Graph Theory 0366-3267

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Homework 2
Due: Dec. 28, 2014

1. Prove that in a connected graph G every two paths of maximum length share a vertex.

2. For a connected graph $G = (V, E)$ and a positive integer k, let G^k be the graph with vertex set V, where two vertices are connected by an edge if and only if their distance in G is at most k. Prove: if G is a connected graph on n vertices and $1 \leq k \leq n-1$ is an integer, then G^k is k-connected.

3. Let $k \geq 2$. Prove that every k-connected graph on at least $2k$ vertices contains a cycle of length at least $2k$.

4. Let G be a graph in which every pair of vertices has an odd number of common neighbors. Prove that G is Eulerian.

5. A mouth eats its way through a $3\times3\times3$ cube of cheese by eating all the $1\times1\times1$ subcubes. If it starts at a corner subcube and always moves on to an adjacent subcube (sharing a face of area 1), can it do this and eat the center subcube last?

6. Let G be a connected graph on n vertices with $\delta(G) = d$. Prove that G contains a path of length $\min\{2d, n-1\}$.

7. Let $t(n, H_n)$ be the maximum number of edges in a graph G on n vertices, not containing a Hamilton cycle H_n. Prove: $t(n, H_n) = \binom{n-1}{2} + 1$. (You need to prove both lower and upper bounds for $t(n, H_n)$.)

8. Let G be a graph of connectivity $\kappa(G)$ with independence number $\alpha(G)$. Assume $\kappa(G) \geq \alpha(G) - 1$. Prive that G contains a Hamilton path.