On saturation games

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Abstract

A graph $G = (V, E)$ is said to be *saturated* with respect to a monotone increasing graph property $\mathcal{P}$, if $G \notin \mathcal{P}$ but $G \cup \{e\} \in \mathcal{P}$ for every $e \in \binom{V}{2} \setminus E$. The *saturation game* $(n, \mathcal{P})$ is played as follows. Two players, called Mini and Max, progressively build a graph $G \subseteq K_n$, which does not satisfy $\mathcal{P}$. Starting with the empty graph on n vertices, the two players take turns adding edges $e \in \binom{V(K_n)}{2} \setminus E(G)$, for which $G \cup \{e\} \notin \mathcal{P}$, until no such edge exists (i.e. until G becomes $\mathcal{P}$-saturated), at which point the game is over. Max's goal is to maximize the length of the game, whereas Mini aims to minimize it. The *score* of the game, denoted by $s(n, \mathcal{P})$, is the number of edges in G at the end of the game, assuming both players follow their optimal strategies.

We prove lower and upper bounds on the score of games in which the property the players need to avoid is being k-connected, having chromatic number at least k, and admitting a matching of a given size. In doing so we demonstrate that the score of certain games can be as large as the Turán number or as low as the saturation number of the respective graph property, and also that the score might strongly depend on the identity of the first player to move.

1   Introduction

Let n be a positive integer, let $\mathcal{P}$ be a monotone increasing property of graphs on n vertices and let $G = ([n], E)$ be a graph which does not satisfy $\mathcal{P}$. An edge $e \in \binom{[n]}{2} \setminus E$ is called *legal* *with respect to G and $\mathcal{P}$* if $G \cup \{e\} \notin \mathcal{P}$. A graph $G = ([n], E)$ is said to be *saturated with respect to $\mathcal{P}$* if $G \notin \mathcal{P}$ and there are no legal edges with respect to G and $\mathcal{P}$. Given a graph $H \notin \mathcal{P}$ with vertex set $[n]$, the *saturation game $(H, \mathcal{P})$* is played as follows. Two players, called Mini and Max, progressively build a graph G, where $H \subseteq G \subseteq K_n$, so that G does not satisfy $\mathcal{P}$. Starting with $G = H$, the two players take turns adding edges which are legal with respect

to the current graph G and the property $\mathcal{P}$ until no such edge exists, at which point the game is over. Max's goal is to maximize the length of the game, whereas Mini aims to minimize it. The *score* of the game, denoted by $s(H, \mathcal{P})$, is the number of edges in G at the end of the game (recall that with some abuse of notation we use G to denote the graph built by both players at any point during the game) when both players follow their optimal strategies. In fact, we would only be interested in the case $H = \overline{K}_n$, where $\overline{K}_n$ is the empty graph on n vertices, but we generalize the definition of the game for the purpose of simplifying the presentation of some of our proofs. We abbreviate $s(\overline{K}_n, \mathcal{P})$ to $s(n, \mathcal{P})$. Note that we did not specify which of the two players starts the game. Since the score of a saturation game might depend on this information, whenever studying a specific game we will consider its score in two cases – when Mini is the first player and when Max is the first player. Where we do not explicitly specify the identity of the first player, our related results hold in both cases.

Straightforward bounds on the score of a saturation game stem from the corresponding *saturation number* and *Turán number*. Given a monotone increasing graph property $\mathcal{P}$, the saturation number of $\mathcal{P}$, denoted by $sat(n, \mathcal{P})$, is the minimum possible size of a saturated graph on n vertices with respect to $\mathcal{P}$. Saturation numbers have attracted a lot of attention since their introduction by Erdős, Hajnal and Moon [7]; many related results and open problems can be found in the survey [8]. Similarly, the Turán number of $\mathcal{P}$, denoted by $ex(n, \mathcal{P})$, is the maximum possible size of a saturated graph on n vertices with respect to $\mathcal{P}$. The theory of Turán numbers is a cornerstone of Extremal Combinatorics; many related results and open problems can be found e.g. in [3]. It is immediate from the definition of the saturation game $(n, \mathcal{P})$ that $sat(n, \mathcal{P}) \leq s(n, \mathcal{P}) \leq ex(n, \mathcal{P})$.

Results on scores of saturation games are quite scarce. For example, let $\mathcal{K}_3$ denote the property of containing a triangle. A well-known theorem of Mantel (see e.g. [16]) asserts that $ex(n, \mathcal{K}_3) = \lfloor n^2/4 \rfloor$. Moreover, since a star is saturated with respect to $\mathcal{K}_3$ and, on the other hand, no disconnected graph is, it follows that $sat(n, \mathcal{K}_3) = n - 1$ (this also follows from a more general result of Erdős, Hajnal and Moon [7]). In contrast to these exact results, very little is known about $s(n, \mathcal{K}_3)$. The best known lower bound, due to Füredi, Reimer and Seress [9], is of order $n \log n$. In the same paper, Füredi et al. attribute an upper bound of $n^2/5$ to Erdős; however, the proof is lost. Biró, Horn and Wildstrom [2] have recently improved the upper bound of $\lfloor n^2/4 \rfloor$ which follows from Mantel's Theorem to $\frac{26}{121} n^2 + o(n^2)$. Additional saturation-type games were recently studied in [5] and [14].

We begin our study of saturation games with games in which both players are required to keep the connectivity of the graph below a certain threshold. For every positive integer k we would like to determine $s(n, \mathcal{C}_k)$, where $\mathcal{C}_k$ is the property of being k-vertex-connected and spanning. It is easy to see that $ex(n, \mathcal{C}_k) = \binom{n-1}{2} + k - 1$ holds for every positive integer $k \leq n$. Very recently, it was shown in [4] that $s(n, \mathcal{C}) = \binom{n-2}{2} + 1$ for every $n \geq 6$. Our first result shows that $s(n, \mathcal{C}_k)$ is almost as large as $ex(n, \mathcal{C}_k)$ for every fixed positive integer k.

Theorem 1.1 $s(n, \mathcal{C}_k) \geq \binom{n}{2} - 5kn^{3/2}$ *for every positive integer k and sufficiently large n.*

Using a different proof technique, for every $k \geq 5$ we can improve the error term in the bound given in Theorem 1.1.

Theorem 1.2 $s(n, \mathcal{C}_k) \geq \binom{n}{2} - (k-1)(2k-4)[n - (k-1)(2k-3)]$ *for every $k \geq 5$ and sufficiently large n.*

Remark 1.3 *The lower bounds on $s(n, \mathcal{C}_k)$ given in Theorems 1.1 and 1.2 are not as tight as the lower bound on $s(n, \mathcal{C})$ given in [4], which is matching the upper bound. Since $\mathcal{C}_k \subseteq \mathcal{C}$ for every $k \geq 1$, it may seem at first sight like $s(n, \mathcal{C}_k) \geq s(n, \mathcal{C}) = \binom{n-2}{2} + 1$ should hold as well. But as we will see later (see Remark 1.9 below), such an implication is not true in general.*

We now move on to study saturation games in which both players are required to keep the chromatic number of the graph below a certain threshold. For every integer $k \geq 2$ we would like to determine $s(n, \chi_{>k})$, where $\chi_{>k}$ is the property of having chromatic number at least $k+1$ (obviously $s(n, \chi_{>1}) = 0$). It is easy to see that if H is a graph on $n \geq k$ vertices which is saturated with respect to $\chi_{>k}$, then H is complete k-partite. From this it easily follows that $sat(n, \chi_{>k}) = (k-1)(n-1) - \binom{k-1}{2}$ and $ex(n, \chi_{>k}) = \sum_{0 \leq i < j \leq k-1} \lfloor \frac{n+i}{k} \rfloor \cdot \lfloor \frac{n+j}{k} \rfloor = (1 - 1/k + o(1)) \binom{n}{2}$. Very recently, it was shown in [4] that $s(n, \chi_{>2})$ is equal to the trivial upper bound, that is, $s(n, \chi_{>2}) = ex(n, \chi_{>2}) = \lfloor n^2/4 \rfloor$.

Our first result regarding colorability games shows that, in contrast to the $(n, \chi_{>2})$ game, Mini does have a strategy to ensure that $s(n, \chi_{>3})$ is smaller than $ex(n, \chi_{>3})$ by a non-negligible fraction.

Theorem 1.4 $s(n, \chi_{>3}) \leq 21n^2/64 + O(n)$.

Additionally, we prove that for every sufficiently large k, Max has a strategy to ensure that $s(n, \chi_{>k})$ is not much smaller than $ex(n, \chi_{>k})$.

Theorem 1.5 *There exists a real number C such that $s(n, \chi_{>k}) \geq (1 - C \log k/k) \binom{n}{2}$ holds for every positive integer k and sufficiently large n.*

Lastly, we study saturation games in which both players are required to keep the size of every matching in the graph below a certain threshold. Starting with the property $\mathcal{PM}$ of admitting a perfect matching, it is easy to see that $ex(n, \mathcal{PM}) = \binom{n-1}{2}$ for every even n. Moreover, using Tutte's well-known necessary and sufficient condition for the existence of a perfect matching [15], Mader [12] characterized all graphs which are saturated with respect to $\mathcal{PM}$. Using this characterization, it is not hard to show that $sat(n, \mathcal{PM}) = \Theta(n^{3/2})$. We prove that $s(n, \mathcal{PM})$ is almost as large as $ex(n, \mathcal{PM})$.

Theorem 1.6 *Let $n \geq 8$ be an even integer, then $s(n, \mathcal{PM}) \geq \binom{n-4}{2}$.*

We then move on to study $s(n, \mathcal{M}_k)$, where $\mathcal{M}_k$ is the property of admitting a matching of size k for some $k \leq n/2$. It was proved by Erdős and Gallai in [6] that

$$ex(n, \mathcal{M}_k) = \max \left\{ (k-1)(n-1) - \binom{k-1}{2}, \binom{2k-1}{2} \right\}.$$

Applying the Berge-Tutte formula [1], Mader [12] also characterized all graphs which are saturated with respect to $\mathcal{M}_k$, for every $1 \leq k \leq n/2$. Using this characterization, it is not hard to derive that $sat(n, \mathcal{M}_k) = 3(k-1)$ if $k \leq n/3$, $sat(n, \mathcal{M}_k) = \Theta(n^2/(n-2k))$ if $n/3 \leq k \leq n/2 - \sqrt{n}$ and $sat(n, \mathcal{M}_k) = \Theta(n^{3/2})$ if $n/2 - \sqrt{n} \leq k \leq n/2$. Our next result

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shows that, at least when k is not too large with respect to n, the score $s(n, \mathcal{M}_k)$ varies in order of magnitude, depending on the parity of k and the identity of the first player. This is in stark contrast to all of our results mentioned until now (where changing the identity of the first player might affect the score of the game, but only by a negligible margin). Note that, among other results and using different terminology, $s(n, \mathcal{M}_2)$ was determined in [14].

Theorem 1.7 *Let $k \geq 2$ be an integer. If Max is the first player and k is even, or Mini is the first player and k is odd, then $s(n, \mathcal{M}_k) \geq n - 1$. In all other cases $s(n, \mathcal{M}_k) \leq \binom{2k-1}{2}$.*

Remark 1.8 *It follows from Theorem 1.7 that that if k is fixed then, depending on the parity of k and the identity of the first player, either $s(n, \mathcal{M}_k) = \Theta(sat(n, \mathcal{M}_k))$ or $s(n, \mathcal{M}_k) = \Theta(ex(n, \mathcal{M}_k))$.*

Remark 1.9 *It follows from Theorem 1.7 that scores of saturation games are not monotone in the following sense. There are monotone increasing graph properties $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{P}_1 \subseteq \mathcal{P}_2$ and yet $s(n, \mathcal{P}_1) < s(n, \mathcal{P}_2)$. Also, there are monotone increasing graph properties $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $sat(n, \mathcal{P}_1) < sat(n, \mathcal{P}_2)$ and $ex(n, \mathcal{P}_1) < ex(n, \mathcal{P}_2)$, but $s(n, \mathcal{P}_1) > s(n, \mathcal{P}_2)$.*

1.1 Notation and preliminaries

For the sake of simplicity and clarity of presentation, we do not make a particular effort to optimize some of the constants obtained in our proofs. We also omit floor and ceiling signs whenever these are not crucial. Throughout the paper, log stands for the natural logarithm. We say that a graph property $\mathcal{P}$ holds *asymptotically almost surely*, or a.a.s. for brevity, if the probability of satisfying $\mathcal{P}$ tends to 1 as the number of vertices n tends to infinity. Our graph-theoretic notation is standard and follows that of [16]. In particular, we use the following.

