Graph and Hypergraph Coloring (0366-4817)

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The following notes are based on the lectures of a course on graph and hypergraph coloring, given by Prof. Michael Krivelevich at the School of Mathematical Sciences, Tel Aviv University, Spring 2022. Despite our best efforts, there are probably still some typos and mistakes left. Hence, corrections and other feedback would be greatly appreciated and can be sent to krivelev@tauex.tau.ac.il.

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1 Introduction and Basic Notions and Definitions

The course will deal with various aspects of graph and hypergraph coloring, presenting the most important and fundamental results on this subject. We start with some basic definitions.

1.1 Vertex Coloring

Definition 1.1. *Let $G = (V, E)$ be a (simple) graph. A function $f : V \to [k]$ is called a k-*coloring *of G if $f(u) \neq f(v)$ for every $(u, v) \in E$.*

Definition 1.2. *A graph G is said to be k-*colorable *if it has a k-coloring.*

Definition 1.3. *The* chromatic number *of a graph G, denoted by $\chi(G)$, is the smallest k such that G is k-colorable.*

Remark 1.4. *If $f : V \to [k]$ is a k-coloring of a graph G, then for every $1 \leq i \leq k$, the set $V_i = \{v \in V \mid f(v) = i\}$ is* independent.

An easy consequence of the above remark is that G is k-colorable if and only if there is a partition of its vertex set V into k independent sets.

Definition 1.5. *A graph G is called* bipartite *if $\chi(G) \leq 2$.*

Theorem 1. *A graph G is bipartite if and only if it does not contain an odd cycle.*

1.2 Edge Coloring

Definition 1.6. *Let $G = (V, E)$ be a graph (or a multigraph). A function $f : E \to [k]$ is called a k-*edge-coloring *of G if $f(e_1) \neq f(e_2)$ for every $e_1, e_2 \in E$ such that $e_1 \cap e_2 \neq \emptyset$.*

Definition 1.7. *A graph G is said to be k-*edge-colorable *if it has a k-edge-coloring.*

Definition 1.8. *The* chromatic index *of a graph G, denoted by $\chi'(G)$, is the smallest k such that G is k-edge-colorable.*

Example:

$$\chi'(K_n) = \begin{cases} n-1 & , \ n \text{ even} \\ n & , \ n \text{ odd} \end{cases} .$$

Remark 1.9. *If $f : E \to [k]$ is a k-edge-coloring of a graph G, then for every $1 \leq i \leq k$, the set $E_i = \{e \in E \mid f(e) = i\}$ is a* matching.

Going back to the previous example, as the size of the maximum matching in K_{2n+1} is $\nu(K_{2n+1}) = n$, we have that $\chi'(K_{2n+1}) \geq \binom{2n+1}{2}/n = 2n + 1$. (We point out that one can think of an edge-coloring of K_n as a representation of the pairings of an n-player round-robin tournament (i.e., where every player plays against all others). Then, for even n, every round (in some paring system) is a matching of size $n/2$ and there are $n - 1$ rounds, while for odd n, we will have an extra round (as in each round one player is free).)

2 Coloring Infinite Graphs

Throughout this course we will almost always consider only finite graphs. The main reason/justification for this is the following theorem of Erdős and De Bruijn.

Theorem 2 (Erdős–De Bruijn '48)**.** *Let $k \in \mathbb{N}$ and G be an infinite graph. Then, if every finite $G_0 \subsetneq G$ is k-colorable then $\chi(G) \leq k$.*

We will see two proofs: for the case where $V(G)$ is countable, and for the general case.

Proof 1. We assume that $|V(G)| = \aleph_0$. We would use the following well-known lemma.

Lemma 2.1 (Kőnig's Infinity Lemma '27)**.** *Let G be an infinite graph such that $V(G) = V_0 \dot\cup V_1 \dot\cup V_2 \dot\cup \dots$, where every V_i is finite and non-empty. In addition, for every $i \geq 1$ and every $v \in V_i$ there exists $u \in V_{i-1}$ such that $\{u, v\} \in E(G)$. Then, G contains an infinite ray $(v_0, v_1, v_2, \dots)$ such that $v_i \in V_i$ and $\{v_i, v_{i+1}\} \in E(G)$ for all $i \geq 0$*

Proof. From the assumptions of the lemma we can conclude that for every $i \geq 0$ and every $v \in V_i$ there exists a (finite) path $(v, u_{i-1}, \dots, u_0)$ such that $u_j \in V_j$. Let P be the set of all such paths (i.e., for every $v \in V(G)$). As G is infinite, we get that P is infinite. Thus, as V_0 is finite, by the infinite pigeonhole principle, there exists $v_0 \in V_0$ such that infinitely many paths from P start at v_0, and let us denote them by P_0. As P_0 is infinite and V_1 is finite, there exists $v_1 \in V_1$ which belongs to infinitely many paths from P_0, and we will denote them by P_1. Continuing in this manner, we will get an infinite ray $(v_0, v_1, v_2, \dots)$, as required. ■

We can now prove the theorem. As $V(G)$ is countable, we can enumerate its vertices by $(v_0, v_1, v_2, \dots)$. We denote $G_n = G[\{v_0, \dots, v_n\}]$, and let C_n be the set of all k-colorings of G_n. We now define an auxiliary graph Γ as follows. We have $V(\Gamma) = V_0 \dot\cup V_1 \dot\cup V_2 \dot\cup \dots$ such that $V_i = C_i$ (i.e., every vertex of V_i corresponds to a k-coloring of G_i). By our assumption that every finite subgraph of G is k-colorable, we get that V_i is non-empty and it is clearly finite. Note that for $i < j$ and $c \in V_j$, the restriction of c to G_i is in V_i. Therefore, we will have an edge $\{c, c'\}$ in Γ for $c \in V_i$ and $c' \in V_{i-1}$ if c' is a restriction of c to G_{i-1}. Now, as we have mentioned before, every V_i is finite and non-empty, and also it is easy to see that every $v \in V_i$ has a neighbor in V_{i-1}. Thus, the conditions of Lemma 2.1 are satisfied, and so we have an infinite ray $(c_0, c_1, \dots)$ in Γ. This ray defines a (legal) k-coloring c of G, by $c(v_n) = c_n(v_n)$ for $n \geq 0$, and so G is k-colorable. ■

We now give a general proof for Theorem 2.

