Graph Theory 0366-3267

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Homework 2
Due: Dec. 19, 2012

1. Prove that in a connected graph G every two paths of maximum length share a vertex.

2. Prove that a graph G on at least three vertices is 2-connected if and only if for every ordered triple x, y, z of distinct vertices, G contains a path P from x to z passing through y.

3. Let Q_k be the k-dimensional cube defined as follows: $V(Q_k) = \{0,1\}^k$, $(\mathbf{x} = (x_1, \ldots, x_k), \mathbf{y} = (y_1, \ldots, y_k)) \in E(Q_k)$ iff $\mathbf{x}$ and $\mathbf{y}$ differ in exactly one coordinate. Prove: $\kappa(Q_k) = \kappa'(Q_k) = k$.

4. Let $k \geq 2$. Prove that every k-connected graph on at least $2k$ vertices contains a cycle of length at least $2k$.

5. Let d be a positive integer. Prove that every $2d$-regular connected graph G with an even number of edges contains a spanning d-regular subgraph.

6. A mouth eats its way through a $3 \times 3 \times 3$ cube of cheese by eating all the $1 \times 1 \times 1$ subcubes. If it starts at a corner subcube and always moves on to an adjacent subcube (sharing a face of area 1), can it do this and eat the center subcube last?

7. Let $t(n, H_n)$ be the maximum number of edges in a graph G on n vertices, not containing a Hamilton cycle H_n. Prove: $t(n, H_n) = \binom{n-1}{2} + 1$. (You need to prove both lower and upper bounds for $t(n, H_n)$.)

8. Let G be a graph of connectivity $\kappa(G)$ with independence number $\alpha(G)$. Assume $\kappa(G) \geq \alpha(G) - 1$. Prove that G contains a Hamilton path.