Graph Theory 0366-3267
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Fall Semester 2013

Homework 3
Due: Dec. 29, 2013

1. Let $G = (A \cup B, E)$ be a bipartite graph with a matching saturating A. Prove that there exists a vertex $a \in A$ such that every edge of G incident to a belongs to a matching of size $|A|$ in G. (**Hint:** recall the two cases in the proof of Hall's theorem.)
2. Let $A = (a_{i,j})$ be an n by n real matrix, where $n > 1$, $a_{i,j} \geq 0$ for all i, j and the sum of elements in each row of A and the sum of elements in each column of A is exactly 1. Show that there is a permutation σ of $1, 2, \ldots, n$ so that $a_{i,\sigma(i)} > \frac{1}{n^2}$ for all $1 \leq i \leq n$.
3. Prove the so called defect version of Hall's theorem: Let $G = (A \cup B, E)$ be a bipartite graph. Define

$$\delta(A) = \max_{S \subseteq A} (|S| - |N(S)|) \ .$$

Prove that $\nu(G) = |A| - \delta(A)$, where $\nu(G)$ is the matching number of G.
4. Let G be a connected graph with an even number of edges. Prove, using Tutte's Theorem, that the set of edges of G can be partitioned into pairwise disjoint pairs, where each pair forms a path of length 2.
5. Let G be a graph on n vertices. Prove: $\chi(G) \cdot \chi(\bar{G}) \geq n$.
6. Let G be a graph in which every pair of odd cycles has a common vertex. Show that $\chi(G) \leq 5$.
7. Let G be a union of k forests. Prove that $\chi(G) \leq 2k$.
8. Let G be a graph in which every edge belongs to at most k cycles. Show that $\chi(G) \leq k+2$.