $\delta(G[A_i, B_j]) \geq \frac{6\delta}{7d'}$ for all $i \neq j$ in $[d+1]$. In addition, $|A_i|, |B_i| \geq \frac{6|A|}{7d'}$ for all $1 \leq i \leq d+1$. Fix $i \neq j$ and let $G' = G[A_i, B_j]$. Repeating the argument in the proof of Lemma 3.7, we observe that by Lemma 3.5, G' is an $(x/3)$-expander; and by Lemma 3.6, it is connected. Let $G^* = G[A_i, B_i]$. Since G^* is $(2x/3)$-bi-connector, it follows from Lemma 3.9 that it contains a matching of size at least $\frac{6|A|}{7d'} - \frac{2x}{3} > \frac{3|A|}{4d'} > d$. Thus, by Lemma 2.11, G admits a strong bipartite d-rigid partition. $\qquad\square$

4 Pseudorandom and random graphs

Throughout this section, we will make routine use of the following standard upper bound on binomial coefficients.

Lemma 4.1. *For any integers $1 \leq k \leq n$,*

$$\binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{en}{k}\right)^k.$$

4.1 Jumbled graphs

Roughly speaking, pseudorandom graphs are graphs with properties that one expects to find in random graphs. One such property, captured by the notion of "jumbledness", originally introduced by Thomason [64, 65][5] is that the graph has essentially the same edge density between any two (sufficiently large) vertex sets. Formally, we say that a graph $G = (V, E)$ is (p, β)-**jumbled** if it satisfies, for all $A, B \subseteq V$,

$$|e(A, B) - p|A||B|| \leq \beta\sqrt{|A||B|}.$$

In the next two lemmas we show that jumbled graphs are thin connectors.

Lemma 4.2. *A (p, β)-jumbled graph is $(x, (px + \beta)/2)$-thin for every positive integer x.*

Proof. Let x be a positive integer, and let $A \subseteq V$ with $|A| \leq x$. Then,

$$2|E(A)| = e(A, A) \leq p|A|^2 + \beta|A| \leq |A|(px + \beta). \qquad\square$$

Lemma 4.3. *A (p, β)-jumbled graph is K-connector for every $K > \beta/p$.*

Proof. Let $A, B \subseteq V$ with $A \cap B = \varnothing$ and $|A| = |B| = K > \beta/p$. Then,

$$|E(A, B)| = e(A, B) \geq p|A||B| - \beta\sqrt{|A||B|} = K(pK - \beta) > 0. \qquad\square$$

Proposition 4.4. *For every Γ there exists $C = C(\Gamma) > 1$ such that the following holds. Let $d \geq 2$, and let $G = (V, E)$ be a (p, β)-jumbled graph with minimum degree $\delta \geq \max\{9d\beta, Cd \log d\}$ and maximum degree $\Delta \leq \Gamma\delta$. Then G admits a strong d-rigid partition.*

Proof. Let $x = 11\beta/(7p)$. Note that $\delta/(7d) \geq 9\beta/7 = (px + \beta)/2$. Hence, by Lemma 4.2, G is $(x, \delta/(7d))$-thin. Since $2x/3 > \beta/p$, by Lemma 4.3 G is a $(2x/3)$-connector. Let $C = C(\Gamma)$ be the constant from Proposition 3.8. Since $\delta \geq Cd \log d$ and $\Delta \leq \Gamma\delta$, by Proposition 3.8, G admits a strong d-rigid partition. $\qquad\square$

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[5]Here, we do not use the original definition of Thomason, but rather a similar definition that appears in [17].

To prove Theorem 1.7, we use the Expander Mixing Lemma, stated below.

Theorem 4.5 ([3]; see also [32, Lemma 2.5])**.** *Every (n, k, λ)-graph is $(k/n, \lambda)$-jumbled.*

We restate Theorem 1.7 before showing how it now follows immediately:

Theorem 1.7 (Pseudorandom graphs)**.** *There exists $C > 1$ such that if G is an (n, k, λ)-graph with $k \geq \max\{9d\lambda, Cd \log d\}$ then G admits a strong d-rigid partition.*

Proof of Theorem 1.7. Let G be an (n, k, λ)-graph. By the Expander Mixing Lemma (Theorem 4.5) G is $(k/n, \lambda)$-jumbled. The result follows from Proposition 4.4 by taking $C = C(1)$. $\qquad\square$

4.2 Spread measures

Following (and slightly extending) terminology from [5], we say that a distribution $\mathcal{G}$ of n-vertex graphs is $(p; Z)$-**spread** if for every set E_0 of pairs of vertices with $|E_0| = m \leq Z$ we have that $\mathbb{P}(E_0 \subseteq E(\mathcal{G})) \leq p^m$. We say that $\mathcal{G}$ is p-**spread** if it is $(p; \binom{n}{2})$-spread. This section is dedicated to proving that spread measures of sufficiently high density are (locally and relatively) "sparse". The following lemma, which will be instrumental in the next few sections, makes this concept precise.