For a graph G, let $V(G)$ and $E(G)$ denote its sets of vertices and edges respectively, and let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For a set $U \subseteq V(G)$ and a vertex $w \in V(G)$, let $N_G(w, U) = \{u \in U : wu \in E(G)\}$ denote the set of neighbors of w in U and let $d_G(w, U) = |N_G(w, U)|$. For disjoint sets $U, W \subseteq V(G)$ let $N_G(W, U) = \bigcup_{w \in W} N_G(w, U)$. We abbreviate $N_G(w, V(G))$ to $N_G(w)$, and $N_G(W, V(G) \setminus W)$ to $N_G(W)$. The minimum degree of a graph G is denoted by $\delta(G)$. Often, when there is no risk of confusion, we omit the subscript G from the notation above. For a set $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by the vertices of S. A connected component C of a graph G is said to be *non-trivial* if it contains an edge. The size of a maximum matching in a graph G is denoted by $\nu(G)$.

Assume that some saturation game $(H, \mathcal{P})$ is in progress, where H is a graph on n vertices. The edges of $K_n \setminus G$ are called *free* (recall that at any point during the game, we use G to denote the graph built by both players up to that point). A *round* of the game consists of a move by the first player and a counter move by the second player. We say that a player follows the *trivial strategy* if in every move he claims an arbitrary legal edge.

We end this subsection by proving the following lemma which asserts that, without any saturation restrictions, either player can build a long path that includes all vertices of positive degree. This lemma will be useful for the connectivity and the matching games we will study. We refer to the strategy described in the proof of the lemma as *the long path strategy*.

Lemma 1.10 *Let $n \geq 3$ and $1 \leq \ell \leq n-2$ be integers. Then starting with the empty graph on n vertices, either player can ensure that, immediately after his ith move for some i, the graph G will contain a path P such that the following three properties are satisfied:*

(a) *The length of P is either ℓ or $\ell + 1$;*

(b) *If $u \in V(G) \setminus V(P)$, then $d_G(u) = 0$;*

(c) *At least one of the endpoints of P has degree one in G.*

Proof We prove our claim by induction on ℓ. For convenience we denote the player who wishes to build the path P by $\mathcal{A}$ and the other player by $\mathcal{B}$. For $\ell = 1$ the correctness of our claim is obvious as, in his first move, $\mathcal{A}$ can build a path of length 1 if he is the first player and of length 2 otherwise. Assume our claim holds for some $1 \leq \ell < n-2$; we will prove it holds for $\ell + 1$ as well. First, $\mathcal{A}$ builds a path P_ℓ which satisfies properties (a), (b) and (c) for ℓ; the induction hypothesis ensures that $\mathcal{A}$ has a strategy to do so. If P_ℓ is of length $\ell + 1$ then there is nothing to prove, so assume P_ℓ is of length ℓ. Let $P_\ell = (u_0, \ldots, u_\ell)$ and assume without loss of generality that $d_G(u_0) = 1$. Let xy denote the edge $\mathcal{B}$ claims in his subsequent move. We distinguish between the following four cases:

(1) If $\{x, y\} \subseteq V(P_\ell)$, then $\mathcal{A}$ claims $u_\ell z$ for some isolated vertex z.

(2) If $\{x, y\} \cap V(P_\ell) = \emptyset$, then $\mathcal{A}$ claims $u_\ell x$.

(3) If $x \in \{u_0, u_\ell\}$ and $y \notin V(P_\ell)$, then $\mathcal{A}$ claims yz for some isolated vertex z.

(4) If $x \in V(P_\ell) \setminus \{u_0, u_\ell\}$ and $y \notin V(P_\ell)$, then $\mathcal{A}$ claims $u_\ell y$.

It is easy to see that in all of the four cases above, $\mathcal{A}$ can follow the proposed strategy and, by doing so, he builds a path which satisfies conditions (a), (b) and (c) for $\ell + 1$. $\qquad \square$

The rest of this paper is organized as follows. In Section 2 we prove Theorems 1.1 and 1.2. In Section 3 we prove Theorems 1.4 and 1.5. In Section 4 we prove Theorems 1.6 and 1.7. Finally, in Section 5 we present some open problems.

2 Connectivity games

In this section we study connectivity games, that is, saturation games in which both players are required to keep the connectivity of the graph below a certain threshold.

Proof of Theorem 1.1 Since $s(n, \mathcal{C}_1)$ was determined in [4], we can assume that $k \geq 2$. We present a strategy for Max; it is divided into the following three stages:

Stage I: Max follows the long path strategy until G contains a path $P = (u_0, \ldots, u_\ell)$ of length $\ell \in \{n - k - \lceil \sqrt{n} \rceil - 1, n - k - \lceil \sqrt{n} \rceil\}$ which includes all vertices of positive degree in G. He then proceeds to Stage II.

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Stage II: Let t and r be the unique integers satisfying $\ell + 1 = \lceil 4\sqrt{n} \rceil t + r$ and $0 \leq r < \lceil 4\sqrt{n} \rceil$. Let $v_1, \ldots, v_k$ be k arbitrary vertices of $V(G) \setminus V(P)$ and let $F = \{v_i u_{\lceil 4\sqrt{n} \rceil j} : 1 \leq i \leq k,\ 1 \leq j \leq t\}$. In each of his moves in this stage, Max claims an arbitrary free edge of F. Once no such edge exists, this stage is over and Max proceeds to Stage III.

Stage III: Throughout this stage, Max follows the trivial strategy.

Our first goal is to prove that Max can indeed follow the proposed strategy. This is obvious for Stage III and follows for Stage I by Lemma 1.10 (note that there are isolated vertices at the end of Stage I so G is certainly not k-connected at that point). Since every vertex of $V(G) \setminus V(P)$ is isolated at the end of Stage I and at most $2|F| + 1 \leq 2k\sqrt{n}/4 < k(k + \sqrt{n} - 1)/2$ edges are claimed by both players throughout Stage II, it follows that, at the end of Stage II, there exists a vertex $u \in V(G) \setminus V(P)$ such that $d_G(u) < k$; in particular, G is not k-connected at that point. Hence, Max can follow Stage II of the proposed strategy.

At the end of the game, G is saturated with respect to k-connectivity, that is, G is not k-connected but $G + uv$ is k-connected for every $u, v \in V(G)$ such that $uv \notin E(G)$. Let $S \subseteq V(G)$ be a cut set of G of size $k - 1$. Since $|S| < k$, it follows that $\{v_1, \ldots, v_k\} \setminus S \neq \emptyset$; assume without loss of generality that $v_1 \notin S$. Let A denote the connected component of $G \setminus S$ which contains v_1 and let $B = V(G) \setminus (A \cup S)$. We claim that $|B| \leq 5k\sqrt{n}$. This is obvious if $B \subseteq (V(G) \setminus V(P)) \cup \{u_0, \ldots, u_{\lceil 4\sqrt{n} \rceil - 1}, u_{\lceil 4\sqrt{n} \rceil t + 1}, \ldots, u_\ell\}$. Assume then that there exists a vertex $u_i \in B \cap \{u_{\lceil 4\sqrt{n} \rceil}, \ldots, u_{\lceil 4\sqrt{n} \rceil t}\}$; let $1 \leq j < t$ denote the unique index such that $\lceil 4\sqrt{n} \rceil j < i < \lceil 4\sqrt{n} \rceil (j+1)$ (note that, for every $1 \leq j \leq t$, we have $u_{\lceil 4\sqrt{n} \rceil j} \notin B$ as $v_1 u_{\lceil 4\sqrt{n} \rceil j} \in E(G)$ holds by Stage II of the proposed strategy). Since $u_i \in B$, it follows that there is no path between u_i and v_1 in $G \setminus S$. It follows that $|S \cap \{u_{\lceil 4\sqrt{n} \rceil j}, \ldots, u_{\lceil 4\sqrt{n} \rceil (j+1)}\}| \geq 2$. Since this is true for every vertex of $B \cap \{u_{\lceil 4\sqrt{n} \rceil}, \ldots, u_{\lceil 4\sqrt{n} \rceil t}\}$, it follows that $B \cap \{u_{\lceil 4\sqrt{n} \rceil}, \ldots, u_{\lceil 4\sqrt{n} \rceil t}\}$ is the union of at most $|S| - 1$ subpaths of P, each of length at most $4\sqrt{n}$. We conclude that $|B| \leq 5k\sqrt{n}$ as claimed. Since, as noted above, G is saturated with respect to k-connectivity, it follows that $xy \notin E(G)$ if and only if $x \in A$ and $y \in B$ (or vice versa). Hence $e(G) = \binom{n}{2} - |A||B| \geq \binom{n}{2} - 5k\sqrt{n}(n - k + 1 - 5k\sqrt{n}) \geq \binom{n}{2} - 5kn^{3/2}$ as claimed. □

Proof of Theorem 1.2 We present a strategy for Max; it is divided into the following three stages:

Stage I: Let $r = \frac{n}{2k-3}$, let $t = n - r$, let $V_0 = \{v_1, \ldots, v_r\}$ be a subset of $V(G)$ and let $V(G) \setminus V_0 = \{u_1, \ldots, u_t\}$. Max's goal in this stage is to ensure that for every set $B \subseteq V \setminus V_0$, by the end of the stage $|N_G(B, V_0)| \geq |B|r/t$ will hold. He does so in the following way. In each of his moves in this stage, Max claims $u_i v_{\lceil ir/t \rceil}$ where i is the smallest positive integer for which $u_i v_{\lceil ir/t \rceil}$ is free. As soon as all edges of $\{u_i v_{\lceil ir/t \rceil} : 1 \leq i \leq t\}$ are claimed, Max proceeds to Stage II.

Stage II: Let H be a k-connected graph on r vertices such that $e(H)$ is minimal among all such graphs. Max ensures that V_0 will contain a copy of H and then proceeds to Stage III.

Stage III: Throughout this stage, Max follows the trivial strategy.

Our first goal is to prove that Max can indeed follow the proposed strategy. This is obvious for Stage III. For Stages I and II this follows since at most $2(t + \lceil kr/2 \rceil) < kn/2$ edges are claimed by both players during these two stages, where the inequality follows by the definition of r since $k \geq 5$. Since there is no k-connected graph on n vertices and strictly less than $kn/2$ edges, it follows that Max can claim any free edge throughout Stages I and II.

At the end of the game, G is saturated with respect to k-connectivity, that is, G is not k-connected but $G + uv$ is k-connected for every $u, v \in V(G)$ such that $uv \notin E(G)$. Let $S \subseteq V(G)$ be a cut set of G of size $k - 1$. It follows by Stage II of Max's strategy that $V_0 \setminus S$ is contained in one connected component of $G \setminus S$; let A denote this component and let $B = V(G) \setminus (A \cup S)$. We claim that $|B| \leq t(k-1)/r$. Indeed, suppose for a contradiction that $|B| > t(k-1)/r$. It follows by Stage I of Max's strategy that $|N_G(B, V_0)| \geq \lceil |B| r/t \rceil \geq k$. Since $|S| < k$ it follows that $N_G(B, V_0 \setminus S) \neq \emptyset$ contrary to S being a cut set. Since, as noted above, G is saturated with respect to k-connectivity, it follows that $xy \notin E(G)$ if and only if $x \in A$ and $y \in B$ (or vice versa). Hence $e(G) = \binom{n}{2} - |A||B| \geq \binom{n}{2} - \frac{t(k-1)}{r}\left(n - k + 1 - \frac{t(k-1)}{r}\right) = \binom{n}{2} - (k-1)(2k-4)[n - (k-1)(2k-3)]$ as claimed. $\square$

3 Colorability games

In this section we study colorability games, that is, saturation games in which both players are required to keep the chromatic number of the graph below a certain threshold.