Proof 2. Assume for contradiction that G is not k-colorable. Let G be a *maximal*[1] counterexample. Then:

1. $\chi(G) > k$.
2. For every finite $G_0 \subsetneq G$, $\chi(G_0) \leq k$.
3. For every $e \notin E(G)$ there exists a finite subgraph $G(e) \subsetneq G$ such that $G(e) + e$ is *not* k-colorable (we note that both vertices of e must be in $G(e)$).

Claim 2.2. *The non-adjacency relation in G is an equivalence relation.*

[1] Here we implicitly use Zorn's lemma (or, equivalently, the axiom of choice).

Proof. The reflexivity and the symmetry are obvious (as G is simple and undirected), and so it is left to prove transitivity. That is, we need to prove that if $\{u,v\},\{v,w\} \notin G$ then also $\{u,w\} \notin G$. Due to the maximality of G there exist:

- Finite $G_1 \subsetneq G$ such that $\chi(G_1 + \{u,v\}) > k$.
- Finite $G_2 \subsetneq G$ such that $\chi(G_2 + \{v,w\}) > k$.

Let us look on $H = G_1 \cup G_2 \cup \{u,v\}$. This is a finite graph and let us assume that $\chi(H) \leq k$. Let c be a k-coloring of H. We have that $c(u) = c(v)$, as otherwise c will be a legal k-coloring of $G_1 + \{u,v\}$. Similarly, we have that $c(v) = c(w)$, as otherwise c will be a legal k-coloring of $G_2 + \{v,w\}$. Thus, $c(u) = c(w)$ and so $\{u,w\} \notin G$, and the claim follows. ■

From the claim we get that the non-adjacency relation induces a partition of $V(G)$ into equivalence classes, every one of which is an independent set. Denote by I the collection of the equivalence classes (i.e., the quotient set). We denote the vertices of an equivalence class $\alpha \in I$ by V_α. As $\chi(G) > k$, clearly $|I| > k$. Let us choose $k+1$ equivalence classes $V_1, \ldots, V_{k+1}$, and let $v_i \in V_i$ be their representatives. We get that all the vertices $\{v_1, \ldots, v_{k+1}\}$ are neighbors, and so form a $(k+1)$-clique. Thus, we have that $K_{k+1} \subsetneq G$, and as $\chi(K_{k+1}) = k+1 > k$, we get a contradiction to assumption (2) on G. ■

From now on we will discuss only finite graphs (unless stated explicitly otherwise).

3 Simple Bounds on the Chromatic Number

1. If $G_1 \subset G_2$ then $\chi(G_1) \leq \chi(G_2)$.
2. $\omega(G) \leq \chi(G)$.
   Remark: There is no function $f : \mathbb{N} \to \mathbb{N}$ such that for all G, $\chi(G) \leq f(\omega(G))$.
3. For all G, $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$.
   Proof: Assume that $\chi(G) = k$. Then there is a partition of $V(G)$ into $V_1, \ldots, V_k$ such that every V_i is an independent set. Thus, $|V_i| \leq \alpha(G)$ for all i. We have that $|V(G)| = \sum_{i=1}^{k} |V_i| \leq k \cdot \alpha(G)$, and so $k \geq |V(G)|/\alpha(G)$.

We also have the following theorem, that connects the chromatic number of a graph and its complement.

Theorem 3 (Nordhaus–Gaddum '56)**.** *For every graph G on n vertices we have*

$$\chi(G) + \chi(\overline{G}) \leq n + 1.$$

Moreover, the bound is tight, e.g., for $G = K_n$.

Proof. The proof will be by induction on n.
Base: $n = 1$ - trivial.
Step: We assume that the claim is true for graphs on $r \leq n$ vertices. Let G be a graph on $n+1$ vertices, and let $v \in V(G)$ be some vertex. We denote $G' = G - \{v\}$. Denote $k = \chi(G'), \ell = \chi(\overline{G'})$. By the assumption, $k + \ell \leq n + 1$. We have that $\chi(G) \leq \chi(G') + 1 = k + 1$ and $\chi(\overline{G}) \leq \chi(\overline{G'}) + 1 = \ell + 1$. We now consider two cases. If $k + \ell < n + 1$ then $\chi(G) + \chi(\overline{G}) \leq k + 1 + \ell + 1 = k + \ell + 2 \leq n + 2$, as required. So it is left to prove the case $k + \ell = n + 1$. As $|V(G)| = n + 1$ we have that $d_G(v) + d_{\overline{G}}(v) = n$. As we assumed that $k + \ell = n + 1$ we get that $d_G(v) < k$ or $d_{\overline{G}}(v) < \ell$. Now, if $d_G(v) < k$ then there exists a

color $1 \le i \le k$ such that v has *no* neighbor colored i (in the k-coloring of G'), and thus we can color v by color i, which implies that $\chi(G) = k$, and so we get $\chi(G) + \chi(\overline{G}) \le k + \ell + 1 = n + 2$. The case of $d_{\overline{G}}(v) < \ell$ is symmetric. ■

Question: What can we say about $\chi(G) \cdot \chi(\overline{G'})$?
We can see that $\chi(G) \cdot \chi(\overline{G'}) \ge n$ from Theorem 3, and we can also derive it directly. Assume that $\chi(G) = k$. Then we have $\alpha(G) \ge n/k$, and thus $\omega(\overline{G}) \ge n/k$ and so $\chi(\overline{G}) \ge n/k$, and thus $\chi(G) \cdot \chi(\overline{G'}) \ge n$.