Lemma 4.6. *For every $\Lambda > 0$ there exists $k > 0$ such that the following holds. Suppose $p \geq k/n$, and let $\mathcal{G}$ be a $(p; Z)$-spread distribution of n-vertex graphs. Set $x = 2\Lambda e^{-(1+1/\Lambda)} \log(np)/p$ and $y = \Lambda \log(np)$, and assume that $xy \leq Z$. Then, $G \sim \mathcal{G}$ is **whp** (x, y)-thin.*

Proof. Let $\eta = 2\Lambda e^{-(1+1/\Lambda)}$. Since $\mathcal{G}$ is $(p; Z)$-spread, for every edge set $F \subseteq E(K_n)$ of size $m \leq Z$, the probability that $F \subseteq E(G)$ is at most p^m. For every $1 \leq a \leq x$, consider the event $\mathcal{L}_a$ that there exists a vertex set A of size a that spans at least ay edges, where $ay \leq xy \leq Z$. By the union bound, we can sum over the $\binom{n}{a}$ choices for A, and for each A, over the $\binom{\binom{a}{2}}{ay}$ potential edge sets F of size ay contained in A. Therefore, using Lemma 4.1,

$$\mathbb{P}(\mathcal{L}_a) \leq \binom{n}{a}\binom{\binom{a}{2}}{ay}p^{ay} \leq \left(\frac{en}{a} \cdot \left(\frac{eap}{2y}\right)^y\right)^a.$$

Take $k > \exp(1/\Lambda)$, so $y \geq \Lambda \log k > 1$. Write $f(a) = (en/a) \cdot (eap/(2y))^y$ and note that f is increasing in a. Recalling that $x = \eta \log(np)/p$ and $xp/y = \eta/\Lambda$, we have

$$f(a) \leq f(x) = \frac{en}{x} \cdot \left(\frac{exp}{2y}\right)^y = \frac{enp}{\eta \log(np)} \cdot \left(\frac{e\eta}{2\Lambda}\right)^{\Lambda \log(np)}$$
$$= \frac{enp}{\eta \log(np)} \cdot \left(e^{-1/\Lambda}\right)^{\Lambda \log(np)} = \frac{enp}{\eta \log(np)} \cdot (np)^{-1} = \frac{e}{\eta \log(np)}.$$

Taking $k > \exp(2e/\eta) = \exp\left(e/\Lambda \cdot e^{1+1/\Lambda}\right) > \exp(1/\Lambda)$, $np \geq k$ implies $f(a) \leq e/(\eta \log k) < 1/2$. Note that $f(\sqrt{x})/f(x) = (1/\sqrt{x})^{y-1} \ll 1$, hence, for $a \leq \sqrt{x}$, $f(a) \leq f(\sqrt{x}) \ll f(x) < 1/2$. Then, by the union bound over all $1 \leq a \leq x$, the probability that there exists a set of size at most x that spans at least ay edges is at most

$$\sum_{a=1}^{x} \mathbb{P}(\mathcal{L}_a) \leq \sum_{a=1}^{x}(f(a))^a \leq \sum_{a=1}^{\sqrt{x}}(f(a))^a + \sum_{a=\sqrt{x}}^{x}(f(a))^a$$
$$\leq \sum_{a=1}^{\infty}(f(\sqrt{x}))^a + \sum_{a=\sqrt{x}}^{\infty}(f(x))^a \leq \frac{f(\sqrt{x})}{1 - f(\sqrt{x})} + \frac{(1/2)^{\sqrt{x}}}{1 - 1/2} = o(1),$$

as required. $\qquad\square$

4.3 Random regular graphs

To prove Theorem 1.8, we show that random regular graphs are **whp** thin connectors, and apply Proposition 3.8. We cast our arguments in the *configuration model*. Let us lay formal grounds. Let $K_{n \cdot k}$ be the complete graph on the vertex set $[n] \times [k]$. The vertices of $K_{n \cdot k}$ are called **half-edges**, and the vertex sets $V_i = \{(i, j) : j \in [k]\}$, $i \in [n]$, are called **clusters**. For a subgraph $G \subseteq K_{n \cdot k}$, denote by $\pi(G)$ the multigraph obtained from G by contracting each cluster V_i into a single vertex i. Assume that nk is even, and denote by $\mathcal{M}_{n,k}$ the uniform distribution over perfect matchings in $K_{n \cdot k}$. Denote further by $\mathcal{G}_{n,k}^*$ the distribution of $\pi(M)$ for $M \sim \mathcal{M}_{n,k}$. We will repeatedly use the following result (see, e.g., [35, Theorem 9.9]): if $\mathcal{G}_{n,k}^*$ satisfies a property **whp** then $\mathcal{G}_{n,k}$ satisfies that property **whp** as well.

To prove that random regular graphs are typically thin, we show that they (or, rather, $\mathcal{G}_{n,k}^*$) are typically well-spread.

