0366.3267 Graph Theory

Fall Semester 2024

Homework assignment 1

Due date: Sunday, December 8, 2024

Problem 1. Prove that for every $n \geqslant 1$, the number of graphs with vertex set $\{1, \ldots, n\}$ and all degrees even is $2^{\binom{n-1}{2}}$.

Problem 2. Let $n \geqslant 7$. Show that every n-vertex graph with at least $5n - 14$ edges contains a subgraph with minimum degree 6.

Problem 3. Prove that every graph G with m edges admits a bipartition $V(G) = V_1 \cup V_2$ such that the number of edges of G crossing between V_1 and V_2 is at least $m/2$.

Problem 4. Let $k \geqslant 2, g \geqslant 3$ be integers, and let G be a graph of minimum degree k and girth g. Show that G contains a cycle of length at least $(g-2)(k-1)+2$.

Problem 5. Let T be a tree with k edges, and let G be a graph of minimum degree at least k. Prove: T is a subgraph of G.

Problem 6. Prove that the graph obtained from K_n by deleting one edge has exactly $(n-2)n^{n-3}$ spanning trees.

Problem 7. Compute the number of spanning trees of the complete bipartite graph $K_{m,n}$.

The exercices below are for you to practice — please do NOT submit their written solutions:

Exercise 1. Show that a graph is bipartite if and only if it contains no odd cycles.

Exercise 2. **(a)** Show that every graph with at least two vertices has two vertices of equal degree. **(b)** For every $n \geqslant 2$, construct an n-vertex graph G with exactly one pair of vertices of equal degree.

Exercise 3. Characterize all graphs G on $n \geqslant 3$ vertices such that for every $v \in V(G)$, the graph $G - v$ is a tree.

Exercise 4. Let $d_1, \ldots, d_n$ be positive integers. Prove that there exists a tree with degree sequence $d_1, \ldots, d_n$ if and only if

$$d_1 + \ldots + d_n = 2n - 2.$$

Exercise 5. Show that every tree with maximum degree Δ has at least Δ leaves.