4 The Gallai–Hasse–Roy–Vitaver Theorem

Theorem 4 (Gallai–Hasse–Roy–Vitaver 1960s)**.** *Let G be a graph and D be an orientation of the edges of G. Then, $\chi(G) \le 1 + \ell(D)$ where $\ell(D)$ is the length of a longest directed path in D (where the length of a path is its number of edges). Moreover, there exists an orientation D such that $\chi(G) = 1 + \ell(D)$.*

Proof. Let D' be a maximal (with respect to inclusion) acyclic subgraph (i.e., that has no directed cycles) of D. Note that $V(D') = V(D) = V(G)$. If $\overrightarrow{(u,v)} \in E(D) \setminus E(D')$ it means that it closes a directed cycle in D', which implies that there is a directed path from v to u in D'. We define $f : V(G) \to [1 + \ell(D)]$ as follows: $f(v)$ will have the value of 1 + the length of a longest path in D' that ends at v. Let $P = (v_1, v_2, \ldots, v_t)$ be a directed path in D', and let P' be some path that ends at v_1. As D' is acyclic, we have that $V(P) \cap V(P') = \{v_1\}$. We have that for all $2 \le i \le t$, $f(v_1) < f(v_i)$, as we can append to every path P' that ends at v_1 the prefix $(v_2, \ldots, v_t)$ of P. We conclude that the function f is monotonically (strictly) increasing along every path. We will now show that f is a legal $(1 + \ell(D))$-coloring of G. Let $(u, v) \in E(D)$. If $(u, v) \in E(D')$ then, as along every path in D' the function f is monotonically (strictly) increasing, we have that $f(u) \ne f(v)$. Otherwise, if $(u, v) \notin E(D')$ then, as we said before, there is a path P from v to u in D', and thus we again have $f(u) \ne f(v)$. Thus, f is a legal $(1 + \ell(D))$-coloring, and so $\chi(G) \le 1 + \ell(D))$. It is left to prove that this is tight. Assume that $\chi(G) = k$, and let $(V_1, \ldots, V_k)$ be a k-coloring of G. We will orient the edges of G as follows: for every $1 \le i < j \le k$, if $u \in V_i, v \in V_j$ and $\{u, v\} \in E(G)$, then we will orient this edge from v to u (i.e., $\overrightarrow{(v,u)}$). Denote this orientation by D'. Now, clearly all paths in D' have length at most $k - 1$, and thus $1 + \ell(D') \le k = \chi(G)$, which completes the proof. ■

5 Degrees and Coloring

5.1 Coloring and Degeneracy

Definition 5.1. *Let G be a graph and let d be a non-negative integer. G is d-*degenerate *if every subgraph $G_0 \subseteq G$ contains a vertex of degree at most d. Equivalently, G is d-*degenerate *if and only if for every $V_0 \subseteq V(G)$ we have $\delta(G[V_0]) \le d$.*

We note that it trivially holds that if $\Delta(G) = d$ then G is d-degenerate.

Definition 5.2. *The* degeneracy *of a graph G, denoted by* degen(G)*, is the smallest value of d for which G is d-degenerate.*

Examples:

1. For $G = K_n$ we have $\text{degen}(G) = n - 1$.
2. For $G = K_4 \dot\cup K_3 \dot\cup K_2 \dot\cup K_1$ we have that $\text{degen}(G) = 3$.
3. For $G = K_{1,t}$ (i.e., a star) we have that $\text{degen}(G) = 1$.

As we have mentioned before, $\text{degen}(G) \le \Delta(G)$.

Claim 5.3. *A graph G is d-degenerate if and only if there exists an ordering $\sigma = (v_1, \dots, v_n)$ of $V(G)$ such that for every $1 \le i \le n$ it holds that v_i has at most d neighbors in $\{v_1, \dots, v_{i-1}\}$ (i.e., that appear before v_i in σ).*

Proof. $(\Longrightarrow)$: Assume that G is d-degenerate. We will define σ vertex by vertex (inductively), starting from v_n and going backwards. Assume that we have already defined the vertices $v_{i+1}, \dots, v_n$ and now we want to choose v_i. Let $V_i = V(G) \setminus \{v_{i+1}, \dots, v_n\}$. As G is d-degenerate, $G[V_i]$ contains a vertex $v \in V_i$ of degree at most d (this is also true for $V_n = V(G)$). We define $v_i = v$. By our construction, v_i will have at most d neighbors among the vertices which precede it in σ (i.e., $V_i \setminus \{v\}$). Continuing in this manner, we obtain the desired ordering.
$(\Longleftarrow)$: Assume that we have an ordering of the vertices $\sigma = (v_1, \dots, v_n)$ as stated in the claim. Let $\emptyset \ne V_0 \subseteq V(G)$, and let $v \in V_0$ be the last vertex of V_0 according to σ. Then, by the definition of σ, v has at most d neighbors among the vertices which precede it, and thus at most d neighbors in V_0. Therefore, $\delta(G[V_0]) \le d$, as required. ■

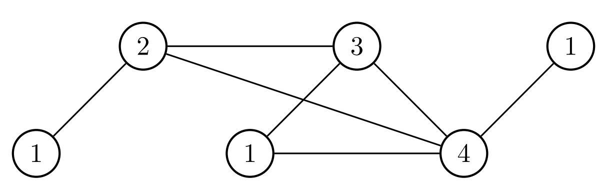
Definition 5.4. *If G is a d-degenerate graph and σ is an ordering of $V(G)$ such that every vertex v has at most d neighbors among the vertices which precede it in σ (exists by Claim 5.3), then σ is called a* d-degenerate ordering *(or a* degeneracy ordering*).*

Connection to coloring: Let G be a graph. We color $V(G)$ vertex by vertex.
Question: In which order should we color? Answer: We may try the degeneracy ordering.

5.2 Greedy Coloring

Definition 5.5. *Given a graph G and an ordering σ of $V(G)$, the* greedy coloring *colors G according to σ as follows: if $\sigma = (v_1, \dots, v_n)$, then we color according to σ such that v_i gets the least color which does not appear on its neighbors (we may assume that the colors are represented by positive integers).*

Example:



In the example above, we color the graph greedily according to the ordering of the vertices from left to right. This gives a 4-coloring of the graph, as can be seen in the figure. However, it is easy to check that the chromatic number of the above graph is 3.

Remarks:

1. For every graph G there exists an ordering σ of $V(G)$ such that the greedy coloring according to σ colors G in exactly $\chi(G)$ colors. To see this, we assume that $\chi(G) = k$ and denote by $V_1, \ldots, V_k$ the color classes of a k-coloring of G. Now, if a vertex in V_i for $i > 1$ has no neighbors in V_1 then we move it to V_1; if a vertex in V_i for $i > 2$ has no neighbors in V_2 then we move it to V_2, etc. It is easy to see that after this process we still get a legal coloring, such that every vertex in V_2 has a neighbor in V_1, every vertex in V_3 has neighbors both in V_1 and V_2, etc. Now we can color the graph greedily in k colors using the following order: we first color the vertices of V_1 (in some arbitrary order), then we color the vertices of V_2 (in some arbitrary order), etc. It is easy to see that this corresponds to the definition of a greedy coloring.