Lemma 4.7. *For every $\varepsilon \in (0, 1)$ and every $k \geq 3$, $\mathcal{G}_{n,k}^*$ is $(p; Z)$-spread for $p = \varepsilon^{-1} k/n$ and $Z = (1 - \varepsilon)kn/2$.*

Proof. Let $G^* \sim \mathcal{G}_{n,k}^*$, and let E_0 be a set of pairs of vertices of G^* of size at most $(1 - \varepsilon)kn/2$. We think of E_0 as a (simple) subgraph of K_n. Note that the number of matchings M_0 of size m of $K_{n \cdot k}$ for which $\pi(M_0) = E_0$ is at most $(k^2)^m$. Fix one such matching $M_0 = \{e_1, \ldots, e_m\}$. Let $M \sim \mathcal{M}_{n,k}$, and note that for every $i \in [m]$,

$$\mathbb{P}(e_i \in M \mid e_1, \ldots, e_{i-1} \in M) \leq \frac{1}{kn - 2m} \leq \frac{1}{\varepsilon kn}.$$

Thus, $\mathbb{P}(M_0 \subseteq M) \leq (\varepsilon kn)^{-m}$. We couple $G^* \sim \mathcal{G}_{n,k}^*$ with M by letting $G^* = \pi(M)$. By the union bound, $\mathbb{P}(E_0 \subseteq E(G^*)) \leq (\varepsilon^{-1} k/n)^m$. $\qquad\square$

Lemma 4.8. *For every sufficiently large constant k, $G \sim \mathcal{G}_{n,k}$ is **whp** (x, y)-thin for $x = 6 \log k \cdot n/k$ and $y = 36 \log k$.*

Proof. We prove the statement for $G \sim \mathcal{G}_{n,k}^*$ and deduce it for $\mathcal{G}_{n,k}$. Let $\Lambda = 18$. By Lemma 4.7, for $\varepsilon = 1/2$, $\mathcal{G}_{n,k}^*$ is $(p; Z)$-spread for $p = 2k/n$ and $Z = kn/4$. Note that $x = 6 \log k \cdot n/k \leq 12 \log(np)/p \leq 2\Lambda e^{-(1+1/\Lambda)} \log(np)/p$, and $y = 36 \log k \geq 18 \log 2k = \Lambda \log(np)$. Note also that $xy < Z$ for sufficiently large k. Thus, by Lemma 4.6, for sufficiently large k, $G^* \sim \mathcal{G}_{n,k}^*$ is **whp** (x, y)-thin. $\qquad\square$

We proceed to prove that $\mathcal{G}_{n,k}$ is typically a good connector.

Lemma 4.9. *Suppose $k \geq 3$, and let $G \sim \mathcal{G}_{n,k}$. Then, **whp** G is a K-connector for $K = 4 \log k \cdot \frac{n}{k}$.*

Proof. As before, we prove the statement for $G \sim \mathcal{G}_{n,k}^*$ and deduce it for $\mathcal{G}_{n,k}$. We prove the result for $K = \lceil 4 \log k \cdot n/k \rceil$, which implies the stated result.

Let $\tilde{V} = [n] \times [k]$ denote the vertex set of $K_{n \cdot k}$, and let $U, W \subseteq \tilde{V}$ be disjoint vertex sets with $|U| = a$ and $|W| = b$. We sample $M \sim \mathcal{M}_{n,k}$ as follows. Let M_0 be the empty matching. For $i = 1, \ldots, \lfloor a/2 \rfloor$, let $V_{i-1} = V(M_{i-1}) \subseteq \tilde{V}$ be the set of vertices matched by M_{i-1}, and let $U_i = U \cap V_{i-1} \subseteq U$ be the set of vertices in U matched by M_{i-1}. Pick a vertex u_i in $U \setminus U_i$ arbitrarily, and let v_i be a vertex chosen uniformly at random from $\tilde{V} \setminus (V_{i-1} \cup \{u_i\})$. Set $e_i = \{u_i, v_i\}$ and define $M_i = M_{i-1} \cup \{e_i\}$. Finally, we extend $M_{\lfloor a/2 \rfloor}$ uniformly at random to obtain M.

Let $\mathcal{A}_i$ be the event that $\{v_1, \ldots, v_{i-1}\} \cap W = \varnothing$. Observe that $\mathbb{P}(v_i \in W \mid \mathcal{A}_i) \geq b/(kn)$. Let $\mathcal{B}_{U,W}$ denote the event that M contains no edge between U, W. By the above,

$$\mathbb{P}(\mathcal{B}_{U,W}) \leq \mathbb{P}(\{v_1, \ldots, v_{\lfloor a/2 \rfloor}\} \cap W = \varnothing) = \prod_{i=1}^{\lfloor a/2 \rfloor} \mathbb{P}(v_i \notin W \mid \mathcal{A}_i) \leq \left(1 - \frac{b}{kn}\right)^{a/2}.$$

Now, we couple G with M by setting $G = \pi(M)$. We deduce from the discussion above and the union bound that the probability p that there exist two vertex sets A, B of G with $|A| = |B| = K$ and $E_G(A, B) = \varnothing$ satisfies

$$p \leq \binom{n}{K}^2 \left(1 - \frac{kK}{kn}\right)^{kK/2} \leq \left(\frac{en}{K}\right)^{2K} \exp\left(-\frac{kK^2}{2n}\right) = \left[\frac{e^2n^2}{K^2} \cdot \exp\left(-\frac{kK}{2n}\right)\right]^K.$$

Plugging in $K = C \log k \cdot n/k$, where $C \geq 4$, we obtain

$$p \leq \left(\frac{e^2 k^{2-C/2}}{C^2 \log^2 k}\right)^K \leq \left(\frac{1}{\log^2 k}\right)^K = o(1),$$

where we used $C \geq 4$ and $k \geq 3$. Thus, G is **whp** a K-connector. $\qquad\square$

