Problem 1 (F20A). Suppose that $G$ is an $n$-vertex, triangle-free graph. Show that $\nu(G) = n - \chi(G^c)$. (Recall that $\nu(G)$ is the largest size of a matching in $G$.)

Problem 2 (F20A). Show that every 4-colourable graph with $m$ edges contains a bipartite subgraph with at least $2m/3$ edges.

Problem 3 (F20A). Show that every 3-connected graph contains a subdivision of $K_4$.

Problem 4 (F20A). Show that, for every $n \geq 5$, the largest number of edges in a nonbipartite triangle-free graph with $n$ vertices is $\lceil (n - 1)^2/4 \rceil + 1$.

Problem 5 (F20B). Prove that a graph $G$ is $3^k$-colourable if and only if $G$ is the union of $k$ three-colourable graphs.

Problem 6 (F20B). Let $G$ be a connected graph that contains neither $K_3$ nor $P_4$ (the path with four vertices) as an induced subgraph. Prove that $G$ is a complete bipartite graph.

Problem 7 (F20B). Suppose that every edge of a graph $G$ appears in at most one cycle. Prove that $\chi(G) \leq 3$.

Problem 8 (F20B). Denote by $m \cdot K_2$ the graph comprising $m$ vertex-disjoint edges (the unique 1-regular graph with $2m$ vertices). Show that $R(m \cdot K_2, m \cdot K_2) = 3m - 1$ for every $m \geq 1$. Recall that $R(H, H)$ denotes the smallest integer $n$ such that every red/blue-colouring of the edges of $K_n$ contains a monochromatic copy of $H$.

Problem 9 (F19A). Show that every 3-regular $n$-vertex graph contains a bipartite subgraph with $n$ edges.

Problem 10 (F19A). Prove that every graph $G$ with $\chi(G) \geq 3$ contains a cycle of length at least $\chi(G)$.

Problem 11 (F19A). Suppose that $G$ is a 4-regular graph with an even number of vertices and $\kappa'(G) > 2$.\footnote{Recall that $\kappa'(G)$ is the edge connectivity of $G$.} Prove that $G$ contains a perfect matching.

Problem 12 (F19A). Prove the following assertion: For every $n \geq 6$, the largest number of edges in an $n$-vertex graph without two edge-disjoint cycles is $n + 3$.

Problem 13 (F19B). Prove that the graph obtained from $K_n$ by deleting one edge has exactly $(n - 2)n^{n-3}$ spanning trees.

Problem 14 (F19B). Prove that every $n$-vertex triangle-free graph has chromatic number at most $2\sqrt{n}$.

Problem 15 (F19B). Suppose that $G$ is bipartite. Prove that $\chi(G^c) = \omega(G^c)$.\footnote{Recall that $G^c$ denotes the complement of $G$ and $\omega(H)$ is the largest order of a clique in $H$.}

Problem 16 (F19B). Suppose that $G$ is a regular graph with $n$ vertices. Show that either $G = K_n$ or the largest clique in $G$ has at most $n/2$ vertices.

Problem 17 (F18A). Suppose that a graph $G$ contains two edge-disjoint spanning trees. Show that $G$ contains a spanning Eulerian subgraph, that is, a spanning subgraph that has an Eulerian tour.
Problem 18 (F18A). Let $G$ be a bipartite graph with bipartition $V(G) = X \cup Y$ and fix some $A \subseteq X$ and $B \subseteq Y$. Suppose that $G$ contains a matching that covers every vertex of $A$ and also a matching that covers every vertex of $B$. Show that $G$ contains a matching that covers every vertex in $A \cup B$.

Problem 19 (F18A). Suppose that a graph $G$ does not contain three pairwise vertex-disjoint odd cycles. Show that $\chi(G) \leq 8$.

Problem 20 (F18A). Suppose that $G$ is a 2-connected simple plane graph. Prove that $G$ is bipartite if and only if the boundary of every face of $G$ is an even cycle.

Problem 21 (F18B). Let $G$ be a connected graph, let $T_1$ and $T_2$ be (the edge sets of) two spanning trees of $G$, and let $e \in T_1 \setminus T_2$. Show that

(a) There exists $f \in T_2 \setminus T_1$ such that $(T_1 \setminus \{e\}) \cup \{f\}$ is a spanning tree of $G$.