2. It might be the case that σ was chosen in such a way, that the graph will be colored in much more colors than $\chi(G)$. For example: $G = K_{n,n} - M$, where M is a perfect matching. Denoting $V(G) = X \dot{\cup} Y$ with $X = \{u_1, \ldots, u_n\}$ and $Y = \{v_1, \ldots, v_n\}$, and taking $M = \{\{u_i, v_i\} \mid i \in [n]\}$, we have that $\{u_i, v_i\} \notin E(G)$ for all i. Then we choose the order $\sigma = (u_1, v_1, u_2, v_2, \ldots, u_n, v_n)$. It is easy to see that the result of coloring the vertices greedily according to σ gives $\mathrm{col}(u_i) = \mathrm{col}(v_i) = i$ for all i. Thus, the graph will be colored by n colors using the greedy coloring according to σ, while its chromatic number is 2.

We will prove next lesson that $\chi(G) \leq 1 + \Delta(G)$.

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Lecture 2

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1 Greedy Coloring

We present a more general setup for a greedy coloring. Let $G = (V, E)$ be a graph and $V_0 \subseteq V$. Let $c_0 : G[V_0] \to [k]$ be a legal coloring of $G[V_0]$, and let $\sigma = (v_1, \ldots, v_t)$ be a permutation of $V \setminus V_0$. We shall extend c_0 to a legal coloring of G by applying a greedy coloring according to the order given by σ.

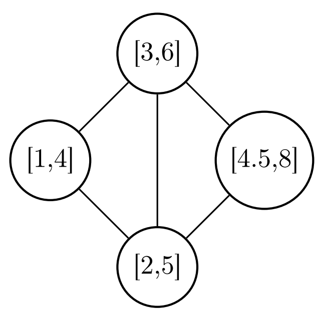
We now present several examples of using greedy coloring.

1.1 Interval Graphs

Definition 1.1. *Let $\mathcal{I} = \{I_1, \ldots, I_n\}$ be a collection of closed intervals in $\mathbb{R}$. We define the* interval graph *$G = G(\mathcal{I})$ as follows:*

1. *The vertex set V consists of the intervals in $\mathcal{I}$.*
2. *The edge set E is defined as: $\{I_i, I_j\} \in E$ if and only if $I_i \cap I_j \neq \emptyset$.*

Example: Let $\mathcal{I} = \{[1, 4], [2, 5], [3, 6], [4.5, 8]\}$. Then the graph $G(\mathcal{I})$ is



Theorem 1. *Let $G = G(\mathcal{I})$ be an interval graph. Then $\chi(G) = \omega(G)$.*

Proof. Let us order the intervals in $\mathcal{I}$ in the ascending order according to their left endpoints (if there are several intervals with the same left endpoint, we will order between them arbitrarily). Denote the resulting order by σ. Now we will color the vertices of G using the greedy coloring according to σ. Let us assume that the largest color used by this coloring is k, and let $I \in \mathcal{I}$ be an interval that got color k. Denote $I = [a, b]$. As I got color k in the greedy coloring, there are intervals $I_1, \ldots, I_{k-1}$ in G that got colors $1, \ldots, k-1$, respectively, such that all of them intersect with I (i.e., its neighbors in G). Hence, by the definition of σ, all the intervals $I_1, \ldots, I_{k-1}$ contain a. Thus, all the intervals $I, I_1, \ldots, I_{k-1}$ intersect with one another (in a), and so form a k-clique. Therefore, $\chi(G) \leq \omega(G)$. The opposite inequality, $\chi(G) \geq \omega(G)$, holds trivially. ■

1.2 Coloring and Degeneracy

Reminder: A graph G is *d-degenerate* if for every $V_0 \subseteq V$ we have $\delta(G[V_0]) \leq d$. The *degeneracy* of G, denoted by $\text{degen}(G)$, is the smallest d for which G is d-degenerate.
We proved: G is d-degenerate if and only if there exists an ordering $\sigma = (v_1, \ldots, v_n)$ of $V(G)$ such that for every $1 \leq i \leq n$ it holds that v_i has at most d neighbors that appear before v_i in σ (called a *d-degenerate order*).
Obviously, $\text{degen}(G) \leq \Delta(G)$.

Theorem 2 (Szekeres-Wilf '68)**.** *For every graph G, it holds that $\chi(G) \leq 1 + \text{degen}(G)$.*

Proof. Assume that G is d-degenerate, and let σ be a d-degenerate order. We will color G greedily according to σ. As every $v \in V(G)$ has at most d neighbors that precede it in σ, when we get to color v, there is at least one color from $[d+1]$ that is left for v (i.e., not used by any of its neighbors). Thus, we will not use more than $d+1$ colors, and so $\chi(G) \leq 1 + \text{degen}(G)$. ■

Corollary 3. $\chi(G) \leq 1 + \Delta(G)$.

2 Brooks' Theorem

Question: Is the bound in Corollary 3 tight?
The answer is yes, e.g., K_n, C_{2n+1}.
We note that it is enough to consider only connected graphs, as $\chi(G) = \max\{\chi(G_i)\}$, where G_i's are the connected components of G.