We recall Theorem 1.8 before turning to its proof:

Theorem 1.8 (Random regular graphs). *There exists $C > 0$ such that for every fixed $d \geq 2$ and $k \geq Cd \log d$, $G \sim \mathcal{G}_{n,k}$ **whp** admits a strong d-rigid partition.*

Proof of Theorem 1.8. Let $C_1 = C(1)$ be the constant from Proposition 3.8, and let $\Lambda = 36$. Let C be a constant satisfying $C \geq \max\{C_1, e\}$ and $C \geq 28\Lambda \log C$. Pick an integer $k = C'd \log d$ with $C' \geq C$ that is sufficiently large for Lemma 4.8 to hold. Let $x = 6 \log k \cdot n/k$ and $y = 36 \log k$. By Lemma 4.8, G is **whp** (x, y)-thin. Note that $k \geq e \cdot 2 \log 2 > 3$, and therefore, by Lemma 4.9, G is **whp** a $(2x/3)$-connector. Note that $C' \geq 28\Lambda \log C'$ and, in particular, $\Lambda \leq C'/28$. Thus,

$$y = \Lambda \log k = \Lambda \log(C'd \log d) \leq \Lambda \log C' + 2\Lambda \log d$$

$$\leq C'/28 + C' \log d/14 \leq C' \log d/14 + C' \log d/14 = C' \log d/7 \leq \frac{k}{7d}.$$

So G is **whp** $(x, k/(7d))$-thin. Hence, since $k \geq Cd \log d \geq C_1 d \log d$, by Proposition 3.8, G **whp** admits a d-rigid partition. $\qquad\square$

4.4 Binomial random graphs

We begin with several lemmas about expansion properties of $G(n,p)$. The following is a well-known statement about the degrees of $G(n,p)$.

Lemma 4.10. *For every $\varepsilon > 0$ there exists $\xi \in (0, 1)$ such that the following holds. Let $p \geq (1+\varepsilon) \log n/n$, and let $G \sim G(n,p)$. Then, **whp**, $(1 - \xi)np \leq \delta(G) \leq \Delta(G) \leq enp$.*

Proof. The degree of a vertex is distributed as a binomial random variable with $n' = n - 1$ attempts and success probability p. Thus, by Chernoff bounds (Theorem 3.1), setting $n' = n - 1$,

$$\mathbb{P}(\deg(v) \geq enp) \leq \mathbb{P}(\deg(v) \geq en'p) \leq \exp(-n'p\varphi(e - 1)) \leq (n')^{-(1+\varepsilon)} \ll n^{-1},$$

$$\mathbb{P}(\deg(v) \leq (1 - \xi)np) \leq \mathbb{P}(\deg(v) \leq (1 - \xi^2)n'p) \leq \exp\left(-n'p\varphi(-\xi^2)\right) \leq (n')^{-(1+\varepsilon)\varphi(-\xi^2)},$$

where the last expression is $o(n^{-1})$ if $\xi < 1$ is close enough to 1 (and thus ξ^2 is close enough to 1), since $\varphi(x) \to 1$ as $x \to -1$. The result follows by applying the union bound over all n vertices. $\qquad\square$

Since $G(n,p)$ is (trivially) p-spread, we obtain the following.

Lemma 4.11. *There exists a sufficiently large constant k such that if $p \geq k/n$, then $G \sim G(n,p)$ is **whp** (x, y)-thin for (i) $x = 3\log(np)/p$ and $y = 5\log(np)$; (ii) for $x = 7\log(np)/p$ and $y = 11\log(np)$.*

Proof. The result follows from Lemma 4.6 since $G(n, p)$ is p-spread, and by noting that $3 \le 2 \cdot 5e^{-(1+1/5)}$ and $7 \le 2 \cdot 11e^{-(1+1/11)}$. $\hfill\square$

Lemma 4.12. *Suppose $p \ge 4/n$, and let $G \sim G(n, p)$. Then, **whp** G is a K-connector for $K = 2\log(np)/p$.*

Proof. Let A, B be two disjoint vertex sets with $|A| = |B| = K$. The probability that G contains no edge between A and B is $(1 - p)^{K^2}$. Thus, by the union bound over all sets A, B, the probability that the assertion of the claim fails is at most

$$\binom{n}{K}^2 (1 - p)^{K^2} \le \left(\left(\frac{en}{K} \right)^2 e^{-pK} \right)^K = \left(\frac{e}{2\log(np)} \right)^{2K} < 0.99^{2K} = o(1). \hspace{1em}\square$$

We are now ready to prove Theorem 1.10. For ease of reference, we restate it here:

Theorem 1.10 (Binomial random graphs)**.** *For every $\varepsilon > 0$ there exists $c > 0$ such that if $p \ge (1 + \varepsilon)\log n/n$, then $G \sim G(n, p)$ **whp** admits a strong d-rigid partition for $d = cnp/\log(np)$.*

Proof of Theorem 1.10. Let $\xi = \xi(\varepsilon)$ be the constant guaranteed by Lemma 4.10, and let $\Gamma = e/(1 - \xi)$. Write $\delta = \delta(G)$ and $\Delta = \Delta(G)$. Recall that by Lemma 4.10 we have $(1 - \xi)np \le \delta \le \Delta \le enp$ **whp**. In particular, $\Delta \le \Gamma\delta$ **whp**. Let $C = C(\Gamma)$ be the constant from Proposition 3.8. Let $c' = \min\{1/35, 1/C\}$, and let $d = c'\delta/\log(np)$.

