Algebraic Methods in Combinatorics 0366-4933

Michael Krivelevich

Spring Semester 2013

Homework 3

Due: May 29, 2013

1. Let $A \in M_{n\times n}(F)$ be a matrix with non-zero permanent over F. Prove that for every vector $\bar{b} = (b_1, \dots, b_n) \in F^n$ and for every family of sets $S_1, \dots, S_n \subset F$, each of cardinality 2, there is a vector $\bar{x} \in S_1 \times \dots \times S_n$ such that for every i, the i-th coordinate of $A\bar{x}$ differs from b_i.

2. Let $G = ([n], A)$ be a directed graph containing a family of vertex disjoint cycles covering all vertices of G. Prove that for every family of sets $S_1, \dots, S_n \subset \mathbb{R}$, each of cardinality 2, there is a vector $\bar{x} \in S_1 \times \dots \times S_n$ such that for every vertex $i \in [n]$ one has: $\sum_{j:(j,i)\in A(G)} x_j \neq 0$.

3. Let K_r^k denote a complete r-uniform hypergraph on k vertices. An r-uniform hypergraph $H = (V, E)$ is called K_r^k-saturated if H does not contain a copy of K_r^k, but any r-tuple from V missing in H creates a copy of K_r^k when added to H. Prove that if $H = (V, E)$ is an r-uniform K_r^k-saturated hypergraph on n vertices, then $|E| \geq \binom{n}{r} - \binom{n-k+r}{r}$. Provide an example of equality.

4. Let $A_1, \dots, A_m, B_1, \dots, B_m$ be finite sets satisfying: $A_i \cap B_i = \emptyset$, $1 \leq i \leq m$, but $A_i \cap B_j \neq \emptyset$, $1 \leq i < j \leq m$. Prove:

$$\sum_{i=1}^{m} \left(\frac{1}{2}\right)^{|A_i|+|B_i|} \leq 1\,.$$

Hint. There are methods other than algebraic in Combinatorics...

5. Let $n > r$ be positive integers. The Kneser graph $K(n, r)$ is the graph whose vertices are r-tuples of $[n]$, where two r-tuples are connected by an edge if they are disjoint. Prove that if r divides n, then the Shannon capacity $c(K(n, r))$ of $K(n, r)$ satisfies: $c(K(n, r)) = \binom{n-1}{r-1}$.

6. For a graph $G = ([n], E)$, define $\eta(G)$ to be the smallest rank of any matrix $A \in M_{n\times n}(F)$ (over any field F) such that $a_{ii} \neq 0$, $1 \leq i \leq n$, but $a_{ij} = 0$ for every $1 \leq i \neq j \leq n$ for which $(i, j) \notin E(G)$. Prove that $c(G) \leq \eta(G)$, where $c(G)$ is the Shannon capacity of G.
Hint. Familiarize yourself with the Kronecker product of matrices and its properties.