Proof of Theorem 1.4 Since the proof is quite technical, even though it is based on a very simple idea, we begin by briefly describing this idea. Regardless of Mini's strategy, at the end of the game G will be a complete 3-partite graph. Since she would like to minimize the number of its edges, she should try to unbalance its parts. She will do so by making sure one part is small, namely, its size is at most $\lceil n/4 \rceil$. This will be achieved by connecting (by her and Max's edges) an arbitrary vertex v_0 to roughly $3n/4$ vertices. In order to eventually prove Mini can achieve this, we will show that for every vertex x Mini cannot connect to v_0, Max must have "used up" at least 3 of his moves.

We first introduce some notation and terminology that will be used throughout this proof. Let $v_0 \in V(G)$ be an arbitrary vertex. This vertex determines the following partition $V(G) = T_G \cup M_G \cup B_G$: T_G consists of all vertices which receive the same color as v_0 in *any* proper 3-coloring of G, $M_G = N_G(T_G)$ and $B_G = V(G) \setminus (T_G \cup M_G)$. In particular, $T_G = \{v_0\}$, $M_G = \emptyset$ and $B_G = V(G) \setminus \{v_0\}$ hold before the game starts. The vertices of T_G are called *top* vertices, the vertices of M_G are called *middle* vertices and the vertices of B_G are called *bottom* vertices. For any vertex $u \in V(G)$, let $\Gamma_G(u)$ denote the connected component of $G[\{u\} \cup M_G]$ which contains u. A connected component of $G[M_G]$ is called a *middle-component*. Note that at any point during the game, every middle-component is 2-colorable. If $u \in B_G$, $v \in M_G$, $uv \in E(G)$ and C is the middle-component containing v, then u is said to be *attached* to C. When there is no risk of confusion, we omit the subscript G from the above notation.

Note that if x is a top (respectively middle) vertex, then x remains a top (respectively middle) vertex throughout the game. On the other hand, if $x \in B$ and some player claims xy for some top vertex y, then x is *moved* to the middle, that is, x becomes a middle vertex. Moreover, if

$x \in B$ and some player claims xy for some vertex $y \in M \cup B$, then either x remains in B or it is moved to the top.

During the game, Mini will want G to satisfy certain structural properties. In order to describe these we introduce some more definitions, starting with the following two properties of a given graph G on n vertices with the partition $V(G) = T \cup M \cup B$:

(P1) $E(G[B]) = \emptyset$;

(P2) Every middle-component of G has at most one attached vertex.

Next, we define the set of bad edges with respect to G as follows:

$$BAD_G := \left\{ uv \in \binom{V(G)}{2} : \exists x, y \in B, \ x \neq y, \ u \in V(\Gamma(x)), \ \text{and} \ v \in V(\Gamma(y)) \right\}.$$

Note that in the definition of a bad edge, it is possible that $x = u$ or $y = v$.

Finally, we say that a vertex $x \in B$ is *good* if every edge in G with exactly one endpoint in $V(\Gamma(x))$ (if such an edge exists) has its other endpoint in T.

Observation 3.1 *Let G be a graph with the partition $V(G) = T \cup M \cup B$ as described above. The following conditions are equivalent:*

1. *G satisfies properties (P1) and (P2).*

2. *$BAD_G \cap E(G) = \emptyset$.*

3. *Every vertex in B is good.*

We say that the graph G is *good* if it satisfies conditions 1–3 of Observation 3.1. Given this definition, we make another observation.

Observation 3.2 *Let G be a good graph and let $G' = G \cup \{e\}$ for some $e \notin E(G)$. G' is a good graph if and only if $e \notin BAD_G$.*

The last definition we need is that of an almost good graph. A graph G is said to be *almost good* if G is good, or there exists an edge $e \in E(G)$ such that $G \setminus \{e\}$ is a good graph (the edge e is not necessarily unique).

We now state and prove several claims that will be very useful in the remainder of the proof. In all of these claims we assume that at all times the graphs in question are 3-colorable.

Claim 3.3 *Consider a graph G with the corresponding partition $V(G) = T \cup M \cup B$, and let $x \neq v_0$ be some vertex. The following hold:*

(a) If $\Gamma(x)$ is not 2-colorable, then $x \in T$.

(b) If G satisfies property (P1) and $\Gamma(x)$ is 2-colorable, then $x \notin T$.

Proof For (a), since $\Gamma(x)$ is not 2-colorable, in every proper 3-coloring c of G there exists a vertex $v \in V(\Gamma(x))$ such that $c(v) = c(v_0)$. Since $V(\Gamma(x)) \setminus \{x\} \subseteq M$ and no vertex in M can receive the same color as v_0, it follows that $v = x$ and thus $x \in T$.

For (b), we show that there exists a proper 3-coloring of G which does not assign the same color to x and v_0. Indeed, since by definition every edge with at least one endpoint in T must have its other endpoint in M and since B is an independent set by assumption, it follows that $T \cup B$ is an independent set. Since $\Gamma(x)$ is 2-colorable, and so is every middle-component, it follows that $G[\{x\} \cup M]$ is 2-colorable. Let c be some proper coloring of $G[\{x\} \cup M]$ with colors 1 and 2. Extend c to a coloring of G by coloring each vertex in $T \cup B \setminus \{x\}$ with the color 3. This is a proper 3-coloring of G which assigns x and v_0 distinct colors. We conclude that $x \notin T$. $\qquad\square$

Claim 3.4 *If G is a good graph, $e \in BAD_G$ and $G' = G \cup \{e\}$, then $B_{G'} = B_G$.*

Proof As G is a good graph, it follows by the contrapositive of Claim 3.3(a) and by the fact that every middle-component is always 2-colorable, that $G[M_G \cup B_G]$ is also 2-colorable. Since, by the definition of a bad edge, e connects two different components of $G[M_G \cup B_G]$, it follows that $G'[M_G \cup B_G]$ is 2-colorable as well. Thus, there exists a proper coloring of G' which assigns all vertices in T_G the color 1, and all the vertices in $M_G \cup B_G$ the colors 2 and 3. It follows that $T_{G'} \subseteq T_G$, and therefore $T_{G'} = T_G$ since vertices are never moved from the top.

Now, since $T_{G'} = T_G$, and since $e \cap T_G = \emptyset$, it follows by definition of the middle that $M_{G'} = M_G$, and therefore we conclude that $B_{G'} = B_G$. $\qquad\square$

Claim 3.5 *Let G be a good graph and let $G' = G \cup \{e\}$ for some $e \notin E(G)$. If there exists a vertex x such that $x \in B_G$ and $x \in T_{G'}$, then there exists a middle-component $C \subseteq M_{G'}$ such that $d_{G'}(x, C) \geq 2$.*

Proof Since the addition of e to G moves x from the bottom to the top, Claim 3.4 implies that $e \notin BAD_G$. Therefore, by Observation 3.2, G' is a good graph. By Claim 3.3(b), $\Gamma_{G'}(x)$ is not 2-colorable, and thus contains an odd cycle. Since $\Gamma_{G'}(x) \setminus \{x\} \subseteq M_{G'}$ is 2-colorable, this cycle must include x. The two neighbors of x in the cycle belong to the same middle-component $C \subseteq M_{G'}$, as claimed. $\qquad\square$

Claim 3.6 *Let G be an almost good graph with the partition $V(G) = T_G \cup M_G \cup B_G$, and let $G_1 := G \cup \{ab\}$ for some $ab \notin E(G)$ such that $a \in T_G$. Assume that there exists a vertex $x \neq a$ such that $\Gamma_{G_1}(x) \neq \Gamma_G(x)$. Then $b \in B_G$, $b \neq x$, and there exists an edge $ub \in E(G)$ such that $u \in V(\Gamma_G(x))$.*

Proof Since $G \subseteq G_1$ and no vertex can move out of the middle, it follows that $M_G \subseteq M_{G_1}$. Therefore $\Gamma_G(x) \subseteq \Gamma_{G_1}(x)$ and so if $\Gamma_{G_1}(x) \neq \Gamma_G(x)$, then there must exist an edge $uw \in E(G_1)$ such that $u \in V(\Gamma_G(x))$ and $w \in M_{G_1} \setminus V(\Gamma_G(x))$. Recall that $a \neq x$ by assumption. Moreover, $a \neq u$ as $a \notin M_G$. Since no vertex can move out of the top, it follows that $a \in T_{G_1}$ and thus $a \neq w$. Therefore, $uw \in E(G)$. However, $w \notin V(\Gamma_G(x))$ and this can only happen if $w \notin M_G$. Since $w \in M_{G_1}$ we conclude that $w \in B_G$. We will now show that $b = w$, which will complete the proof, as $b \notin V(\Gamma_G(x))$ implies $b \neq x$.

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Suppose for a contradiction that $b \neq w$. Since $w \in M_{G_1}$ there exists a vertex $z \in T_{G_1}$ such that $wz \in E(G_1)$. By our assumption that $b \neq w$, and since $a \neq w$ as previously shown, it follows that $wz \in E(G)$. Since $w \notin M_G$ it follows that $z \notin T_G$ and therefore $z \in B_G$. Clearly the graph G is not good as the edge wz violates property (P1). However, since G is almost good, there exists an edge $e \in E(G)$ such that $G_0 := G \setminus \{e\}$ is a good graph. Note that $w, z \in B_{G_0}$, and therefore $e = wz$ as otherwise property (P1) would have been violated in G_0 as well.

Now, similarly to the proof of Claim 3.4, there exists a proper coloring c of G which assigns all vertices in T_G the color 1 and all the vertices in $M_G \cup B_G$ the colors 2 and 3. This coloring is also a proper coloring of G_1, since $c(a) = 1$ and $c(b) \neq 1$. Since $z \in B_G$, it follows that $c(z) \neq c(v_0)$ contrary to z being an element of T_{G_1}. We conclude that indeed $b = w$ and the proof is complete. $\qquad \square$

The following result is an immediate consequence of Claim 3.6.

Corollary 3.7 *Under the assumptions of Claim 3.6, and using the same terminology, $\Gamma_{G_1}(x)$ is the subgraph of G induced on the vertex set $V(\Gamma_G(x)) \cup V(\Gamma_G(b))$.*

Claim 3.8 *Consider a good graph G with the corresponding partition $V(G) = T_G \cup M_G \cup B_G$. Let $G_1 = G \cup \{e\}$ for some $e \notin E(G)$, let $x, y \in B_{G_1}$ and let $G_2 = G_1 \cup \{v_0 y\}$. Assume that G_2 satisfies property (P1). Then, $x \notin T_{G_2}$.*

Proof Assume first that $\Gamma_{G_2}(x) = \Gamma_{G_1}(x)$. By the assumption that $x \in B_{G_1}$ and by Claim 3.3(a) we deduce that $\Gamma_{G_1}(x)$ is 2-colorable and thus so is $\Gamma_{G_2}(x)$. Since, moreover, G_2 satisfies property (P1) by assumption, it follows by Claim 3.3(b) that $x \notin T_{G_2}$.

Assume then that $\Gamma_{G_2}(x) \neq \Gamma_{G_1}(x)$. As G_1 is an almost good graph, by Claim 3.6 there is an edge of G_1 between $V(\Gamma_{G_1}(x))$ and y. Since G is a good graph, it contains no edges between $V(\Gamma_G(x))$ and y. Therefore, the edge e must have one endpoint in $V(\Gamma_G(x))$ and one endpoint in $V(\Gamma_G(y))$. Let C denote the connected graph $\Gamma_G(x) \cup \Gamma_G(y) \cup \{e\}$. Since $x, y \in B_G$, it follows by Claim 3.3(a) that $\Gamma_G(x)$ and $\Gamma_G(y)$ are 2-colorable, and therefore so is C. Since $\Gamma_{G_1}(x)$ and $\Gamma_{G_1}(y)$ are clearly subgraphs of C, it follows by Corollary 3.7 that $\Gamma_{G_2}(x) = C$. Since G_2 satisfies property (P1), and since $\Gamma_{G_2}(x)$ is 2-colorable, it follows by Claim 3.3(b) that $x \notin T_{G_2}$. $\qquad \square$

Now, we present a strategy for Mini; it is divided into two simple stages. In the first stage Mini claims only edges incident with v_0, aiming to make its degree as large as possible, and in the second stage she plays arbitrarily. For convenience we assume that Max is the first player; if Mini is the first player, then in her first move she claims $v_0 z$ for an arbitrary vertex $z \in B$ and the remainder of the proof is essentially the same.