Let $x = 3\log(np)/p$ and $y = 5\log(np)$. By Lemma 4.11, G is **whp** (x, y)-thin. By Lemma 4.12, G is **whp** a $(2x/3)$-connector. Note that $5\log(np) \le \delta/(7d)$, so G is **whp** $(x, \delta/(7d))$-thin. Furthermore, using the fact that $d \le c'\delta \le c'enp < np$, we obtain $\delta = d\log(np)/c' \ge Cd\log(np) > Cd\log d$. Hence, by Proposition 3.8, G **whp** admits a d-rigid partition. The statement follows for $c = c'(1 - \xi)$. $\hfill\square$

4.5 Random graph processes

In this section we prove Theorem 1.9. Let $2 \le d = d(n) \le c\log n/\log\log n$ for a small constant $c > 0$ to be determined later, and let $G \sim G(n, \tau_d)$. For $\eta > 0$ denote $S_\eta = \{v \in V : \deg_G(v) < \eta\log n\}$. It will be useful to couple G with $G^+ \sim G(n, p)$ for $p = 2\log n/n$, $G^- \sim G(n, p)$ for $p = \log n/n$, and $G^\circ \sim G(n, m)$ for $m = n\log n$, so that, **whp**, G^- is a subgraph of G and G is a subgraph of G^+, G°. Such a coupling is possible since $\tau_d < n\log n$ **whp** (see Lemma 4.10) and by using standard translations between $G(n, p)$ and $G(n, m)$ (see, e.g., [28]).

Claim 4.13. *There exists $\eta > 0$ such that in G, **whp**, the distance between any two vertices in S_η is at least 5, and no vertex in S_η lies in a cycle of length at most 4.*

Proof. From monotonicity, it suffices to prove that **whp** G° does not contain a path of length at most 4 between two potentially identical vertices in S_η. This fact is proved in [42, Lemma 4.5] for a small enough constant $\eta > 0$. $\hfill\square$

We restate Theorem 1.9 prior to presenting its proof:

Theorem 1.9 (Hitting time for d-rigidity)**.** *There exists $c > 0$ for which the following holds. For every $1 \le d = d(n) \le c\log n/\log\log n$, a random graph $G \sim G(n, \tau_d)$ **whp** admits a strong d-rigid partition.*

Proof of Theorem 1.9. For $d = 1$ the claim follows from the known fact that $G(n, \tau_1)$ is **whp** connected (see, e.g., [22]). Therefore, we will assume $d \ge 2$. We also assume that $d \le c\log n/\log\log n$ for $c > 0$ to be chosen later. Let $\eta > 0$ be the constant from Claim 4.13. Let $V^* = V \setminus S_\eta$ and consider the graph $G^* = G[V^*]$. Write $\delta = \delta(G^*)$ and $\Delta = \Delta(G^*)$. Note that by Lemma 4.10, $\Delta \le \Gamma\delta$ for $\Gamma = e/\eta$ **whp**. Let $\alpha = 1/8$ and let $C = C(\alpha, \Gamma)$ be the constant guaranteed by the partition lemma (Lemma 3.3). Let

$c = \min\{\eta/12, \eta/C\}$. Note that $\delta \geq \eta \log n \geq Cc \log n \geq Cd \log \log n > Cd \log d$. Thus, there exists a partition $V^* = V_1 \cup \cdots \cup V_d$ with $e(v, V_i) \geq 7\delta/(8d)$ for every $v \in V^*$ and $i \in [d]$.

We now adjust the partition as follows: for every $v \in S_\eta$, let $v_1, \ldots, v_d$ be d arbitrary neighbours of v. Note that Claim 4.13 implies that the sets $\{v_1, \ldots, v_d\}$ are **whp** disjoint for distinct $v \in S_\eta$, and are subsets of V^*. For every $v \in S_\eta$ and $i \in [d]$, we move v_i to V_i. Note that after the change, $e(v, V_i) \geq 7\delta/(8d) - 1 \geq 6\delta/(7d)$ (for large enough n) for every $v \in V^*$ and $i \in [d]$.

Let $p = \log n/n$ and $x = 3\log(np)/p$. By Lemma 4.12, G^- (and hence G, and hence G^*) is **whp** a $(2x/3)$-connector. Let $p' = 2p$, and note that $x \sim 6\log(np')/p' < 7\log(np')/p'$. Let $y = 11\log(np') \sim 11\log\log n$. By Lemma 4.11, G^+ (and hence G, and hence G^*) is **whp** (x, y)-thin. Noting that $6\delta/(7d) \geq \frac{6\eta}{c}\log\log n \geq 72\log\log n > 6y$, by Lemma 3.7, $G[V_i, V_j]$ is connected for all $1 \leq i, j \leq d$, and thus the partition $V^* = V_1 \cup \cdots \cup V_d$ is a strong d-rigid partition of G^*. By adding S_η to V_1 (say), noting that every $v \in S_\eta$ has a neighbour in each V_i, we obtain a strong d-rigid partition of G. □

4.6 Binomial random bipartite graphs

We state the following lemmas without proofs; they are the bipartite analogues of Lemmas 4.10 to 4.12, and their proofs are essentially identical.

