**Problem 1.** Show that every two paths of maximum length in a connected graph have a vertex in common.

**Problem 2.** Prove: if for every edge $e$ of a connected graph $G$ there are two cycles $C_1, C_2$ in $G$ such that $E(C_1) \cap E(C_2) = \{e\}$, then $G$ is 3-edge-connected.

**Problem 3.** Let $k \geq 2$. Show that every $k$-connected graph with at least $2k$ vertices contains a cycle of length at least $2k$.

**Problem 4.** Let $G$ be a graph in which every pair of vertices has an odd number of common neighbors. Prove that $G$ is Eulerian.

**Problem 5.** Let $d$ be a positive integer. Show that every $2d$-regular connected graph $G$ with an even number of edges contains a spanning $d$-regular subgraph.

**Problem 6.** Let $G$ be a connected graph with $n$ vertices. Prove that $G$ contains a path of length $\min\{2\delta(G), n-1\}$.

**Problem 7.** A tournament is a complete graph in which each edge $uv$ is given a direction, either from $u$ to $v$ or from $v$ to $u$. Show that a tournament must contain a Hamilton path, that is, a directed path through all the vertices. Does it necessarily contain a Hamilton cycle?

**Problem 8.** Let $t(n, H_n)$ be the maximum number of edges in a graph $G$ on $n$ vertices not containing a Hamilton cycle $H_n$. Prove: $t(n, H_n) = \binom{n-1}{2} + 1$. (You need to prove both lower and upper bounds for $t(n, H_n)$.)

The exercises below are for you to practice — please do NOT submit their written solutions:

**Exercise 1.** Let $G$ be a graph and let $A \subseteq V(G)$. Let $H$ be the graph obtained from $G$ by adding to it a new vertex $v$ with $N_H(v) = A$. Show that $\kappa(H) \geq \min\{|A|, \kappa(G)\}$.

**Exercise 2.** Let $Q^d$ be the $d$-dimensional cube defined as follows: $V(Q^d) = \{0, 1\}^d$, $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in V(Q^d)$ are connected by an edge in $Q^d$ if and only if $x$ and $y$ differ in exactly one coordinate. Prove: $\kappa(Q^d) = \kappa'(Q^d) = d$.

**Exercise 3.** Let $G$ be a graph with all degrees even. Prove that the edges of $G$ can be oriented in such a way that every vertex of the resulting directed graph $\overrightarrow{G}$ has its outdegree equal to its indegree.

**Exercise 4.** Let $G$ be a graph of connectivity $\kappa(G)$ and with independence number $\alpha(G)$. Assume $\kappa(G) \geq \alpha(G) - 1$. Show that $G$ contains a Hamilton path.