Stage I: This stage lasts as long as there are vertices in B. Once $B = \emptyset$, this stage is over and Mini proceeds to Stage II. Before each of Mini's moves during Stage I, let G denote the graph at that point and let e denote the last edge claimed by Max. Mini plays as follows:

    (i) If there exists a vertex $x \in B_G$ such that $\{e\} \cap \Gamma_G(x) \neq \emptyset$, then Mini claims $v_0 x$ (if there are several such bottom vertices, then Mini picks one arbitrarily).

    (ii) Otherwise, Mini claims $v_0 z$, where $z \in B_G$ is an arbitrary vertex.

Mini then repeats Stage I.

Stage II: Throughout this stage, Mini follows the trivial strategy.

It remains to prove that Mini can indeed follow the proposed strategy and that, by doing so, she ensures that $e(G) \leq 21n^2/64 + O(n)$ will hold at the end of the game. Starting with the former, note that Mini can clearly follow Stage II of the strategy. As for Stage I, in each of her moves in this stage Mini claims an edge between v_0 and some vertex $u \in B_G$. By definition v_0u is free and $\chi(G \cup \{v_0u\}) \leq 3$. Hence Mini can follow the proposed strategy. In order to prove the latter, we first prove the following four additional claims.

Claim 3.9 *Immediately after each of Mini's moves in Stage I, the current graph G built by both players is good.*

Proof We will prove this claim by induction on the number of moves played by Mini. The claim clearly holds before the game starts. Assume that it holds immediately after Mini's ith move for some non-negative integer i (where $i = 0$ we refer to the initial graph before the game begins); we will prove it holds after her $(i+1)$st move as well (assuming it is played in Stage I). Let G denote the graph immediately after Mini's ith move, let uv denote the edge claimed by Max in his subsequent move, let $G_1 = G \cup \{uv\}$, and let G_2 denote the graph immediately after Mini's $(i+1)$st move.

If $uv \notin BAD_G$, then by Observation 3.2 G_1 is good. Mini then claims an edge e with one endpoint in T_{G_1} (the vertex v_0), and so $e \notin BAD_{G_1}$ by definition. Therefore, applying Observation 3.2 once again we infer that G_2 is good.

Assume then that uv is a bad edge. Therefore, by definition, there exist distinct vertices $x, y \in B_G$ such that $u \in V(\Gamma_G(x))$ and $v \in V(\Gamma_G(y))$. Note that according to her strategy, in her next move Mini claims either v_0x or v_0y (by the induction hypothesis, no other bottom vertex is a candidate); without loss of generality assume that she claims v_0y. In order to prove that G_2 is good, we will show that every vertex of B_{G_2} is good. Consider first a vertex $z \in B_{G_2} \setminus \{x\}$ (note that $z \neq y$ as $y \in M_{G_2}$). Clearly $z \in B_G$ and since G is a good graph, z is a good vertex in G. Since $\{uv, v_0y\} \cap \Gamma_G(z) = \emptyset$, it is easy to see that $\Gamma_{G_2}(z) = \Gamma_{G_1}(z) = \Gamma_G(z)$ and that z is a good vertex in G_2 as well. Now consider x. Since G_1 is an almost good graph, $\Gamma_{G_2}(x) = \Gamma_G(x) \cup \Gamma_G(y) \cup \{uv\}$ by Corollary 3.7. Since x and y are both good vertices in G, it is evident that x is a good vertex in G_2 as well. This concludes the proof of the claim. □

Claim 3.10 *Throughout Stage I, no vertex is moved from B to T as a result of a move by Mini.*

Proof Recall that by assumption Max is the first player to move. Let i be some positive integer, let G denote the graph immediately before Max's ith move, let G_1 denote the graph immediately after Max's ith move, and let G_2 denote the graph immediately after Mini's ith move. Since, by Claim 3.9, G and G_2 are good graphs, and since Mini in her ith move claims v_0y for some $y \in B_{G_1}$, it follows by Claim 3.8 that for every vertex $x \in B_{G_1}$ (including y), $x \notin T_{G_2}$. □

Claim 3.11 *Let x be a bottom vertex which is attached to a middle-component C. If at some point during Stage I x is moved to the top, then from this point until the end of Stage I, immediately after every move of Mini, no bottom vertex will be attached to the unique middle-component containing C.*

Proof We prove this claim by induction on the number of rounds played after x was moved to the top. Consider first the moment at which x is moved to the top. By Claim 3.10 this happens as a result of Max's i_0th move, for some positive integer i_0. Denote the players' graph immediately before this move by G_0 and the graph immediately after this move by G_0'. Since by Claim 3.9, G_0 is a good graph, it is not hard to see (similarly to the proof of Claim 3.5) that in his i_0th move Max claimed an edge $e \subseteq \Gamma_{G_0}(x)$, and thus $V(\Gamma_{G_0'}(x)) = V(\Gamma_{G_0}(x))$. Therefore, there are no edges of G_0' between $V(\Gamma_{G_0'}(x))$ and $B_{G_0'}$, as x itself is in $T_{G_0'}$ by assumption, and it was the only vertex attached to the middle-components of $\Gamma_{G_0}(x)$ (that is, the middle-components of G_0 contained in $\Gamma_{G_0}(x)$) since G_0 is a good graph. In her subsequent move, Mini certainly does not attach any vertex to any of the middle-components of $\Gamma_{G_0}(x)$, nor does she change $\Gamma(x)$, so the claim holds at this point.

Now let $i \geq i_0$ and assume the claim holds immediately after Mini's ith move. Let G_1 be the graph after Mini's ith move, let G_2 be the graph after Max's subsequent move, and let G_3 be the graph after Mini's $(i+1)$st move. Let C be a middle-component of $\Gamma_{G_0}(x)$ and for $j = 1, 2, 3$ let C_j denote the middle-component containing C in G_j. If there is no bottom vertex attached to C_2 in G_2, then by Mini's strategy $C_3 = C_2$ and there will be no such vertex in G_3 either. Assume then that there is such a vertex y. It follows that in his $(i+1)$st move Max claimed an edge uv such that $u \in C_1$ and $v \in V(\Gamma_{G_1}(y))$. Therefore $C_1 \subseteq \Gamma_{G_2}(y)$. Since, by the induction hypothesis, there is no vertex attached to C_1 in G_1 and since y is the only vertex attached to any middle-component of $\Gamma_{G_1}(y)$ in G_1 (by property (P2), as G_1 is a good graph), it follows that there is no bottom vertex $z \neq y$ in G_2 such that $\{uv\} \cap \Gamma_{G_2}(z) \neq \emptyset$. Therefore, by the proposed strategy Mini claims $v_0 y$ in her $(i+1)$st move and thus $C_3 = C_1 \cup \Gamma_{G_1}(y) \cup \{uv\}$. It follows that no bottom vertex is attached to C_3 in G_3. $\square$

Claim 3.12 $|T| \leq \frac{n+3}{4}$ *holds at the end of Stage I.*

Proof Consider the moment at which some vertex x is moved from the bottom to the top (if this never happens, then $|T| = 1$). At this moment we *assign* to x every edge of G which is incident with x and every edge of every middle-component to which x is attached. We claim that any edge of G is assigned to at most one vertex. Indeed, this is evident for the edges incident to the vertex that was moved to the top, and is also true for the edges inside the middle-components it was attached to by Claims 3.10 and 3.11. In addition, it follows by Claims 3.10 and 3.5 that every top vertex, other than v_0, is assigned at least 3 edges. Since throughout Stage I Mini claims only edges which are incident with v_0, all assigned edges were claimed by Max. It thus follows that for every vertex of $T \setminus \{v_0\}$, Mini increased the degree of v_0 by at least 3, that is, $|M| \geq d(v_0) \geq 3(|T| - 1)$. Since $B = \emptyset$ holds by definition at the end of Stage I, it follows that $|T| + |M| = n$ holds at that point. We conclude that $|T| \leq \frac{n+3}{4}$ as claimed. $\square$

We can now complete the proof of Theorem 1.4. Let X, Y and Z denote the color classes in the unique proper 3-coloring of G at the end of the game. It follows by the definition of T, M

and B that without loss of generality $X \cup Y = M$ and $Z = T$. We thus conclude that

$$e(G) = |T|(|X| + |Y|) + |X||Y| \leq \frac{n+3}{4} \cdot \frac{3n-3}{4} + \left(\frac{3n-3}{8}\right)^2 = \frac{21}{64}n^2 + O(n)$$

as claimed. □

Proof of Theorem 1.5 For convenience we assume that Mini is the first player; if Max is the first player, then he makes an arbitrary first move and the remainder of the proof is essentially the same. Let k, C and n be as in the statement of the theorem; by choosing C to be sufficiently large, we can assume that k is large as well, as otherwise the statement of the theorem trivially holds. Since the game in question is a finite, perfect information game, with no chance moves, then exactly one of the following must hold:

(a) Max has a strategy to ensure that $e(G) \geq (1 - C \log k/k) \binom{n}{2}$ will hold at the end of the game against any strategy of Mini.

(b) Mini has a strategy to ensure that $e(G) < (1 - C \log k/k) \binom{n}{2}$ will hold at the end of the game against any strategy of Max.

We present a random strategy for Max, and show that with positive probability (in fact, a.a.s.), $e(G) \geq (1 - C \log k/k) \binom{n}{2}$ will hold at the end of the game. Therefore, (b) cannot hold, which implies (a), thus proving Theorem 1.5.

We will need the following Chernoff type bound for our calculations.

Theorem 3.13 *[10] Let $X \sim Bin(n, p)$ and let $x \geq 7np$. Then $Pr(X \geq x) \leq e^{-x}$.*

The proposed strategy for Max is divided into the following two stages:

Stage I: This stage is over as soon as $\delta(G) \geq k - 1$; at that point Max proceeds to Stage II. For every positive integer i, let $a_i b_i$ denote the edge claimed by Mini in her ith move of this stage and, immediately after Mini's ith move, let $S_i = \{x \in \{a_i, b_i\} : d_G(x) \leq k-2\}$. Max plays his ith move as follows:

   (i) If $S_i = \emptyset$, then Max claims a free edge xy such that $\min\{d_G(x), d_G(y)\} \leq k - 2$ uniformly at random among all such edges; we refer to such moves as being *fully-random*.

   (ii) If $S_i \neq \emptyset$, then, with probability $139/140$, Max makes a fully-random move and, with probability $1/140$, he claims an edge xy such that $x \in S_i$ and subsequently $y \in \{z \in V(K_n) : xz \notin E(G)\}$ are chosen uniformly at random; we refer to such moves as being *semi-random*.

Stage II: Throughout this stage, Max follows the trivial strategy.

Note that if H is a graph with chromatic number $\chi(H) \leq k$ and $u, v \in V(H)$ are two vertices such that $d_H(u) \leq k - 2$, then $\chi(H + uv) \leq k$. It thus follows that Max can play according to the proposed strategy.