Remark 2.1. *If $\Delta(G) = 2$, then G is a collection of cycles and paths, and thus $\chi(G) = 3$ if G contains an odd cycle, and otherwise $\chi(G) = 2$.*

Theorem 4 (Brooks '41)**.** *Let $d \geq 3$ and let G be a connected graph with $\Delta(G) \leq d$. If $K_{d+1} \not\subseteq G$, then $\chi(G) \leq d$.*

Proof. (a variant of the proof by Zając '18) The proof will be by induction on $n = |V(G)|$.
Base: $n \leq d$ – G can be colored by $\leq d$ colors, trivially.
Step: We assume that the claim is true for graphs on $< n$ vertices. Let G be a graph on n vertices. If there exists $v \in V(G)$ such that $d_G(v) < d$, then let us look on $G' = G - \{v\}$. By induction, we can color every connected component of G' by at most d colors. We will now color v. As v has $d_G(v) < d$ colored neighbors, at least one of the d colors is left for v, and so we can color v with this color, extending the coloring of G' to a legal d-coloring of G. Thus, we can assume that G is d-regular. We first handle the following special case: G contains a cycle C that is *not* Hamiltonian and such that there is a vertex $v \in V(C)$ which has *no* neighbors outside of C. As G is connected, we can find two consecutive vertices u, v along C such that v has no neighbors outside of C while u has a neighbor $w \notin V(C)$. By induction, we can color $G[V \setminus V(C)]$ with d colors. Denote by c_0 such a coloring, and assume, without loss of generality, that $c_0(w) = 1$. We will now "walk" along C from v to u (i.e., not using the edge between them), and denote the order of the vertices along this walk by σ. In particular, $\sigma = (v_1, \ldots, v_k)$ where $|V(C)| = k$, $v_1 = v$, $v_k = u$. We will now extend the coloring c_0 to a proper coloring of G, by coloring greedily the vertices of $V(C)$ according to σ. Due to the fact that $v = v_1$ has no neighbors outside of C, v will get color 1. Now, every vertex $v_i \neq v_1, v_k$ has a neighbor that appears after it in σ (and thus at most $d-1$ of its neighbors were already colored when we get to color v_i), and so we can color v_i by one of the d colors (which was not used on its neighbors). When we get to color $v_k = u$, we observe

that v_k has two neighbors, v, w, that both got color 1. Therefore, we can find a non-used color out of the d colors in which we can color u.

We now turn to prove the general case. As G is connected, d-regular ($d \geq 3$) and $G \neq K_{d+1}$, we have that the adjacency relationship in G is *not* transitive. Therefore, there exist $v_1, v_2, v_3 \in V(G)$ such that $(v_1, v_2), (v_2, v_3) \in E(G)$ but $(v_1, v_3) \notin E(G)$. Let $P = (v_1, v_2, v_3, \ldots, v_k)$ be a maximal path (i.e., not contained in a longer path) that starts with v_1, v_2, v_3. From the maximality of P we have that all neighbors of v_k belong to P. We consider two cases:
Case 1: $k < n$: Let v_i be the farthest vertex from v_k along P which is a neighbor of v_k. Then the cycle $C = (v_i, v_{i+1}, \ldots, v_k, v_i)$ is a non-Hamiltonian cycle in G that contains all the neighbors of v_k. Thus, by the previous (special) case we have addressed, G is d-colorable.
Case 2: $k = n$: $P = (v_1, v_2, v_3, \ldots, v_n)$ is a Hamiltonian path. Let $v_j \neq v_1, v_3$ be a neighbor of v_2 in P (here we use the assumption that $d \geq 3$). We will now color G greedily according to the following order σ on $V(P) = V(G)$: $\sigma = (v_1, v_3, v_4, \ldots, v_{j-1}, v_n, v_{n-1}, \ldots, v_j, v_2)$. We observe that every vertex $x \neq v_2$ has a neighbor that appears after it in σ, and so we will be able to color x with one of the d colors. Moreover, v_1, v_3 will get the same color 1 (as they are not neighbors). Now, when we get to color v_2, we observe that in spite of the fact that all of its d neighbors were already colored, two of them, namely v_1 and v_3, got the same color (color 1), and so there is a color that is left for v_2 out of the d colors. Thus, we can color v_2, and therefore complete a proper d-coloring of G, implying that $\chi(G) \leq d$. ■

3 Equitable Coloring

We saw that $\chi(G) \leq 1 + \Delta(G)$ for every graph G.
Question: Can we guarantee the existence of "special" colorings in $\Delta + 1$ colors?

Definition 3.1. *Given a graph G, a k-coloring of G with color classes $V_1, \ldots, V_k$ (where the vertices in V_i are colored by color i) is called* equitable *if $\big||V_i| - |V_j|\big| \leq 1$ for all $1 \leq i \neq j \leq k$.*

Question (Erdős '64): Does every graph G with $\Delta(G) \leq r$ have an equitable $(r+1)$-coloring?

Theorem 5 (Hajnal-Szemerédi '70)**.** *For every graph G with $\Delta(G) \leq r$, there exists an equitable $(r+1)$-coloring.*

Definition 3.2. *Let $k \geq 2$ be an integer. We say that a graph G on n vertices (where $k|n$) contains a* k-factor *if it contains n/k vertex-disjoint copies of K_k.*

Corollary 6 (from Theorem 5)**.** *Let $n, k \geq 2$ be integers such that $k|n$. If G is a graph on n vertices with $\delta(G) \geq \frac{k-1}{k}n$, then G contains a k-factor.*

Proof. Let us look on the complement graph $\overline{G}$ of G. We have that $\Delta(\overline{G}) \leq n - 1 - \frac{k-1}{k}n = n/k - 1$. By Theorem 5, $\overline{G}$ has an equitable $\frac{n}{k}$-coloring. Now, as n/k divides n, all color classes have equal size of k, and each of them is an independent set in $\overline{G}$ and thus a clique in G. Therefore, G contains n/k vertex-disjoint copies of K_k, and so contains a k-factor. ■

Remark 3.3. *Can we have Brooks' theorem analogue for equitable colorings (i.e., equitable Δ-colorings)? The answer is no, as can be seen in the following example. Let $G = K_{r,r}$ where r is odd. Clearly, $\Delta(G) = r$. Now, we claim that G does not have an equitable r-coloring. Indeed, as r divides $2r$, we know that every color class must be of size 2. As every color class is an independent set, both vertices of every color class must belong to one side of the graph. As r is odd, the vertices of each side cannot be partitioned into pairs, and thus G does not have an equitable r-coloring.*