Lemma 4.14. *For every $\varepsilon > 0$ there exists $\xi > 0$ such that the following holds. Let $p \geq (1+\varepsilon)\log n/n$, and let $G \sim G(n, n, p)$. Then, **whp**, $(1-\xi)np \leq \delta(G) \leq \Delta(G) \leq enp$.*

Lemma 4.15. *There exists a sufficiently large constant k such that if $p \geq k/n$, $G \sim G(n, n, p)$ is **whp** (x, y)-thin for $x = 3\log(np)/p$ and $y = 5\log(np)$.*

Lemma 4.16. *Suppose $p \gg 1/n$ and let $G \sim G(n, n, p)$. Then, **whp** G is a K-bi-connector for $K = 2\log(np)/p$.*

Let us recall Theorem 1.12 before we turn to its proof:

Theorem 1.12 (Binomial random bipartite graphs)**.** *For every $\varepsilon > 0$ there exists $c > 0$ such that if $p \geq (1+\varepsilon)\log n/n$, then $G \sim G(n, n, p)$ **whp** admits a strong bipartite d-rigid partition for $d = \min\{cnp/\log(np), 0.8\sqrt{n}\}$.*

Proof of Theorem 1.12. Let $\xi = \xi(\varepsilon)$ be the constant guaranteed by Lemma 4.14, and let $\Gamma = e/(1-\xi)$. Write $\delta = \delta(G)$ and $\Delta = \Delta(G)$. Recall that by Lemma 4.14 we have $(1-\xi)np \leq \delta \leq \Delta \leq enp$ **whp**. In particular, $\Delta \leq \Gamma\delta$ **whp**. Let $C = C(\Gamma)$ be the constant from Proposition 3.10. Let $c' = \min\{1/70, 1/(2C)\}$, and let $d = \min\{c'\delta/\log(np), 0.8\sqrt{n}\}$.

Let $d' = d + 1$, $x = 3\log(np)/p$ and $y = 5\log(np)$. Note that $x = 3\log(np)/p \leq 3en\log(np)/\delta \leq 3c'en/d < n/(7d')$. By Lemma 4.16, G is **whp** a $(2x/3)$-bi-connector. By Lemma 4.15, G is **whp** (x, y)-thin. Note that $5\log(np) \leq \delta/(7d')$, so G is **whp** $(x, \delta/(7d'))$-thin. Furthermore, using the fact that $d \leq c'\delta \leq c'enp \leq np - 1$, we obtain $\delta \geq d\log(np)/c' \geq 2Cd\log(np) > Cd'\log(d')$. Finally, $d' \leq 0.8\sqrt{n} + 1 < \sqrt{3n/4}$. Hence, by Proposition 3.10, G **whp** admits a strong bipartite d-rigid partition. The statement follows for $c = c'(1-\xi)$. □

4.7 Giant rigid component

In this section we prove Theorem 1.13, showing the typical existence of a giant d-rigid component in sparse binomial random graphs. The initial step in this direction is the establishment of a lemma, which posits the presence of a giant "balanced" component.

Lemma 4.17. *For any $\varepsilon \in (0,1)$ and $c \in (0, 1-\varepsilon)$ there exists $k_0 = k_0(\varepsilon, c) > 0$ such that for every $k \geq k_0$, $G(n, k/n)$ contains **whp** an induced subgraph on at least $(1-\varepsilon)n$ vertices, with minimum degree at least ck and maximum degree at most $4k/\varepsilon$.*

For the proof of Lemma 4.17 we use the following generalisation of [2, Lemma 5.2].

Lemma 4.18. *For any $\varepsilon \in (0, 1/2)$ and $c \in (0, 1-2\varepsilon)$ there exists $k_0 = k_0(\varepsilon, c) > 0$ such that for every $k \geq k_0$, $G \sim G(n, k/n)$ satisfies the following property **whp**: every vertex set U with $|U| \geq (1-\varepsilon)n$ contains a subset U' with $|U'| \geq (1-2\varepsilon)n$ such that $\delta(G[U']) \geq ck$.*

Proof. We follow [2]. Write $b = 1 - 2\varepsilon$. By Chernoff bounds (Theorem 3.1) and the union bound, the probability that there exist disjoint vertex sets A, B with $|A| = \varepsilon n$, $|B| \geq bn$ and $e(A, B) < \varepsilon ckn$ is at most

$$2^n \cdot 2^n \cdot \mathbb{P}(\mathrm{Bin}(\varepsilon bn^2, k/n) \leq \varepsilon ckn) = 4^n e^{-\frac{1}{2}(b-c)^2 \varepsilon kn} = o(1),$$

for large enough k. We thus assume from now on that for every disjoint A, B with $|A| = \varepsilon n$ and $|B| \geq bn$ we have that $e(A, B) \geq \varepsilon ckn$. Fix U with $|U| \geq (1-\varepsilon)n$. We build a sequence $U_0, U_1, \ldots, U_m$ of subsets of U in the following way: set $U_0 = U$. For every $i \geq 0$, if U_i has a vertex with degree smaller than ck in $G[U_i]$, we denote that vertex by u_{i+1}, and set $U_{i+1} = U_i \smallsetminus \{u_{i+1}\}$; otherwise, set $m = i$. Let $U' = U_m$. Suppose that $m > \varepsilon n$, and let $A = \{u_1, \ldots, u_{\varepsilon n}\}$. Let $B = U \smallsetminus A$, and note that $|B| \geq bn$. Note also that every $u \in A$ has less than ck neighbours in B. Therefore,

$$\varepsilon ckn \leq e(A, B) < |A| ck = \varepsilon ckn,$$

a contradiction. Thus, $m \leq \varepsilon n$, and $|U'| \geq |U| - \varepsilon n \geq (1-2\varepsilon)n$. The statement follows since $\delta(G[U']) \geq ck$. $\qquad\square$