Fix $r := Cn\log k/k$ and let $U \subseteq V(K_n)$ be an arbitrary set of size r. Assume that for some positive integer i, in her ith move Mini claims an edge $a_i b_i$ such that $S_i \cap U = \emptyset$. Max's ith move xy is said to be *bad with respect to* U if it is semi-random and $\{x, y\} \cap U \neq \emptyset$. The set U is said to be *bad* if throughout Stage I Max makes at least $kr/20$ bad moves with respect to U. We claim that a.a.s. there will be no bad sets. Indeed, fix some U of size r and let B_U be the random variable which counts the number of bad moves with respect to U. Throughout the game Mini can claim at most $(k-2)n$ edges $a_i b_i$ for which $S_i \neq \emptyset$. Hence, it follows by the proposed strategy that $B_U \sim Bin(N, p)$, where $N \leq (k-2)n$ and $p \leq \frac{1}{140} \cdot \frac{r}{n-k+1}$. In particular, $\mathbb{E}(B_U) = Np \leq kr/140$. Therefore, by Theorem 3.13 we have

$$Pr(U \text{ is bad}) = Pr(B_U \geq kr/20) \leq e^{-kr/20}.$$

Thus

$$Pr(\text{there exists a bad set } U) \leq \binom{n}{r} e^{-kr/20} \leq \left[\frac{ek}{C\log k} \cdot e^{-k/20} \right]^r = o(1).$$

Hence, from now on we will assume that there are no bad sets.

Our next goal is to prove that a.a.s. $\alpha(G) \leq Cn\log k/k$ holds at the end of the game. Let $U \subseteq V(G)$ be an arbitrary set of size r. At any point during the game let $X_U = \{x \in U : d_G(x) \geq k-1\}$ and let $Y_U = U \setminus X_U$. Consider the point in time at which $|X_U| \geq |Y_U|$ first occurs; clearly $|U|/2 \leq |X_U| \leq \lceil |U|/2 \rceil + 1$ at this point. Note that such a moment must occur during Stage I, since there are no vertices of degree at most $k-2$ at the end of Stage I; denote this moment by t.

We say that a claimed edge uv is a Y-edge if $\{u, v\} \cap Y_U \neq \emptyset$ holds immediately after uv is claimed (by either player). Let A_U denote the event: "up until the moment t, Max has played at least $kr/20$ fully-random moves in which he claimed Y-edges" and let A_U^c denote its complement. Let I_U denote the event: "at the end of the game, U is an independent set". Clearly $Pr(I_U) = Pr(I_U \wedge A_U) + Pr(I_U \wedge A_U^c)$.

We wish to bound $Pr(I_U)$ from above. Consider first a fully-random move e such that e is a Y-edge, and assume that U is independent immediately before this move. We have

$$Pr(e \subseteq U) \geq \frac{|Y_U||U| - \binom{|Y_U|}{2}}{|Y_U|n} \tag{1}$$

$$\geq \frac{|Y_U|\frac{|U|}{2}}{|Y_U|n} = \frac{r}{2n}. \tag{2}$$

It follows by (1) and by the definition of A_U that

$$Pr(I_U \wedge A_U) = Pr(I_U \mid A_U) \cdot Pr(A_U) \leq Pr(I_U \mid A_U) \leq \left(1 - \frac{r}{2n}\right)^{kr/20}. \tag{3}$$

Next, assume that for some positive integer i, in her ith move Mini has claimed a Y-edge $a_i b_i$; assume without loss of generality that $a_i \in Y_U$. We again assume that U is independent immediately after Mini's ith move; in particular, $b_i \in V(K_n) \setminus U$. According to the proposed strategy, with probability $1/140$ in his subsequent move Max claims an edge xy such that $x \in \{a_i, b_i\}$ and $y \in V(K_n)$. Moreover, $Pr(x = a_i) \geq 1/2$ and, based on the assumption that U is independent, $Pr(y \in U \mid x = a_i) \geq \frac{|U \setminus \{a_i\}|}{n} \geq \frac{r-1}{n}$. Therefore

$$Pr(\{x, y\} \subseteq U) \geq 1/140 \cdot Pr(x = a_i) \cdot Pr(y \in U \mid x = a_i) \geq \frac{r}{290n}. \tag{4}$$

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Since $X_U = \emptyset$ before the game starts, if U is an independent set at the end of the game, it follows that by time t the number of Y-edges claimed by both players is at least $(k-2)|X_U| \geq (k-2)r/2$. Since U is not a bad set, if by time t Max makes at most $kr/20$ fully-random moves in which he claims Y-edges, then the number of Y-edges Mini must claim up to that point is at least $[(k-2)r/2 - kr/20 - kr/20]/2 \geq kr/10$. It thus follows by the proposed strategy, by (4) and by the definition of A_U, that

$$Pr(I_U \wedge A_U^c) = Pr(I_U \mid A_U^c) \cdot Pr(A_U^c) \leq Pr(I_U \mid A_U^c) \leq \left(1 - \frac{r}{290n}\right)^{kr/10} . \qquad (5)$$

Putting inequalities (3) and (5) together we conclude that

$$\begin{aligned}
Pr(I_U) &= Pr(I_U \wedge A_U) + Pr(I_U \wedge A_U^c) \leq \left(1 - \frac{r}{2n}\right)^{kr/20} + \left(1 - \frac{r}{290n}\right)^{kr/10} \\
&\leq \exp\left\{-\frac{r}{2n} \cdot \frac{kr}{20}\right\} + \exp\left\{-\frac{r}{290n} \cdot \frac{kr}{10}\right\} \leq \exp\left\{-\frac{C^2 n \log^2 k}{3000k}\right\} . \qquad (6)
\end{aligned}$$

Using (6), we can now show that a.a.s. $\alpha(G) \leq Cn \log k/k$ holds at the end of the game by the following union bound estimate:

$$\begin{aligned}
Pr(\alpha(G) \geq Cn \log k/k) &\leq \binom{n}{Cn \log k/k} \cdot \exp\left\{-\frac{C^2 n \log^2 k}{3000k}\right\} \\
&\leq \left(\frac{ek}{C \log k}\right)^{Cn \log k/k} \cdot \exp\left\{-\frac{C^2 n \log^2 k}{3000k}\right\} \\
&\leq \exp\left\{\frac{Cn \log^2 k}{k} - \frac{C^2 n \log^2 k}{3000k}\right\} \\
&= o(1),
\end{aligned}$$

where the last equality holds for $C > 3000$. Since $\alpha(G) \leq Cn \log k/k$ holds a.a.s., Max has a deterministic strategy to achieve this, as was shown above.

Once the game is over, G is saturated and thus complete k-partite; let $A_1, \ldots, A_k$ denote its parts. Max can ensure that $|A_i| \leq Cn \log k/k$ holds for every $1 \leq i \leq k$, and therefore

$$e(G) = \frac{1}{2} \sum_{i=1}^{k} |A_i|(n - |A_i|) \geq \frac{1}{2} \sum_{i=1}^{k} |A_i|(n - Cn \log k/k) = \frac{n^2}{2}(1 - C \log k/k) .$$

$\square$

4 Matching games

In this section we study matching games, that is, saturation games in which both players are required to keep the size of every matching in the graph below a certain threshold.

Proof of Theorem 1.6 In order to prove the theorem, we present a strategy for Max. In order to simplify the description of the strategy, we first consider several possible *end-games*. These are described in the following lemmas.

Lemma 4.1 *Let $n \geq 6$ be an even integer and let $G_0 = (V, E)$ be a graph on n vertices. Assume that there exist vertices $x, y \in V$ such that $d_{G_0}(x) = d_{G_0}(y) = 0$ and $G_0 \setminus \{x, y\}$ admits a Hamilton cycle C. If Max is the second player, then $s(G_0, \mathcal{PM}) \geq \binom{n-2}{2}$.*

Proof Max plays according to the following simple strategy which consists of two stages.

Stage I: Let uv denote the last edge claimed by Mini; we distinguish between the following two cases:

    (1) If $\{u, v\} \cap \{x, y\} = \emptyset$, then Max claims an arbitrary free edge ww' such that $\{w, w'\} \cap \{x, y\} = \emptyset$ and repeats Stage I; if this is not possible, then he skips to Stage II.

    (2) Otherwise, if $u \in \{x, y\}$ and $v \in V \setminus \{x, y\}$, then Max claims a free edge uv', where v' is a neighbor of v in C. He then proceeds to Stage II.

Stage II: Throughout this stage, Max follows the trivial strategy.

Note that at any point during the game, the graph $G \cup \{xy\}$ admits a perfect matching; it follows that $xy \notin E(G)$. In particular, the proposed strategy does account for every legal move of Mini. Moreover, if Max never plays according to Case (2) of Stage I, then clearly $ww' \in E(G)$ holds for every $w, w' \in V \setminus \{x, y\}$ at the end of the game. If on the other hand Max does play according to Case (2) of Stage I, then, at the end of the game, $ww' \in E(G)$ holds for every $w, w' \in V \setminus \{z\}$ for some $z \in \{x, y\}$. In either case we conclude that $s(G_0, \mathcal{PM}) \geq \binom{n-2}{2}$ as claimed. $\square$

Lemma 4.2 *Let $n \geq 6$ be an even integer and let $G_0 = (V, E)$ be a graph on n vertices. Assume that there exist vertices $x, y, z \in V$ such that $xy \in E$, $d_{G_0}(x) = d_{G_0}(y) = 1$, $d_{G_0}(z) = 0$ and $G_0 \setminus \{x, y, z\}$ admits a Hamilton cycle C. If Max is the second player, then $s(G_0, \mathcal{PM}) \geq \binom{n-3}{2}$.*

Proof Max plays according to the following simple strategy which consists of two stages.

Stage I: Let uv denote the last edge claimed by Mini; we distinguish between the following three cases:

    (1) If $\{u, v\} \cap \{x, y, z\} = \emptyset$, then Max claims an arbitrary free edge ww' such that $\{w, w'\} \cap \{x, y, z\} = \emptyset$ and repeats Stage I; if this is not possible, then he skips to Stage II.

    (2) Otherwise, if $uv = xz$ (respectively $uv = yz$), then Max claims yz (respectively xz) and proceeds to Stage II.

    (3) Otherwise, if $u \in \{x, y\}$ and $v \in V \setminus \{x, y, z\}$, then Max claims a free edge $u'v'$, where u' is the unique vertex in $\{x, y\} \setminus \{u\}$ and v' is a neighbor of v in C. He then proceeds to Stage II.

Stage II: Throughout this stage, Max follows the trivial strategy.

Note that at any point during the game, for every $w \in V \setminus \{x, y, z\}$, the graph $G \cup \{wz\}$ admits a perfect matching; it follows that $wz \notin E(G)$. In particular, the proposed strategy

does account for every legal move of Mini. Moreover, note that if $\{xz, yz\} \subseteq E(G)$, then $ww' \notin E(G)$ for every $w \in \{x, y, z\}$ and $w' \in V \setminus \{x, y, z\}$. Therefore, if Max never plays according to Case (3) of Stage I, then $ww' \in E(G)$ holds for every $w, w' \in V \setminus \{x, y, z\}$ at the end of the game. If on the other hand Max does play according to Case (3) of Stage I, then $ww' \in E(G)$ holds for every $w, w' \in V \setminus \{z\}$ at the end of the game. In either case we conclude that $s(G_0, \mathcal{PM}) \geq \binom{n-3}{2}$ as claimed. $\qquad\square$

Lemma 4.3 *Let $n \geq 6$ be an even integer and let $G_0 = (V, E)$ be a graph on n vertices. Assume that there exist vertices $w, x, y, z \in V$ such that $wx \in E$, $d_{G_0}(x) = 1$, $d_{G_0}(y) = d_{G_0}(z) = 0$ and $G_0 \setminus \{x, y, z\}$ admits a Hamilton cycle C. If Max is the second player, then $s(G_0, \mathcal{PM}) \geq \binom{n-2}{2}$.*

Proof Max plays according to the following simple strategy which consists of two stages.

Stage I: Let uv denote the last edge claimed by Mini; we distinguish between the following three cases:

    (1) If $\{u, v\} \cap \{y, z\} = \emptyset$, then Max claims an arbitrary free edge ab such that $\{a, b\} \cap \{y, z\} = \emptyset$ and repeats Stage I; if this is not possible, then he skips to Stage II.

    (2) Otherwise, if $u \in \{y, z\}$ and $v \in V \setminus \{x, y, z\}$, then Max claims ux and proceeds to Stage II.

