Random Graphs 0366-4767
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Homework 3
Due: Jan. 2, 2011

1. Notation: We write $G \to H$ if every Red-Blue coloring of the edges of G contains a monochromatic copy of H. Now, the *size Ramsey number* of a graph H, denoted by $\hat{r}(H)$, is the minimal m for which there exists a graph G with m edges such that $G \to H$. For example, $K_6 \to K_3$, $K_5 \not\to K_3$, $\hat{r}(K_3) \leq 15$.

The aim of this (guided) exercise is to prove the following theorem of Beck (1983). Let P_k be a path of length k. Then:

$$\hat{r}(P_k) = O(k) \ .$$

(a) Prove that a graph G with average degree at least d contains a subgraph G' with minimum degree at least $d/2$.
(b) Let $C > 0$ be a sufficiently large constant. Prove that **whp** the random graph $G \sim G(n, p = C/n)$ is such that every subgraph $G_0 \subseteq G$ with $\delta(G_0) \geq C/5$ is an $(n/1000, 2)$-expander.
(c) Prove Beck's theorem.
Hint. Take $G(n, C/n)$, and for any 2-coloring of $E(G)$ consider the majority color. Use the theorem from the class asserting that a $(k, 2)$-expander has a path of length $3k - 1$.
2. Prove that for any $\epsilon > 0$ there exists $C > 0$ such that the random graph $G \sim G(n, C/n)$ has **whp** the following property: for every pair of disjoint sets $A, B \subset [n]$ of cardinality $|A| = |B| = \epsilon n$, there is a path $P = (v_0, \ldots, v_l)$ in G of length $l \geq (1 - 3\epsilon)n$ such that $V(P) \cap A = \{v_0\}$, $V(P) \cap B = \{v_l\}$.
3. Let $k \geq 1$ be an integer, and let $\omega(n)$ be any function tending to infinity arbitrarily slowly with n. Prove that for

$$p(n) = \frac{\ln n + (k-1)\ln\ln n - \omega(n)}{n}$$

the random graph $G \sim G(n, p)$ satisfies **whp**: $\delta(G) \leq k - 1$, while for

$$p(n) = \frac{\ln n + (k-1)\ln\ln n + \omega(n)}{n}$$

the random graph $G \sim G(n, p)$ satisfies **whp**: $\delta(G) \geq k$.
4. Let $3 \leq k(n) \leq n$ be a given integer valued function of n. Prove that **whp** the random graph $G \sim G(n, p)$ with $p(n) = \frac{\ln n + \ln\ln n + \omega(n)}{n}$ contains a cycle of length k.
Hint. Hamiltonicity is (obviously) the key.