0366.3267 Graph Theory

Fall Semester 2024

Homework assignment 3

Due date: Sunday January 26, 2025

Problem 1. Let $G = (A \cup B, E)$ be a bipartite graph with a matching saturating A. Prove that there exists a vertex $a \in A$ such that every edge e containing a belongs to a matching of size $|A|$ in G. (**Hint:** recall the two cases in the proof of Hall's theorem.)

Problem 2. Prove the so-called defect version of Hall's theorem: let G be a bipartite graph with bipartition $V(G) = A \cup B$. Define

$$\delta(A) = \max_{S \subseteq A} \big(|S| - |N(S)|\big).$$

Prove that the maximal size of a matching in G is $|A| - \delta(A)$.

Problem 3. Let G be a connected graph with an even number of edges. Use Tutte's theorem to prove that the set of edges of G can be partitioned into pairwise disjoint paths of length 2.

Problem 4. Let G be a connected graph on n vertices with $\chi(G) = k$. Show that $|E(G)| \geqslant \binom{k}{2} + n - k$.

Problem 5. Suppose that a graph G is a union of k forests. Prove that $\chi(G) \leqslant 2k$.

Problem 6. Suppose that every pair of odd cycles in a graph G has a common vertex. Show that $\chi(G) \leqslant 5$.

Problem 7. Let $G = (V, E)$ be a graph of maximum degree Δ, and let $v \in V$ be a vertex of degree Δ in G. Form a graph G' by adding a new vertex u and an edge (u, v) to G. Find the chromatic index $\chi'(G')$. (Give arguments justifying your claim.)

The exercices below are for you to practice — please do NOT submit their written solutions:

Exercise 1. A square matrix $A = (a_{ij})$ of nonnegative real numbers is called *doubly stochastic* if the entries of each row and each column sum up to 1, that is, for every i and j,

$$\sum_i a_{ij} = \sum_j a_{ij} = 1.$$

A doubly stochastic matrix with all entries in $\{0, 1\}$ is called a *permutation matrix*. Prove the *Birkhoff–von Neumann theorem*, which states that every doubly stochastic matrix is a convex combination of permutation matrices.

Exercise 2. Deduce Hall's theorem from König's theorem.

Exercise 3. Let G be a graph in which every edge belongs to at most k cycles. Show that $\chi(G) \leqslant k + 2$.

Exercise 4. Show: $\chi'(K_{2n}) = 2n - 1$.