Graph Theory 0366-3267
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Homework 4
Due:Jan. 12, 2014

1. Let $G = (V, E)$ be a graph with chromatic number $\chi(G) > 10$ and girth $g > 10$. Prove that the number of vertices of G is bigger than $10 \cdot 9^4$.

2. Let $G = (V, E)$ be a (simple) graph with maximum degree $k > 1$ and exactly $k(k+1)$ edges. Prove that the set of edges of G can be partitioned into $k+1$ pairwise disjoint sets, each forming a matching of size precisely k.

3. Let $G = (V, E)$ be a bipartite graph with minimum degree $\delta \geq 2$. Prove that there is a (not necessarily proper) coloring of the edges of G by δ colors, so that every vertex is incident with at least one edge of each color.

4. Let $G = (V, E)$ be a bipartite graph. Prove that there is a partition of the set of edges E into 3 disjoint parts $E = E_1 \cup E_2 \cup E_3$, $E_1 \cap E_2 = E_2 \cap E_3 = E_3 \cap E_1 = \emptyset$, so that for every vertex v of G and for each $1 \leq i \leq 3$, the degree $d_i(v)$ of v in the graph (V, E_i) satisfies $\lfloor d(v)/3 \rfloor \leq d_i(v) \leq \lceil d(v)/3 \rceil$, where $d(v)$ is the degree of v in G.

5. (i) Let $G = (V, E_1 \cup E_2)$ be a graph, where E_1 and E_2 are (nonempty) matchings. Show that the chromatic number of G is 2.

(ii) $(*)$ Let $G' = (V, E_1 \cup E_2 \cup E_3)$ be a graph, where E_1 and E_2 are (nonempty) matchings and E_3 is the set of edges of a nonempty collection of pairwise vertex disjoint copies of K_4. Prove that the chromatic number of G' is 4.

6. For two graphs H_1 and H_2, the Ramsey number $r(H_1, H_2)$ is the minimum number r so that in any red-blue coloring of the edges of the complete graph K_r on r vertices there is necessarily either a red copy of H_1 or a blue copy of H_2 (or both). Let $K_{1,n}$ denote the star with n edges. Compute the Ramsey number $r(K_{1,n}, K_{1,m})$ for all values of m and n. Note: the formula depends on the parity of m and n.

7. Let P_n denote a path with n vertices. What is the Ramsey number $r(P_n, K_m)$? Prove the required upper and lower bounds to justify the value claimed.

8. Prove that for every k there is a finite integer $n = n(k)$ so that for any coloring of the integers $1, 2, \ldots, n$ by k colors there are **distinct** integers a, b, c and d of the same color satisfying $a + b + c = d$.