0366.3267 Graph Theory

Fall Semester 2024

Homework assignment 2

Due date: Sunday, January 5, 2025

Problem 1. Show that every two paths of maximum length in a connected graph have a vertex in common.

Problem 2. Prove that a graph is 2-connected if and only if for any three vertices x, y, and z, there is a path from x to z that passes through y.

Problem 3. Let $k \geqslant 2$. Show that every k-connected graph with at least $2k$ vertices contains a cycle of length at least $2k$.

Problem 4. Let G be a graph in which every pair of vertices has an odd number of common neighbors. Prove that G is Eulerian.

Problem 5. Suppose that a graph G contains two edge-disjoint spanning trees. Show that G contains a spanning Eulerian subgraph, that is, a spanning subgraph that has an Euler tour.

Problem 6. Let G be a connected graph with n vertices. Prove that G contains a path of length $\min\{2\delta(G), n-1\}$.

Problem 7. A *tournament* is a complete graph in which each edge uv is given a direction, either from u to v or from v to u. Show that a tournament must contain a Hamilton path, that is, a directed path through all the vertices. Does it necessarily contain a Hamilton cycle?

Problem 8. Let $t(n, H_n)$ be the maximum number of edges in a graph G on n vertices not containing a Hamilton cycle H_n. Prove: $t(n, H_n) = \binom{n-1}{2} + 1$. (You need to prove both lower and upper bounds for $t(n, H_n)$.)

The exercices below are for you to practice — please do NOT submit their written solutions:

Exercise 1. For a graph $G = (V, E)$ and a positive integer k, let G^k be the graph with vertex set V, in which two vertices are connected by an edge if and only if their distance in G is at most k. Prove: if G is a connected graph on n vertices and $1 \leqslant k \leqslant n-1$ is an integer, then G^k is k-connected.

Exercise 2. Let Q^d be the d-dimensional cube defined as follows: $V(Q^d) = \{0,1\}^d$, $\mathbf{x} = (x_1, \ldots, x_d), \mathbf{y} = (y_1, \ldots, y_d) \in V(Q^d)$ are connected by an edge in Q^d if and only if $\mathbf{x}$ and $\mathbf{y}$ differ in exactly one coordinate. Prove: **(a)** $\kappa(Q^d) = \kappa'(Q^d) = d$. **(b)** Q^d is Hamiltonian for $d \geqslant 2$.

Exercise 3. Let d be a positive integer. Show that every $2d$-regular connected graph G with an even number of edges contains a spanning d-regular subgraph.

Exercise 4. Let G be a graph of connectivity $\kappa(G)$ and with independence number $\alpha(G)$. Assume $\kappa(G) \geqslant \alpha(G) - 1$. Show that G contains a Hamilton path.