1. Prove that in a connected graph $G$ every two paths of maximum length share a vertex.
2. For a connected graph $G = (V, E)$ and a positive integer $k$, let $G^k$ be the graph with vertex set $V$, where two vertices are connected by an edge if and only if their distance in $G$ is at most $k$. Prove: if $G$ is a connected graph on $n$ vertices and $1 \leq k \leq n - 1$ is an integer, then $G^k$ is $k$-connected.
3. Let $k \geq 2$. Prove that every $k$-connected graph on at least $2k$ vertices contains a cycle of length at least $2k$.
4. Let $G$ be a graph in which every pair of vertices has an odd number of common neighbors. Prove that $G$ is Eulerian.
5. A mouth eats its way through a $3 \times 3 \times 3$ cube of cheese by eating all the $1 \times 1 \times 1$ subcubes. If it starts at a corner subcube and always moves on to an adjacent subcube (sharing a face of area 1), can it do this and eat the center subcube last?
6. Let $G$ be a connected graph on $n$ vertices with $\delta(G) = d$. Prove that $G$ contains a path of length $\min\{2d, n - 1\}$.
7. Let $t(n, H_n)$ be the maximum number of edges in a graph $G$ on $n$ vertices, not containing a Hamilton cycle $H_n$. Prove: $t(n, H_n) = \left(\frac{n-1}{2}\right) + 1$. (You need to prove both lower and upper bounds for $t(n, H_n)$.)
8. Let $G$ be a graph of connectivity $\kappa(G)$ with independence number $\alpha(G)$. Assume $\kappa(G) \geq \alpha(G) - 1$. Prove that $G$ contains a Hamilton path.