Proof of Theorem 5. (Kierstead-Kostochka '08) The proof will be by induction on $|E(G)|$.
Base: $|E(G)| = 0$ – the claim is trivial.
Step: We first observe that we can assume that $r+1$ divides $n = |V(G)|$. Indeed, if $n = (r+1)s - p$ where $1 \leq p \leq r$, we will add to G a clique K_p on a new set of vertices (i.e., disjoint from $V(G)$) and denote the resulting graph by G'. Observe that $\Delta(G') \leq r$. Now, if f is an equitable $(r+1)$-coloring of G', then f colors all the vertices of the clique K_p in different colors, and thus the restriction of f to $V(G)$ (i.e., $f|_{V(G)}$) is an equitable $(r+1)$-coloring of G. So, from now on we assume that $n = (r+1)s$. We now continue with the inductive step. Let $e = (x, y) \in E(G)$ and denote $G' = G - \{e\}$. By induction, G' has an equitable $(r+1)$-coloring $f_0 = (V_1, \ldots, V_{r+1})$. By our assumption, $|V_i| = s$ for every i. We now put back the edge e. If $e \not\subseteq V_i$ for every $1 \leq i \leq r+1$, then f_0 is also an equitable $(r+1)$-coloring of G. Otherwise, there exists a color class V_i such that $e \subseteq V_i$. As $d_G(x) \leq r$ and x has a neighbor in V_i, we conclude that there exists a color class V_j, $1 \leq i \neq j \leq r+1$, such that $d_{V_j}(x) = 0$. We move vertex x to V_j, and get a proper $(r+1)$-coloring which is **nearly-equitable**: every color class, except of two, has size exactly s, one class V^+ has size $s+1$ (the *large* class), and one class V^- has size $s-1$ (the *small* class). Our goal will be to start with a nearly-equitable coloring, and, making simple changes, to obtain an equitable $(r+1)$-coloring. We first introduce some notation. For $X \subseteq V(G)$ and $y \in V(G)$, we denote $N_X(y) = N(y) \cap X$ and $d_X(y) = |N_X(y)|$. If μ is a function on $E(G)$, then for $A, B \subseteq V(G)$, $A \cap B = \emptyset$, we denote $\mu(A, B) = \sum_{e \in E(A,B)} \mu(e)$, where $E(A, B)$ is the set of edges of G joining A and B. In addition, if f is a function on $V(G)$ and $W \subseteq V(G)$, then we denote by $f|_W$ the restriction of f to W.

Given a nearly-equitable coloring $f = (V_1, \ldots, V_{r+1})$ of G, we define an auxiliary directed graph $H = H(G, f)$, as follows: $V(H) = \{V_1, \ldots, V_{r+1}\}$ (i.e., the color classes of f), and $\overrightarrow{(V_i, V_j)} \in E(H)$ if there exists a vertex $v \in V_i$ with $d_{V_j}(v) = 0$ (then moving v to V_j, produces a new proper coloring). In such a case, we say that v is *movable* from V_i to V_j. We denote by V^- the small color class of f ($|V^-| = s-1$), and by V^+ the large color class of f ($|V^+| = s+1$). We call a color class $W \in V(H)$ *accessible* (in H) if there exists a directed path in H from W to V^-. Clearly, V^- itself is accessible.

Lemma 3.4. *If V^+ is accessible in H, then G has an equitable $(r+1)$-coloring.*

Proof. By definition, there exists in H a directed path $P = (V_1, \ldots, V_k)$ where $V_1 = V^+$ and $V_k = V^-$. This means that for every $1 \leq i \leq k-1$ there exists $v_i \in V_i$ without a neighbor in V_{i+1} (i.e., a movable vertex). We now move v_i to V_{i+1} for $1 \leq i \leq k-1$, and obtain a proper $(r+1)$-coloring where $V_1 = V^+$ lost a vertex, $V_k = V^-$ gained a vertex, and rest of the classes maintained their sizes. Therefore, we have obtained an equitable $(r+1)$-coloring. ■

We now introduce some more notation.

$$\begin{aligned}
\mathcal{A} &= \mathcal{A}(f) - \text{all the accessible color classes in } H,\\
A &= \bigcup \mathcal{A} - \text{all the vertices in the accessible classes},\\
B &= V(G) \setminus A,\\
|\mathcal{A}| &= m+1 \text{ (i.e., } m \text{ color classes apart from } V^-),\\
q &= (r+1) - (m+1) = r - m \text{ (} B \text{ is composed of } q \text{ color classes).}
\end{aligned}$$

Let $y \in B$. Since y cannot be moved to any class in $\mathcal{A}$ (as otherwise, there would be a path in H between the color class of y and V^-), we conclude that y has at least one neighbor in every $W \in \mathcal{A}$. This implies that

$$d_A(y) \geq m+1 \Longrightarrow d_B(y) \leq r - (m+1) = q - 1. \tag{1}$$

We can assume that $V^+ \subseteq B$, as otherwise we are done (i.e., have an equitable $(r+1)$-coloring) by Lemma 3.4. We observed that $V^- \in \mathcal{A}$, which implies that $m \geq 0$. If $m = 0$ then $\mathcal{A} = \{V^-\}$, and thus $|E(A,B)| \leq r|V^-| = r(s-1)$. On the other hand, by (1), $d_A(y) \geq 1$ for every $y \in B$, and so $|E(A,B)| \geq |B| = (r+1)s - (s-1) = rs+1$. We got that $rs + 1 \leq |E(A,B)| \leq r(s-1)$, which is a contradiction. Therefore, we can assume that $m > 0$, and so $|\mathcal{A}| \geq 2$ (i.e., $\mathcal{A}$ contains additional classes, apart from V^-).

Definition 3.5. *A color class $V \in \mathcal{A}$ is called* terminal *if every class $W \in \mathcal{A} \setminus \{V\}$ remains accessible (i.e., V^- is reachable from W) after removing V (i.e., in $H - \{V\}$), and otherwise V is* non-terminal.