    (3) Otherwise, if $u \in \{y, z\}$ and $v = x$, then Max claims an arbitrary free edge uu', where $u' \in V \setminus \{x, y, z\}$. He then proceeds to Stage II.

Stage II: Throughout this stage, Max follows the trivial strategy.

Note that at any point during the game, the graph $G \cup \{yz\}$ admits a perfect matching; it follows that $yz \notin E(G)$. In particular, the proposed strategy does account for every legal move of Mini. Moreover, if Max never plays according to Cases (2) and (3) of Stage I, then clearly $ww' \in E(G)$ holds for every $w, w' \in V \setminus \{y, z\}$ at the end of the game. If on the other hand Max does play according to Cases (2) or (3) of Stage I, then without loss of generality $xy \in E(G)$ (otherwise $xz \in E(G)$ and the proof can be completed by an analogous argument). In these cases, Max claims an edge and immediately proceeds to Stage II. Note that starting from that point and until the end of the game, $G \setminus \{z, t\}$ admits a perfect matching for every $t \in V$. Hence $d_G(z) = 0$, and it follows that $ww' \in E(G)$ holds for every $w, w' \in V \setminus \{z\}$ at the end of the game. In either case we conclude that $s(G_0, \mathcal{PM}) \geq \binom{n-2}{2}$ as claimed. $\qquad\square$

Lemma 4.4 *Let $n \geq 8$ be an even integer and let $G_0 = (V, E)$ be a graph on n vertices. Assume that there exist vertices $w_1, w_2, w_3, w_4 \in V$ such that $G_0[\{w_1, w_2, w_3\}] \cong K_3$, $d_{G_0}(w_1) = d_{G_0}(w_2) = d_{G_0}(w_3) = 2$, $d_{G_0}(w_4) = 0$ and $G_0 \setminus \{w_1, w_2, w_3, w_4\}$ admits a Hamilton cycle C. If Max is the second player, then $s(G_0, \mathcal{PM}) \geq \binom{n-4}{2}$.*

Proof Max plays according to the following simple strategy which consists of two stages.

Stage I: Let uv denote the last edge claimed by Mini; we distinguish between the following three cases:

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(1) If $\{u,v\} \cap \{w_1, w_2, w_3, w_4\} = \emptyset$, then Max claims an arbitrary free edge xy such that $\{x,y\} \cap \{w_1, w_2, w_3, w_4\} = \emptyset$ and repeats Stage I; if this is not possible, then he proceeds to Stage II.

(2) Otherwise, if $u = w_4$ and $v \in V \setminus \{w_1, w_2, w_3, w_4\}$, then Max claims uv', where v' is a neighbor of v in C. He then proceeds to Stage II.

(3) Otherwise, if $u \in \{w_1, w_2, w_3\}$ and $v \in V \setminus \{w_1, w_2, w_3, w_4\}$, then Max claims a free edge $u'v'$, where $u' \in \{w_1, w_2, w_3\} \setminus \{u\}$ and v' is a neighbor of v in C. He then proceeds to Stage II.

Stage II: Throughout this stage, Max follows the trivial strategy.

Note that at any point during the game, the graph $G \cup \{w_i w_4\}$ admits a perfect matching for every $1 \leq i \leq 3$; it follows that $w_i w_4 \notin E(G)$. In particular, the proposed strategy does account for every legal move of Mini. Moreover, if Max proceeds from Case (1) to Stage II, then clearly $xy \in E(G)$ holds for every $x, y \in V \setminus \{w_1, w_2, w_3, w_4\}$ at the end of the game. Similarly, if Max proceeds from Case (2) to Stage II, then $xy \in E(G)$ holds for every $x, y \in V \setminus \{w_1, w_2, w_3\}$ at the end of the game. Finally, if Max proceeds from Case (3) to Stage II, then $xy \in E(G)$ holds for every $x, y \in V \setminus \{w_4\}$ at the end of the game. In either case we conclude that $s(G_0, \mathcal{PM}) \geq \binom{n-4}{2}$ as claimed. $\qquad \square$

Lemma 4.5 *Let $n \geq 8$ be an even integer and let $G_0 = (V, E)$ be a graph on n vertices. Assume that there exist vertices $w_1, w_2, w_3, w_4 \in V$ such that $w_3 w_4 \in E$, $d_{G_0}(w_1) = d_{G_0}(w_2) = 0$, $d_{G_0}(w_3) = d_{G_0}(w_4) = 1$ and $G_0 \setminus \{w_1, w_2, w_3, w_4\}$ admits a Hamilton cycle C. If Max is the second player, then $s(G_0, \mathcal{PM}) \geq \binom{n-4}{2}$.*

Proof Max plays according to the following simple strategy which consists of two stages.

Stage I: Let uv denote the last edge claimed by Mini; we distinguish between the following four cases:

(1) If $\{u,v\} \cap \{w_1, w_2, w_3, w_4\} = \emptyset$, then Max claims an arbitrary free edge xy such that $\{x,y\} \cap \{w_1, w_2, w_3, w_4\} = \emptyset$ and repeats Stage I; if this is not possible, then he proceeds to Stage II.

(2) Otherwise, if $u \in \{w_1, w_2\}$ and $v \in V \setminus \{w_1, w_2, w_3, w_4\}$, then Max claims uv', where v' is a neighbor of v in C. He then follows the strategy described in the proof of Lemma 4.2 until the end of the game.

(3) Otherwise, if $u \in \{w_3, w_4\}$ and $v \in V \setminus \{w_1, w_2, w_3, w_4\}$, then Max claims a free edge $u'v'$, where v' is a neighbor of v in C and u' is the unique vertex in $\{w_3, w_4\} \setminus \{u\}$. He then follows the strategy described in the proof of Lemma 4.1 until the end of the game.

(4) Otherwise, if $u \in \{w_1, w_2\}$ and $v \in \{w_3, w_4\}$, then Max claims uv', where v' is the unique vertex in $\{w_3, w_4\} \setminus \{v\}$. He then follows the strategy described in the proof of Lemma 4.4 until the end of the game.

Stage II: Throughout this stage, Max follows the trivial strategy.

Note that at any point during the game, the graph $G \cup \{w_1 w_2\}$ admits a perfect matching; it follows that $w_1 w_2 \notin E(G)$. In particular, the proposed strategy does account for every legal move of Mini. Moreover, if Max never plays according to Cases (2), (3) and (4) of Stage I, then clearly $xy \in E(G)$ holds for every $x, y \in V \setminus \{w_1, w_2, w_3, w_4\}$ at the end of the game. If on the other hand Max does play according to Cases (2), (3) or (4) of Stage I, then it follows by Lemmas 4.2, 4.1 and 4.4, respectively, that $s(G_0, \mathcal{PM}) \geq \binom{n-4}{2}$. In either case we conclude that $s(G_0, \mathcal{PM}) \geq \binom{n-4}{2}$ as claimed. $\qquad \square$

We can now describe Max's strategy for the perfect matching game $(n, \mathcal{PM})$. At any point during Stages I – III, if Max is unable to follow the proposed strategy, then he skips to Stage IV. The proposed strategy is divided into the following four stages.

Stage I: Max follows the long path strategy until G contains a path $P = (u_0, \ldots, u_\ell)$ of length $\ell \in \{n - 5, n - 4\}$ which includes all vertices of positive degree. At that moment, if $\ell = n - 4$, then Max skips to Stage III, otherwise he proceeds to Stage II.

Stage II: Let $V(G) \setminus V(P) = \{w_1, w_2, w_3, w_4\}$. Let uv denote the edge Mini claims in her subsequent move; we distinguish between the following two cases:

    (1) If $\{u, v\} \cap V(P) \neq \emptyset$, then Max plays as follows. If $\{u, v\} \subseteq V(P)$, then Max claims $u_\ell w_4$. Otherwise, assume without loss of generality that $u \notin V(P)$. Max then claims $u_\ell u$ if it is free and $u_0 u$ otherwise. In either case he extends P to a path of length $n - 4$. By abuse of notation and for simplicity of presentation, we denote this path by $P = (u_0, \ldots, u_\ell)$ as well. Max then proceeds to Stage III.

    (2) Otherwise, assume without loss of generality that $u = w_3$ and $v = w_4$. Max claims $u_0 u_\ell$, and then follows the strategy described in the proof of Lemma 4.5 until the end of the game.

Stage III: Let $V(G) \setminus V(P) = \{w_1, w_2, w_3\}$. Let uv denote the edge Mini claims in her subsequent move; we distinguish between the following three cases:

    (1) If $\{u, v\} \subseteq V(P)$, then Max claims a free edge xx' such that $\{x, x'\} \subseteq V(P)$ and repeats Stage III.

    (2) Otherwise, if $\{u, v\} \subseteq \{w_1, w_2, w_3\}$, then Max claims $u_0 u_\ell$ if it is free and an arbitrary free edge xx' such that $\{x, x'\} \subseteq V(P)$ otherwise. He then follows the strategy described in the proof of Lemma 4.2 until the end of the game.

    (3) Otherwise, assume without loss of generality that $u \in V(P)$ and $v \in \{w_1, w_2, w_3\}$. Max claims $u_0 u_\ell$ if it is free and an arbitrary free edge xx' such that $\{x, x'\} \subseteq V(P)$ otherwise. He then follows the strategy described in the proof of Lemma 4.3 until the end of the game.

Stage IV: Throughout this stage, Max follows the trivial strategy.

It remains to prove that Max can indeed follow the proposed strategy and that, by doing so, he ensures that $e(G) \geq \binom{n-4}{2}$ holds at the end of the game. Starting with the former, note that Max can follow Stage I of the proposed strategy by Lemma 1.10 (throughout Stage I there are isolated vertices in G and thus it does not admit a perfect matching). He can follow Stage II

by Lemma 4.5 and by the properties of P, and can follow Stage III by Lemmas 4.2 and 4.3. Finally, it is obvious that he can follow Stage IV of the proposed strategy.

As for the latter, if Max does not reach Stage IV of the proposed strategy, then it follows by Lemmas 4.5, 4.2 and 4.3 that $e(G) \geq \binom{n-4}{2}$ holds at the end of the game. If on the other hand Max does reach Stage IV of the proposed strategy, then it follows by the description of the proposed strategy that $xy \in E(G)$ holds at the end of the game for every $x, y \in V(P)$ and thus $e(G) \geq \binom{n-4}{2}$. $\qquad\square$

Proof of Theorem 1.7 Throughout this proof, we assume that $n \geq 2k$, as otherwise $s(n, \mathcal{M}_k) = \binom{n}{2}$ and so the assertion of the theorem holds trivially. We will use the following terminology: the *parity* of a player is *odd* if he is the first to move and *even* otherwise. Assume first that the parity of Max is opposite to the parity of k. In order to prove that $s(n, \mathcal{M}_k) \geq n - 1$, we present a strategy for Max. Before doing so, we prove the following auxiliary lemma.

Lemma 4.6 *Let $k \geq 2$ be an integer and let $G_0 = (V, E)$ be a graph. Assume that there exists a partition $V = U \cup W$ such that $\nu(G_0) = \nu(G_0 \setminus W) = k - 1$. Assume further that there exist vertices $w_1, w_2 \in W$ and $u \in U$ such that $d_{G_0}(w_1) = d_{G_0}(w_2) = 1$ and $\{uw_1, uw_2\} \subseteq E$. Then Max, as the second player, has a strategy to ensure that at the end of the $(G_0, \mathcal{M}_k)$ game, $d_G(w, U) \geq 1$ will hold for every $w \in W$.*

Proof We present a strategy for Max; it is divided into the following two stages.

Stage I: At any point during this stage, let $I = \{w \in W : d_G(w, U) = 0\}$. If $I = \emptyset$, then this stage is over and Max proceeds to Stage II. Otherwise, Max claims uw, where $w \in I$ is an arbitrary vertex.

Stage II: Throughout this stage, Max follows the trivial strategy.