(b) There exists $f \in T_2 \setminus T_1$ such that $(T_2 \setminus \{f\}) \cup \{e\}$ is a spanning tree of $G$.

Problem 22 (F18B). Let $G$ be an $n$-vertex graph such that $2\delta(G) \leq n$. Show that $G$ contains a matching with $\delta(G)$ edges.

Problem 23 (F18B). Suppose that $G$ is a graph without isolated vertices. Let $\gamma(G)$ be the smallest number of edges in a spanning subgraph of $G$ that has no isolated vertices. Show that

$$\gamma(G) + \nu(G) = |V(G)|,$$

where $\nu(G)$ is the largest size of a matching in $G$.

Problem 24 (F18B). Suppose that $m$ and $n$ are positive integers such that $m - 1$ divides $n - 1$. Show that $R(T, K_{1,n}) = m + n - 1$ for every tree $T$ with $m$ vertices. Recall that $R(G, H)$ is the smallest integer $N$ such that every colouring of the edges of $K_N$ with red and blue contains either a subgraph isomorphic to $G$ whose all edges are red or a subgraph isomorphic to $H$ whose all edges are blue.

Problem 25 (F16A). Suppose that $G$ is a $k$-connected graph, where $k \geq 2$. Show that every set of $k$ vertices of $G$ lies on a common cycle.

Problem 26 (F16A). Show that every graph $G$ is a union of $\lceil \log_2 \chi(G) \rceil$ bipartite graphs.

Problem 27 (F16A). Let $G$ be a bipartite graph with $\delta(G) = \Delta(G) \geq 2$. Show that $\kappa(G) \neq 1$.

Problem 28 (F16A). Let $T$ be a tree with $2k$ leaves. Prove that $T$ contains $k$ pairwise edge-disjoint paths joining distinct leaves (so that each leaf is an endpoint of one of the paths).

Problem 29 (F16B). Let $G$ be a non-bipartite graph with $n$ vertices. Show that $G$ has an odd cycle of length at most $\max \{3, 2n/\delta(G)\}$.

Problem 30 (F16B). Let $G$ be a connected simple graph having neither $K_3$ nor $P_4$ (the path with four vertices) as an induced subgraph. Prove that $G$ is a complete bipartite graph.

Problem 31 (F16B). Prove or disprove the following statement: Every tree has at most one perfect matching.

Problem 32 (F16B). Recall that $R(G, H)$ denotes the smallest integer $n$ such that every red/blue-colouring of the edges of $K_n$ contains either a red copy of $G$ or a blue copy of $H$. Let $T$ be a tree with $t$ edges. Prove that $R(K_{s+1}, T) = st + 1$. 
Problem 33 (F17A). Show that every connected graph $G$ contains a spanning tree with at least $\Delta(G)$ leaves.

Problem 34 (F17A). Show that
\[ \text{ex}(n, C_5) \leq \left( \frac{n + 2}{2} \right)^2. \]
In other words, show that an $n$-vertex graph without a cycle of length 5 has at most $(n + 2)^2/4$ edges.

Problem 35 (F17A). Suppose that $v_1, \ldots, v_n$ are distinct unit (that is, of length 1) vectors in $\mathbb{R}^3$. Prove that there are at most $4n^{5/3}$ pairs $\{i, j\} \subseteq \{1, \ldots, n\}$ such that $v_i$ and $v_j$ are orthogonal.

Problem 36 (F17A). Without invoking the four-colour theorem, prove that every planar graph without a triangle is four-colourable.

Problem 37 (F17B). Let $k \geq 1$ be an integer and suppose that a connected graph $G$ has $2k$ vertices of odd degree. Prove that the edge set of $G$ can be partitioned into $k$ walks.

Problem 38 (F17B). Let $G$ be a 2-connected graph and let $x$ and $y$ be two distinct vertices of $G$. Suppose that each $z \in V(G) \setminus \{x, y\}$ has degree at least $k$. Prove that $G$ contains a path of length at least $k$ with endpoints $x$ and $y$.

Problem 39 (F17B). For every even integer $n$, determine the largest number of edges in an $n$-vertex graph that does not contain a perfect matching.

Problem 40 (F17B). Prove that the edge set of every graph with $n$ vertices can be covered by at most $\lceil n^2/4 \rceil$ edges and triangles.