A trivial observation is that V^- is non-terminal. If $W \in \mathcal{A}$ is a non-terminal class, then it partitions $\mathcal{A} - \{W\}$ into two parts: $\mathcal{S}_W$ – classes in $\mathcal{A} - \{W\}$ that are still accessible (to V^-) after the removal of W, and $\mathcal{T}_W$ – rest of the classes in $\mathcal{A} - \{W\}$ (i.e., that are no longer accessible to V^-). Note that $\mathcal{T}_W \neq \emptyset$. Let us now choose a non-terminal class $U \in \mathcal{A}$ such that the set $\mathcal{A}' := \mathcal{T}_U \neq \emptyset$ is minimal. By the minimality of U we get that every class in $\mathcal{A}' = \mathcal{T}_U$ is terminal (as if there was a non-terminal class $W \in \mathcal{A}'$, then we would have $\mathcal{T}_W \subseteq \mathcal{T}_U - \{W\}$, as clearly every class in $\mathcal{S}_U \cup \{U\}$ has a path to V^- not using W). Let us denote $|\mathcal{A}'| = t$ and $A' = \bigcup \mathcal{A}'$. Then every $x \in A'$ has a neighbor in every class $W \in \mathcal{A} \setminus (\mathcal{A}' \cup \{U\})$ (i.e., x is not movable to any class in $\mathcal{A} \setminus (\mathcal{A}' \cup \{U\}) = \mathcal{S}_U$), as otherwise there would be an edge (in H) from the class of x to W (and so a path to V^-). We thus have:

$$d_A(x) \geq m - t. \tag{2}$$

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1 Equitable Coloring

Reminder: In the last lesson we have started to prove the following theorem on the existence of equitable $(\Delta+1)$-colorings.

Theorem 1 (Hajnal-Szemerédi '70)**.** *For every graph G with $\Delta(G) \leq r$, there exists an equitable $(r+1)$-coloring.*

We now continue with the proof of the theorem.

Proof of Theorem 1 (cont.). (Kierstead-Kostochka '08) Using the notation we have introduced last time, we recall that by choosing a non-terminal class $U \in \mathcal{A}$ such that the set $\mathcal{A}' := \mathcal{T}_U \neq \emptyset$ is minimal, we get that every class in $\mathcal{A}' = \mathcal{T}_U$ is terminal. In addition, denoting $|\mathcal{A}'| = t$ and $A' = \bigcup \mathcal{A}'$, we showed that every $x \in A'$ has a neighbor in every class $W \in \mathcal{A} \setminus (\mathcal{A}' \cup \{U\})$, and thus:

$$d_A(x) \geq m - t. \tag{2}$$

We call an edge $(z, y) \in E(G)$ such that $z \in W$, $W \in \mathcal{A}'$, and $y \in B$, a *solo edge* if $N_W(y) = \{z\}$. In this case, z, y are called *solo vertices*, and also *special neighbors* of each other. For such $z \in W \in \mathcal{A}'$, $y \in B$, let S_z be the set of all special neighbors of z in B, and let S^y be the set of all special neighbors of y in A'. Now, let $y \in B$. We know that y has a neighbor in every class from $\mathcal{A}$. Denote by m_0 the number of color classes in $\mathcal{A}$ having at least 2 neighbors of y. Then we have:

$$r - d_B(y) \geq d_A(y) \geq m_0 \cdot 2 + (m + 1 - m_0) \cdot 1 \Longrightarrow m_0 \leq r - (m + 1 + d_B(y)).$$

Hence there are at least $t - m_0$ color classes in $\mathcal{A}'$ getting exactly one edge from y, implying that

$$|S^y| \geq t - m_0 \geq t - r + m + 1 + d_B(y) = t - q + 1 + d_B(y). \tag{3}$$

Lemma 1.1. *Assume there exists a color class $W \in \mathcal{A}'$ such that no solo vertex in W is movable to a color class in $\mathcal{A} \setminus \{W\}$. Then $q + 1 \leq t$, and every vertex $y \in B$ is a solo vertex.*

Proof. Let S be the set of solo vertices in W, and denote $D = W \setminus S$. Recalling that every $y \in B$ has a neighbor in W, we conclude that every vertex in $N_B(S)$ has at least one neighbor in W, and every vertex in $B \setminus N_B(S)$ has at least two neighbors in W. It follows that

$$|E(W, B)| \geq |N_B(S)| + 2(|B| - |N_B(S)|) = 2|B| - |N_B(S)|.$$

Since no vertex in $z \in S$ is movable to other color class in $\mathcal{A} \setminus \{W\}$ (by lemma's assumption), we have that $d_A(z) \geq m \Longrightarrow d_B(z) \leq r - m = q$. Also, by (2), every vertex $x \in W$ satisfies $d_B(x) \leq t + q$. Hence, $|E(W, B)| \geq 2|B| - q|S|$ but also $|E(W, B)| \leq q|S| + (t + q)|D| = q|W| + t|D| = qs + t|D|$. This implies that $2|B| - q|S| \leq |E(W, B)| \leq qs + t|D|$. We then have

$$|E(W, B)| \geq 2|B| - q|S|$$

$$\begin{aligned}&= 2(qs+1) - q|S| \\ &= 2(qs+1) - q(|W| - |D|) \\ &= 2(qs+1) - q(s - |D|) \\ &= qs + q|D| + 2.\end{aligned}$$

Recalling that we also have $|E(W,B)| \leq qs + t|D|$, we derive that $qs + t|D| \leq qs + q|D| + 2$, and thus $t \geq q+1$. Moreover, by (3), for every $y \in B$ we have $|S^y| \geq t - q + 1 + d_B(y) > 1$, and it follows that y is a solo vertex. ■

Lemma 1.2. *There exists a solo vertex $z \in W \in \mathcal{A}'$ such that either z is movable to a color class in $\mathcal{A} \setminus \{W\}$ or z has two non-adjacent special neighbors in B.*

Proof. Suppose not. Then, by Lemma 1.1, every vertex in B is solo. Moreover, by the lemma's assumption, for every solo vertex $z \in A'$, its special neighbors in B form a clique. Define the following weight function μ on $E(A', B)$: for $(x,y) \in E(A',B)$ ($x \in A', y \in B$),

$$\mu(x,y) = \begin{cases} \frac{q}{|S_x|} & , \ (x,y) \text{ is a solo edge} \\ 0 & , \ \text{otherwise} \end{cases}.$$

For $z \in A'$, we have $\mu(z,B) = |S_z| \cdot \frac{q}{|S_z|} = q$ if z is a solo vertex and $\mu(z,B) = 0$ otherwise. Hence, $\mu(A',B) \leq q|A'| = q \cdot s \cdot t$. On the other hand, consider $y \in B$. Let $c_y = \max\{|S_z| \mid z \in S^y\}$, say, $c_y = |S_z|$ for $z \in S^y$. We assume that S_z is a clique, implying $d_B(y) \geq c_y - 1$. Also, by (1), $d_B(y) \leq q-1$, and so $c_y \leq q$. Then we have

$$\mu(A', y) = \sum_{z \in S^y} \frac{q}{|S_z|} \geq |S^y| \cdot \frac{q}{c_y} \underset{(3)}{\geq} (t - q + c_y)\frac{q}{c_y} = (t-q)\frac{q}{c_y} + q \geq (t-q)\frac{c_y}{c_y} + q = t.$$

Hence, $\mu(A',B) \geq t|B| = t(qs+1) > t \cdot q \cdot s$, and so we obtain a contradiction. ■

We are finally ready to prove Theorem 1. The proof will proceed by a *double* induction, on $|E(G)|$ and then on q. We define as before:

- A nearly-equitable $(r+1)$-coloring f, large color class V^+, small color class V^-, auxiliary digraph $H = H(G,f)$.
- The families $\mathcal{A}$, $\mathcal{A}'$, etc.