It is evident that, if Max is able to follow the proposed strategy, then $d_G(w, U) \geq 1$ holds for every $w \in W$ at the end of the game. It thus suffices to prove that he can indeed do so. We will prove this by induction on the size of I in the beginning of the game. If $|I| = 0$, then there is nothing to prove, as clearly Max can follow Stage II of the proposed strategy. Assume that our claim holds if $|I| \leq m$ for some non-negative integer m; we will prove it holds for $m + 1$ as well. Let xy denote the edge Mini claims in her first move. Since no edge with both endpoints in W is legal (with respect to G_0 and $\mathcal{M}_k$), we can assume without loss of generality that $x \in U$. In particular, $|\{x, y\} \cap \{w_1, w_2\}| \leq 1$ and thus we can assume that $y \neq w_1$. If $I = \emptyset$ holds immediately after this move, then there is nothing to prove; hence, let $z \in I$ be an arbitrary vertex and assume that Max claims uz in his first move. Suppose for a contradiction that this is not a legal move, that is, that $H := G_0 \cup \{xy, uz\}$ admits a matching of size k. Since this matching must contain uz, and no matching of H can cover both z and w_1, it follows that $\nu(H \setminus \{w_1\}) = k$. On the other hand, $\nu(H \setminus \{z\}) = k - 1$ holds by assumption. This is a contradiction as clearly $H \setminus \{w_1\}$ is isomorphic to $H \setminus \{z\}$. Immediately after Max's first move, $z \in W \setminus I$ and thus $|I| \leq m$. Moreover, $d_H(w_1) = d_H(z) = 1$ and $\{uw_1, uz\} \subseteq E(H)$.

By the induction hypothesis, we conclude that Max can follow the proposed strategy until the end of the game. □

We can now describe Max's strategy for the k-matching saturation game $(n, \mathcal{M}_k)$. At any point during the game, if Max is unable to follow the proposed strategy, then he forfeits the game. The proposed strategy is divided into the following three stages.

Stage I: Max follows the long path strategy until G contains a path $P = (u_0, \ldots, u_\ell)$ of length $\ell \in \{2k-4, 2k-3\}$ which includes all vertices of positive degree. At that moment, if $\ell = 2k - 4$, then Max proceeds to Stage II, otherwise he skips to Stage III.

Stage II: Let wv denote the edge Mini claims in her subsequent move; we distinguish between the following two cases:

(1) If $\{w, v\} \cap V(P) \neq \emptyset$, then Max plays as follows. If $\{w, v\} \subseteq V(P)$, then Max claims $u_\ell z$ for an arbitrary vertex $z \in V(G) \setminus V(P)$. Otherwise, assume without loss of generality that $w \notin V(P)$. Max then claims $u_\ell w$ if it is free and $u_0 w$ otherwise. In either case he extends P to a path of length $2k - 3$. By abuse of notation and for simplicity of presentation, we denote this path by $P = (u_0, \ldots, u_\ell)$ as well. Max then proceeds to Stage III.

(2) Otherwise, assume without loss of generality that $d_G(u_0) = 1$ (recall property (c) in Lemma 1.10). Max claims $u_1 z$ for an arbitrary isolated vertex z, and then follows the strategy described in the proof of Lemma 4.6, with $U = \{w, v, u_1, \ldots, u_\ell\}$, $u = u_1$ and $\{w_1, w_2\} = \{z, u_0\}$, until the end of the game.

Stage III: Let wv denote the edge Mini claims in her subsequent move; we distinguish between the following two cases:

(1) If $\{w, v\} \subseteq V(P)$, then Max claims a free edge xy such that $\{x, y\} \subseteq V(P)$ and repeats Stage III.

(2) Otherwise, assume without loss of generality that $w \in V(P)$ and $v \notin V(P)$. Max claims wz for some arbitrary isolated vertex z, and then follows the strategy described in the proof of Lemma 4.6, with $U = V(P)$, $u = w$ and $\{w_1, w_2\} = \{z, v\}$, until the end of the game.

It remains to prove that Max can indeed follow the proposed strategy and that, by doing so, he ensures that $e(G) \geq n - 1$ holds at the end of the game. Starting with the former, note that Max can follow Stage I of the proposed strategy by Lemma 1.10 (throughout Stage I there are at most $2k - 2$ vertices of positive degree in G and thus $\nu(G) < k$). An analogous argument shows that he can follow Case (1) of Stage II. Max can make his first move in Case (2) of Stage II, as $n \geq 2k$ and immediately after this move, there are exactly $2k$ vertices of positive degree in G but no matching of G covers both z and u_0. Moreover, he can follow the remainder of Case (2) of Stage II by Lemma 4.6. Next, consider Stage III. Mini cannot claim an edge wv such that $\{w, v\} \cap V(P) = \emptyset$ as no such edge is legal. Therefore, Cases (1) and (2) of Stage III account for every legal move of Mini. Suppose for a contradiction that at some point during the game Max forfeits the game while attempting to follow Case (1) of Stage III. Since every free edge with both endpoints in $V(P)$ is clearly legal, it follows that no such edges remain. Therefore, the total number of edges played thus far is $\binom{2k-2}{2} = (k-1)(2k-3)$ and it is Max's

turn to play. Since Max's parity is opposite to that of k, this is a contradiction. Moreover, Max can make his first move in Case (2) of Stage III, as immediately after this move, there are exactly $2k$ vertices of positive degree in G but no matching of G covers both z and v. Finally, he can follow the remainder of Case (2) of Stage III by Lemma 4.6.

In order to prove that $e(G) \geq n - 1$ holds at the end of the game, we examine the graph G at the end of the game. If the game ends when Max plays according to Case (2) of Stage III, then G is connected and thus $e(G) \geq n - 1$. Otherwise, the game ends when Max plays according to Case (2) of Stage II. Suppose for a contradiction that $e(G) < n - 1$ holds at the end of the game; in particular, G must be disconnected. It thus follows by the description of the proposed strategy, that G consists of exactly two connected components, $C_1 \supseteq V(P)$ and $C_2 \supseteq \{w, v\}$. Since P admits a matching of size $k - 2$ and $\nu(G) < k$, it follows that C_2 is either a star or a triangle. Since $e(G) \geq n - 1$ holds in the latter case, we can assume that C_2 is a star. However, any edge xy, where x is the center of the star and $y \in C_1$, is still legal in this case, contrary to our assumption that the game is over.

Next, assume that the parity of Mini is opposite to the parity of k. Since the case $k = 2$ was considered in [14], we can assume that $k \geq 3$. In order to prove the theorem, we present a strategy for Mini. In order to simplify the description of the strategy, we first consider several possible *end-games* which are described in the following lemmas. Since these lemmas and their proofs are quite similar to those of Lemmas 4.1 – 4.5, we will omit some of the details. Though this is not always necessary, in each of these lemmas we assume that Mini is the second player.

Lemma 4.7 *Let $k \geq 3$ be an integer and let $G_0 = (V, E)$ be a graph on $n \geq 6$ vertices. Assume that there exists a non-trivial connected component C_1 of G_0 such that $G_0[C_1]$ admits a Hamilton cycle C and that $d_{G_0}(u) = 0$ for every $u \in V \setminus C_1$.*

(a) *If $|C_1| = 2k - 1$, then $s(G_0, \mathcal{M}_k) \leq \binom{2k-1}{2}$.*

(b) *If $|C_1| = 2k - 2$ and $\binom{2k-2}{2} - e(G_0)$ is even, then $s(G_0, \mathcal{M}_k) \leq \binom{2k-1}{2}$.*

Proof Part (a) is trivial since, throughout the $(G_0, \mathcal{M}_k)$ game, the only legal edges are those with both endpoints in C_1. Hence, $e(G) \leq \binom{2k-1}{2}$ will hold at the end of the game no matter how Mini plays. As for (b), Mini plays according to the following simple strategy.

Stage I: Let uv denote the last edge claimed by Max; we distinguish between the following two cases:

   (1) If $\{u, v\} \subseteq C_1$, then Mini claims an arbitrary free edge ww' such that $\{w, w'\} \subseteq C_1$ and repeats Stage I.

   (2) Otherwise, assume without loss of generality that $v \in C_1$ and $u \notin C_1$. Mini claims uv', where v' is a neighbor of v in C. She then proceeds to Stage II.

Stage II: Throughout this stage, Mini follows the trivial strategy.

Since no edge xy such that $\{x, y\} \in V \setminus C_1$ is legal, it follows that the proposed strategy does account for every legal move of Max. Moreover, since Mini is the second player and $\binom{2k-2}{2} - e(G_0)$ is even, it follows that she can play according to Case (1) of Stage I. Hence, at

some point during the game, Max must claim an edge uv such that $|\{u,v\} \cap C_1| = 1$. By Case (2) of Stage I and by the analysis of Part (a) of the lemma, we conclude that $e(G) \leq \binom{2k-1}{2}$ will hold at the end of the game. $\qquad\square$

Lemma 4.8 *Let $k \geq 3$ be an integer and let $G_0 = (V, E)$ be a graph on $n \geq 6$ vertices. Assume that there are two non-trivial connected component C_1 and C_2 of G_0, where $|C_1| = 2k - 3$. Assume further that $G_0[C_1]$ admits a Hamilton cycle C and that $d_{G_0}(u) = 0$ for every $u \in V \setminus (C_1 \cup C_2)$.*

(a) *If $G_0[C_2] \cong K_3$, then $s(G_0, \mathcal{M}_k) \leq \binom{2k-1}{2}$.*

(b) *If $G_0[C_2] \cong K_2$ and $\binom{2k-3}{2} - e(G_0[C_1])$ is even, then $s(G_0, \mathcal{M}_k) \leq \binom{2k-1}{2}$.*

Proof Part (a) is trivial since, throughout the $(G_0, \mathcal{M}_k)$ game, the only legal edges are those with both endpoints in C_1. Hence, by following the trivial strategy, Mini ensures that $e(G) \leq 3 + \binom{2k-3}{2} \leq \binom{2k-1}{2}$ will hold at the end of the game. As for (b), Mini plays according to the following simple strategy.

Stage I: Let uv denote the last edge claimed by Max; we distinguish between the following two cases:

    (1) If $\{u,v\} \subseteq C_1$, then Mini claims an arbitrary free edge ww' such that $\{w, w'\} \subseteq C_1$ and repeats Stage I.

    (2) Otherwise, assume without loss of generality that $u \in C_2$. Let u' be the unique vertex in $C_2 \setminus \{u\}$. If $v \in C_1$, Mini claims $u'v'$, where v' is a neighbor of v in C. Otherwise, Mini claims $u'v$. In either case, she then proceeds to Stage II.

Stage II: Throughout this stage, Mini follows the trivial strategy.

Since every legal edge either has two endpoints in C_1 or one endpoint in C_2, the proposed strategy does account for every legal move of Max. Moreover, since Mini is the second player and $\binom{2k-3}{2} - e(G_0[C_1])$ is even, it follows that she can play according to Case (1) of Stage I. Hence, at some point during the game, Max must claim an edge uv such that $|\{u,v\} \cap C_1| \leq 1$. If $|\{u,v\} \cap C_1| = 0$, then by Case (2) of Stage I and by the analysis of Part (a) of the lemma, we conclude that $e(G) \leq \binom{2k-1}{2}$ will hold at the end of the game. If $|\{u,v\} \cap C_1| = 1$, then by Case (2) of Stage I and by Lemma 4.7(a), we conclude that $e(G) \leq \binom{2k-1}{2}$ will hold at the end of the game. $\qquad\square$

Lemma 4.9 *Let $k \geq 4$ be an integer and let $G_0 = (V, E)$ be a graph on $n \geq 8$ vertices. Assume that there are two non-trivial connected component C_1 and C_2 of G_0, where $|C_1| = 2k - 4$. Assume further that $G_0[C_1]$ admits a Hamilton cycle C, that $d_{G_0}(u) = 0$ for every $u \in V \setminus (C_1 \cup C_2)$, and that $\binom{2k-4}{2} - e(G_0[C_1])$ is even.*

(a) *If $G_0[C_2] \cong K_3$, then $s(G_0, \mathcal{M}_k) \leq \binom{2k-1}{2}$.*

(b) *If $G_0[C_2] \cong K_2$, then $s(G_0, \mathcal{M}_k) \leq \binom{2k-1}{2}$.*

Proof Starting with (a), Mini plays according to the following simple strategy.