Proof of Lemma 4.17. Let B be the set of vertices of degree larger than $4k/\varepsilon$, and let $m = |E(G)|$. Then, deterministically, $|B| \leq 2m/(4k/\varepsilon) = \varepsilon m/(2k)$. Note that m is distributed as a binomial random variable with $\binom{n}{2}$ attempts and success probability k/n; thus, by Chernoff bounds (Theorem 3.1), $\mathbb{P}(m \geq kn) \leq \exp(-0.6kn) = o(1)$. Thus, **whp**, $|B| \leq \varepsilon n/2$. Let $U = V(G) \smallsetminus B$, so $|U| \geq (1-\varepsilon/2)n$. By Lemma 4.18, for large enough k, there exists **whp** $U' \subseteq U$ with $|U'| \geq (1-\varepsilon)n$ satisfying $\delta(G[U']) \geq ck$. Since $U' \subseteq U = V(G) \smallsetminus B$, we also have $\Delta(G[U']) \leq 4k/\varepsilon$. $\qquad\square$

Using the fact that $G(n, p)$ is typically $(p, \Theta(\sqrt{np}))$-jumbled (see [44]), we can readily infer from Proposition 4.4 that for any $\varepsilon \in (0, 1)$, $G(n, \Omega_\varepsilon(d^2/n))$ includes a d-rigid component comprising at least $(1-\varepsilon)n$ vertices. To further refine the dependence of p on d, we apply the same methodology used in previous sections. This involves showing that the (nearly spanning) induced subgraph found in Lemma 4.17 is a "thin connector".

We recall Theorem 1.13 before turning to its proof:

Theorem 1.13 (Binomial random bipartite graphs). *For every $\varepsilon \in (0, 1)$ there exists $C > 0$ such that for every fixed $d \geq 2$, $G \sim G(n, Cd \log d/n)$ contains **whp** a d-rigid component with at least $(1-\varepsilon)n$ vertices.*

Proof of Theorem 1.13. Let $G \sim G(k/n)$, for k to be determined later. Let $\varepsilon \in (0, 1)$ and set $c = (1-\varepsilon)/2$. By Lemma 4.17, there exists k_0 such that $k \geq k_0$ implies the existence of a set U of size at least $(1-\varepsilon)n$ with $ck \leq \delta(G[U]) \leq \Delta(G[U]) \leq 4k/\varepsilon$. Write $G' = G[U]$ and $\delta = \delta(G')$. Let $\Gamma = 4/(c\varepsilon)$, and let $C = C(\Gamma)$ be the constant from Proposition 3.8. Let $x = 3 \log(np)/p$ and $y = 5 \log(np)$. By Lemma 4.11, there exists k_1 such that $k \geq k_1$ implies that G (and hence G') is **whp** (x, y)-thin. By Lemma 4.12, for $k \geq 4$, G (and hence G') is **whp** a $(2x/3)$-connector. Take C to be sufficiently large

so that $k := Cc^{-1}d\log d \geq \max\{k_0, k_1, 4\}$ and that $C \geq 35\log k / \log d$, and set $p = k/n$. Note that $\delta \geq Cd\log d$ and that $5\log(np) = 5\log k \leq C\log d/7 \leq \delta/(7d)$, so G' is **whp** $(x, \delta/(7d))$-thin. Thus, by Proposition 3.8, G' **whp** admits a strong d-rigid partition, and is therefore (by Corollary 1.4) d-rigid. $\square$

5 Dense graphs

As we mentioned in the introduction, Theorem 1.14 follows immediately from the following more general statement.

Theorem 5.1. *Let $\ell = \ell(n)$ satisfy $3\log n \leq \ell < n$. Let $G = (V, E)$ be an n-vertex graph in which every pair of vertices has at least ℓ common neighbours. Then, G admits a strong d-rigid partition for $d = \ell/(3\log n)$.*

For the proof of Theorem 5.1, we will need the following simple lemma.

