Problem 1. Let $G$ be a bipartite graph with bipartition $V(G) = A \cup B$ having a perfect matching. Suppose that all degrees at side $A$ are at least $k$, for a positive integer $k$. Prove that $G$ contains at least $k!$ perfect matchings. (Hint: recall the two cases in the proof of Hall’s theorem.)

Problem 2. Prove the so-called defect version of Hall’s theorem: let $G$ be a bipartite graph with bipartition $V(G) = A \cup B$. Define

$$
\delta(A) = \max_{S \subseteq A} (|S| - |N(S)|).
$$

Prove that the maximal size of a matching in $G$ is $|A| - \delta(A)$.

Problem 3. Let $G$ be a connected graph with an even number of edges. Use Tutte’s theorem to prove that the set of edges of $G$ can be partitioned into pairwise disjoint paths of length 2.

Problem 4. Let $G$ be a graph on $n$ vertices. Prove that $\chi(G) \cdot \chi(\overline{G}) \geq n$.

Problem 5. Suppose that a graph $G$ is a union of $k$ forests. Prove that $\chi(G) \leq 2k$.

Problem 6. Let $G$ be a graph in which every edge belongs to at most $k$ cycles. Show that $\chi(G) \leq k + 2$.

Problem 7. Let $G = (V, E)$ be a graph of maximum degree $\Delta$. Prove that there is a $(\Delta + 1)$-edge coloring of $G$ in which every color class has $\left\lfloor \frac{|E|}{\Delta+1} \right\rfloor$ or $\left\lceil \frac{|E|}{\Delta+1} \right\rceil$ edges.

The exercises below are for you to practice — please do NOT submit their written solutions:

Exercise 1. A square matrix $A = (a_{ij})$ of nonnegative real numbers is called doubly stochastic if the entries of each row and each column sum up to 1, that is, for every $i$ and $j$,

$$
\sum_i a_{ij} = \sum_j a_{ij} = 1.
$$

A doubly stochastic matrix with all entries in $\{0, 1\}$ is called a permutation matrix. Prove the Burkhoff-von Neumann theorem, which states that every doubly stochastic matrix is a convex combination of permutation matrices.

Exercise 2. Deduce Hall’s theorem from König’s theorem.

Exercise 3. Suppose that every pair of odd cycles in a graph $G$ has a common vertex. Show that $\chi(G) \leq 5$.

Exercise 4. Show: $\chi'(K_{2n}) = 2n - 1$. 