Stage I: Let uv denote the last edge claimed by Max; we distinguish between the following two cases:

(1) If $\{u, v\} \subseteq C_1$, then Mini claims an arbitrary free edge ww' such that $\{w, w'\} \subseteq C_1$ and repeats Stage I.

(2) Otherwise, assume without loss of generality that $v \in C_1$ and $u \notin C_1$. If $u \in C_2$, Mini claims $u'v'$, where u' is some vertex of $C_2 \setminus \{u\}$ and v' is a neighbor of v in C. Otherwise, Mini claims uv', where v' is a neighbor of v in C. In either case, she then proceeds to Stage II.

Stage II: Throughout this stage, Mini follows the trivial strategy.

Since every legal edge has at least one endpoint in C_1, it follows that the proposed strategy does account for every legal move of Max. Moreover, since Mini is the second player and $\binom{2k-4}{2} - e(G_0[C_1])$ is even, it follows that she can play according to Case (1) of Stage I. Hence, at some point during the game, Max must claim an edge uv such that $|\{u, v\} \cap C_1| = 1$. If $|\{u, v\} \cap C_2| = 1$, then by Case (2) of Stage I and by Lemma 4.7(a) and its proof, we conclude that $e(G) \leq \binom{2k-1}{2}$ will hold at the end of the game. Otherwise, by Case (2) of Stage I and by Lemma 4.8(a) and its proof, we conclude that $e(G) \leq \binom{2k-1}{2}$ will hold at the end of the game.

As for (b), Mini plays according to the following simple strategy. Let uv denote the last edge claimed by Max; we distinguish between the following three cases:

(1) If $\{u, v\} \subseteq C_1$, then Mini claims an arbitrary free edge ww' such that $\{w, w'\} \subseteq C_1$.

(2) Otherwise, if $\{u, v\} \cap C_1 = \emptyset$, assume without loss of generality that $u \in C_2$ and $v \in V \setminus (C_1 \cup C_2)$. Mini claims $u'v$, where u' is the unique vertex in $C_2 \setminus \{u\}$ and then follows the strategy described in the proof of Part (a) of the lemma until the end of the game.

(3) Otherwise, assume without loss of generality that $v \in C_1$ and let v' be a neighbor of v in C. If $u \in C_2$, Mini claims $u'v'$, where u' is the unique vertex in $C_2 \setminus \{u\}$ and then follows the strategy described in the proof of Lemma 4.7(b) until the end of the game. Otherwise, Mini claims uv' and then follows the strategy described in the proof of Lemma 4.8(b) until the end of the game.

Since every legal edge has at least one endpoint in $C_1 \cup C_2$, it follows that the proposed strategy does account for every legal move of Max. Moreover, since Mini is the second player and $\binom{2k-4}{2} - e(G_0[C_1])$ is even, it follows that she can play according to Case (1). Hence, at some point during the game, Max must claim an edge uv such that $|\{u, v\} \cap C_1| \leq 1$. If $|\{u, v\} \cap C_1| = 0$, then by Case (2) and by Part (a) of the lemma, we conclude that $e(G) \leq \binom{2k-1}{2}$ will hold at the end of the game. Otherwise, by Case (3) and by Lemmas 4.7(b) and 4.8(b), we conclude that $e(G) \leq \binom{2k-1}{2}$ will hold at the end of the game. $\qquad\square$

Lemma 4.10 *Let $k \geq 3$ be an integer and let $G_0 = (V, E)$ be a graph on $n \geq 6$ vertices. Assume that there exists a non-trivial connected component C_1 of G_0 of order $2k - 3$ such that $G_0[C_1]$ admits a Hamilton cycle C. Let $x \in V \setminus C_1$ and assume that $d_{G_0}(x) \leq 1$ and $d_{G_0}(u) = 0$ for every $u \in V \setminus (C_1 \cup \{x\})$.*

(a) If $\binom{2k-2}{2} - e(G_0)$ is even and there exists a vertex $w \in C_1$ such that $wx \in E$, then $s(G_0, \mathcal{M}_k) \leq \binom{2k-1}{2}$.

(b) If $\binom{2k-3}{2} - e(G_0)$ is odd and $d_{G_0}(x) = 0$, then $s(G_0, \mathcal{M}_k) \leq \binom{2k-1}{2}$.

Proof Starting with (a), Mini plays according to the following simple strategy.

Stage I: Let uv denote the last edge claimed by Max; we distinguish between the following two cases:

   (1) If $\{u, v\} \subseteq C_1 \cup \{x\}$, then Mini claims a free edge xw', where w' is a neighbor of w in C. She then follows the strategy described in the proof of Lemma 4.7(b) until the end of the game.

   (2) Otherwise, assume without loss of generality that $d_{G_0}(u) = 0$. If $v = x$, Mini claims uz, where $z \in C_1$ is an arbitrary vertex and otherwise she claims ux. In either case, she then proceeds to Stage II.

Stage II: Throughout this stage, Mini follows the trivial strategy.

Since Mini is the second player and $\binom{2k-2}{2} - e(G_0)$ is even, it follows that Mini can play according to Case (1) of Stage I and thus, by Lemma 4.7(b), ensure that $e(G) \leq \binom{2k-1}{2}$ will hold at the end of the game. If Mini plays according to Case (2) of Stage I, then after her first move, every legal edge has both endpoints in $C_1 \cup \{u, x\}$ and thus $e(G) \leq \binom{2k-1}{2}$ will hold at the end of the game.

As for (b), Mini plays according to the following simple strategy. Let uv denote the last edge claimed by Max; we distinguish between the following three cases:

(1) If $\{u, v\} \subseteq C_1$, then Mini claims an arbitrary free edge ww' such that $d_{G_0}(w) = d_{G_0}(w') = 0$ and then follows the strategy described in the proof of Lemma 4.8(b) until the end of the game.

(2) Otherwise, if $\{u, v\} \cap C_1 = \emptyset$, then Mini claims an arbitrary free edge ww' such that $\{w, w'\} \subseteq C_1$ and then follows the strategy described in the proof of Lemma 4.8(b) until the end of the game.

(3) Otherwise, assume without loss of generality that $v \in C_1$ and $d_{G_0}(u) = 0$. Mini claims uv', where v' is a neighbor of v in C and then follows the strategy described in the proof of Lemma 4.7(b) until the end of the game.

Since $\binom{2k-3}{2} - e(G_0)$ is odd, it follows that Mini can play according to the proposed strategy. Moreover, it follows by Lemmas 4.8(b) and 4.7(b) that $e(G) \leq \binom{2k-1}{2}$ will hold at the end of the game. $\square$

We can now describe Mini's strategy for the k-matching saturation game $(n, \mathcal{M}_k)$. At any point during the game, if Mini is unable to follow the proposed strategy, then she forfeits the game. The proposed strategy is divided into the following three stages.

Stage I: Mini follows the long path strategy until G contains a path $P = (u_0, \ldots, u_\ell)$ of length $\ell \in \{2k - 5, 2k - 4\}$ which includes all vertices of positive degree. At that moment, if $\ell = 2k - 5$, then Mini proceeds to Stage II, otherwise she skips to Stage III.

Stage II: Let uv denote the edge Max claims in his subsequent move; we distinguish between the following two cases:

(1) If $\{u, v\} \cap V(P) \neq \emptyset$, then Mini plays as follows. If $\{u, v\} \subseteq V(P)$, then Mini claims $u_\ell z$ for an arbitrary vertex $z \in V(G) \setminus V(P)$. Otherwise, assume without loss of generality that $u \notin V(P)$. Mini then claims $u_\ell u$ if it is free and $u_0 u$ otherwise. In either case she extends P to a path of length $2k - 4$. By abuse of notation and for simplicity of presentation, we denote this path by $P = (u_0, \ldots, u_\ell)$ as well. Mini then proceeds to Stage III.

(2) Otherwise, Mini claims the edge $u_0 u_\ell$, and then plays according to the strategy described in the proof of Lemma 4.9(b) until the end of the game.

Stage III: Let uv denote the edge Max claims in his subsequent move. Mini claims $u_0 u_\ell$ if it is free and an arbitrary edge ww' such that $\{w, w'\} \subseteq V(P)$ otherwise; we then distinguish between the following three cases:

(1) If $|\{u, v\} \cap V(P)| = 0$, then Mini plays according to the strategy described in the proof of Lemma 4.8(b) until the end of the game.

(2) If $|\{u, v\} \cap V(P)| = 1$, then Mini plays according to the strategy described in the proof of Lemma 4.10(a) until the end of the game.

(3) If $|\{u, v\} \cap V(P)| = 2$, then Mini plays according to the strategy described in the proof of Lemma 4.10(b) until the end of the game.

It follows by Lemma 1.10 that Mini can play according to Stage I of the proposed strategy. Lemma 1.10 also ensures that $u_0 u_\ell$ is free if Mini wishes to follow Case (2) of Stage II (the only possible exception is the case $\ell = 1$, but this can only occur if $k = 3$ and Mini is the first player; this case is excluded by our assumption that the parity of Mini is opposite to the parity of k). Mini can play according to the remainder of the proposed strategy by our assumption that the parity of Mini is opposite to the parity of k.

Finally, it follows by Lemmas 4.9(b), 4.8(b), 4.10(a) and 4.10(b) that $e(G) \leq \binom{2k-1}{2}$ will hold at the end of the game. $\qquad\square$

5 Concluding remarks and open problems

In this paper we proved lower and upper bounds on the scores of several natural saturation games, namely, connectivity, colorability and matching games. Other natural graph properties could be considered; one interesting example is Hamiltonicity. Let $\mathcal{H}$ denote the graph property of admitting a Hamilton cycle. It was proved by Ore [13] that $ex(n, \mathcal{H}) = \binom{n-1}{2} + 1$. On the other hand, it is known (see, e.g., [8]) that if n is not too small, then $sat(n, \mathcal{H}) = \lceil 3n/2 \rceil$. Our attempts to determine $s(n, \mathcal{H})$ lead us to make the following conjecture.

Conjecture 5.1 $s(n, \mathcal{H}) = \Theta(n^2)$.

All games considered in this paper require Max and Mini to avoid certain large structures. Another interesting line of research would be to avoid small structures. Given a fixed graph H, let $\mathcal{F}_H$ denote the graph property of admitting a copy of H. It follows from the celebrated Stone-Erdős-Simonovits Theorem (see, e.g., [3]) that $ex(n, \mathcal{F}_H) = \Theta(n^2)$ holds for every non-bipartite graph H. On the other hand, it was proved by Kászonyi and Tuza [11] that $sat(n, \mathcal{F}_H) = O(n)$ for every graph H. As noted in the introduction, very little is known about $s(n, \mathcal{F}_H)$, even in the case $H = K_3$. Several simpler cases were considered in [4].

For most graph properties $\mathcal{P}$ considered in this paper, we have shown that the score of the $(n, \mathcal{P})$ saturation game is very close to the trivial upper bound $ex(n, \mathcal{P})$. A bold exception are the k-matching games under some assumptions on the parity of k and the identity of the first player. It is not hard to find examples of properties $\mathcal{P}$ for which the trivial lower bound $s(n, \mathcal{P}) \geq sat(n, \mathcal{P})$ is in fact tight. For example, as shown in Theorem 1.7, if Mini is the first player, then $s(n, \mathcal{M}_2) = 3 = sat(n, \mathcal{M}_2)$. In fact, there are infinitely many such examples. For every integer $k \geq 2$, let α_k denote the property of having independence number less than k. If $G \in \alpha_k$ then clearly G admits an independent set I of size k and $uv \in E(G)$ whenever $\{u, v\} \setminus I \neq \emptyset$. It follows that $sat(n, \alpha_k) = ex(n, \alpha_k) = \binom{n}{2} - \binom{k}{2}$ and thus $s(n, \alpha_k) = \binom{n}{2} - \binom{k}{2}$ as well. It would be interesting to find less obvious examples of the tightness of the trivial lower bound.

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