Lemma 5.2. *Let $G = (V, E)$ be an n-vertex graph in which every pair of vertices has at least ℓ common neighbours. Let U be a uniformly random subset of V of size t. Then, with probability at least $1 - n^2 e^{-\ell t/n}$, every pair of vertices in V has a common neighbour in U.*

Proof. Fix a pair of distinct vertices $u, v \in V$. Let p be the probability that U is disjoint from the set of common neighbours of u, v. Then,

$$p \leq \frac{\binom{n-\ell}{t}}{\binom{n}{t}} = \frac{\binom{n-t}{\ell}}{\binom{n}{\ell}} = \prod_{i=1}^{\ell}\left(1 - \frac{t}{n - \ell + i}\right) \leq \left(1 - \frac{t}{n}\right)^{\ell} \leq e^{-\ell t/n}.$$

The result follows by the union bound. $\square$

Proof of Theorem 5.1. Let $\ell = \ell(n)$ and let $d = \ell/(3\log n)$. Partition our vertex set V at random into sets $V_1, \ldots, V_d$, each of size $n/d = 3n\log n/\ell$. By Lemma 5.2, for every $1 \leq i \leq d$, the probability that there is a pair of vertices in V without a common neighbour in V_i is at most $n^2 e^{-3\log n} = 1/n$. By the union bound, the probability that there exist $i \in [d]$ and a pair of vertices in V without a common neighbour in V_i is at most $d/n < 1$. Therefore, there exists a partition $V_1, \ldots, V_d$ such that every pair of vertices in V has a common neighbour in V_i for every $1 \leq i \leq d$.

We are left to show that $V_1, \ldots, V_d$ is a strong d-rigid partition of G. Indeed, for all $1 \leq i \leq d$, $G[V_i]$ is connected, since every pair of distinct vertices $u, v \in V_i$ has a common neighbour in V_i. Similarly, let $i \neq j$, and let $u, v \in V_i \cup V_j$. If $u, v \in V_i$, then they have a common neighbour in V_j, so there is a path between them in $G[V_i, V_j]$. If $u \in V_i$ and $v \in V_j$, we divide into two cases: if u and v are adjacent, then we are done. Otherwise, let w be a neighbour of u in V_j, and let z be a common neighbour of v and w in V_i. Then (u, w, z, v) is a path from u to v in $G[V_i, V_j]$. So, $G[V_i, V_j]$ is connected for all $i \neq j$. Therefore, $V_1, \ldots, V_d$ is a strong d-rigid partition of G. $\square$

Let us recall Theorem 1.14 before we prove it:

Theorem 1.14. *Let $\ell = \ell(n)$ satisfy $3\log n/2 \leq \ell < n/2$. Let $G = (V, E)$ be an n-vertex graph with $\delta(G) \geq \frac{n}{2} + \ell$. Then, G admits a strong d-rigid partition for $d = 2\ell/(3\log n)$.*

Proof of Theorem 1.14. Let $u, v \in V$. Then, the number of common neighbours of u and v is at least $\deg(u) + \deg(v) - n \geq 2\ell$. Therefore, by Theorem 5.1, G admits a strong d-rigid partition for $d = 2\ell/(3\log n)$. $\square$

For the proof of Theorem 1.15, we use the following well-known result concerning the existence of k-connected subgraphs. Denote by $\bar{d}(G)$ the average degree of G.

Theorem 5.3 (Mader [53] (see also [21])). *For every $k \geq 1$, if $\bar{d}(G) \geq 4k$, then G contains a $(k+1)$-connected subgraph H with $\bar{d}(H) > \bar{d}(G) - 2k$.*

Let us recall Theorem 1.15 before we prove it:

Theorem 1.15. *There exists $c > 0$ such that for every $0 \leq d < c\sqrt{n}/\log n$, if G is an n-vertex graph with $\delta(G) \geq \frac{n}{2} + d$ then G is $(d+1)$-rigid.*

Proof of Theorem 1.15. Let C be the constant from Theorem 1.3, take $c = \min\{\sqrt{1/8C}, 1/9\}$, and let $k = C(d+1)^2\log^2 n$. Thus, $k \leq n/8$. Since $\bar{d}(G) \geq \delta(G) \geq n/2 + d \geq 4k$, by Theorem 5.3, G contains a subgraph H which is k-connected and has $\bar{d}(H) \geq n/2 + d - 2k \geq n/4$. Thus, by Theorem 1.3, H is $(d+1)$-rigid and $|V(H)| \geq n/4$. Let V_1 be a maximum-sized vertex set for which $G[V_1]$ is $(d+1)$-rigid. If $V_1 = V$, we are done, so assume otherwise. Write $n_1 = |V_1| \geq n/4$. Let $V_2 = V \smallsetminus V_1$ and write $G_2 = G[V_2]$ and $n_2 = |V_2| \leq 3n/4$. Observe that every $v \in V_2$ has at most d neighbours in V_1 (as otherwise $V_1 \cup \{v\}$ would have been a larger $(d+1)$-rigid component). Thus, $\delta(G_2) \geq \delta(G) - d \geq n/2$. In particular, $n_2 > n/2$, and therefore $n_2 > n_1$. Set $\ell = \frac{3}{2}(d+1)\log n_2$, and note that $\ell \leq \sqrt{n_2}/6 \leq n_2/6$. Thus, $\delta(G_2) \geq n/2 \geq 2n_2/3 = \left(\frac{1}{2} + \frac{1}{6}\right)n_2 \geq \frac{1}{2}n_2 + \ell$. By Theorem 1.14, G_2 is $(d+1)$-rigid, contradicting the maximality of V_1; thus $V_1 = V$. $\qquad\square$

Let us remark that applying Villányi's recent result [66, Theorem 1.1] instead of Theorem 1.3 allows for the condition on d in Theorem 1.15 to be extended to $d = O(\sqrt{n})$.

Acknowledgements

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