Now, if $V^+ \in \mathcal{A}$, then we are done be Lemma 2.4 (from the previous lecture). This takes care of the base case $q = 0$. Otherwise, we apply Lemma 1.2. We set a color class $W \in \mathcal{A}'$, a solo vertex $z \in W$, and a vertex $y_1 \in S_z$ such that:

1. z is movable to a color class in $\mathcal{A} \setminus \{W\}$.
2. z is not movable but has another neighbor $y_2 \in S_z$ such that $(y_1, y_2) \notin E(G)$.

Define $B^- = B \setminus \{y_1\}$. Since there are some edges between A and B (by (1)), we can apply induction to the induced subgraph $G[B^-]$. Notice that $|B^-| = |B| - 1 = qs$ and $\Delta(G[B^-]) \leq q-1$. By induction, the graph $G[B^-]$ has an equitable coloring g in q colors. Let us also define $A^+ = A \cup \{y_1\}$. We now consider two cases.

Case 1: z is movable to $X \in \mathcal{A} \setminus \{W\}$. Then move z to X, move y_1 to $W \setminus \{z\}$ (we can do it as y_1 is

a special neighbor of z). We get a nearly-equitable $(m+1)$-coloring φ of $G[A^+]$. Since $W \in \mathcal{A}'(f)$, W is terminal in $H(G,f)$, and thus $V^+(\varphi) = X \cup \{z\}$ is accessible to V^-. Then, by Lemma 2.4 (from the previous lecture), $G[A^+]$ has an equitable $(m+1)$-coloring φ'. Combining φ' and g, we get an equitable $(r+1)$-coloring of G.
Case 2: z is *not* movable to any color class in $\mathcal{A}$. Then z has at least m neighbors in A. Also, as $(z, y_1) \in E(G)$ we have that $d_{A^+}(z) \geq m+1$. This in turn implies that $d_{B^-}(z) \leq r-(m+1) = q-1$. Hence, there is a color class Y in g such that $d_Y(z) = 0$. We move z to Y and y_1 to W, and obtain a nearly-equitable coloring ψ' of G. Also, observe that y_2 is movable to the color class containing y_1 in a coloring of A^+ (as z was moved from it and $(y_1, y_2) \notin E(G)$). This means that the color class of y_2 becomes accessible to V^-. Hence, the parameter q of the new coloring is strictly smaller then the original q. Hence, the (secondary) induction (on q) applies, and the proof is complete. ■

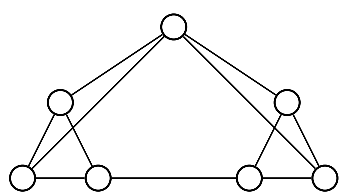
2 Color-Critical Graphs

Definition 2.1. *A graph G is called k-*critical *if:*

1. $\chi(G) = k$.
2. $\chi(G') < k$ *for every proper subgraph* $G' \subsetneq G$.

Examples:

1. K_n – n-critical.
2. C_{2n+1} – 3-critical.
3. The following graph is 4-critical (verify it!):



Question: Why are critical graphs important?

Observation 2.2. *If $\chi(G) \geq k$ then G contains a k-critical graph.*

2.1 Characterization of k-Critical Graphs

1. $k = 1$: $G = K_1$.
2. $k = 2$: $G = K_2$.
3. $k = 3$: $G = C_{2n+1}$.
4. $k = 4$: No simple characterization is known.

2.2 Basic Properties of Color-Critical Graphs

Proposition 2.3. *Let G be a k-critical graph. Then:*

1. *$\forall v \in V(G)$, $\chi(G-\{v\}) = k-1$. Moreover, in every $(k-1)$-coloring c of $G-\{v\}$, the neighborhood $N_G(v)$ carries all $k-1$ colors.*

2. *$\forall e = (u, v) \in E(G)$, $\chi(G - \{e\}) = k - 1$. Moreover, in every $(k - 1)$-coloring c of $G - \{e\}$, we have $c(u) = c(v)$.*

Proof. Item 1: The first part is trivial. Now, if there is a $(k - 1)$-coloring c of $G - \{v\}$ which misses some color $i \in [k - 1]$ on $N_G(v)$, we can complete c to a $(k - 1)$-coloring of G by assigning $c(v) = i$, which is a contradiction.
Item 2: The first part is trivial. Now, if there is a $(k - 1)$-coloring c of $G - \{e\}$ such that $c(u) \neq c(v)$, then c is actually a $(k - 1)$-coloring of G, which is a contradiction. ■

Corollary 2. *Let G be a k-critical graph on n vertices. Then, $\delta(G) \geq k - 1$, and so $|E(G)| \geq \frac{k-1}{2} \cdot n$.*

Remark 2.4. *By Brooks' theorem, if $G \neq K_k, C_{2n+1}$ and G is k-critical, then G has a vertex of degree $> k - 1$, which implies that $|E(G)| > \frac{k-1}{2} \cdot n$.*

2.3 Connectivity Properties of Color-Critical Graphs

Proposition 2.5. *Let G be a k-critical graph, $k \geq 3$. Then G is 2-vertex-connected.*

Proof. If $\kappa(G) = 0$, then G has more than one connected component. By criticality, each connected component of G is $(k-1)$-colorable. But then $\chi(G) = \max\{\chi(G_i) \,|\, G_i \text{ is a conn. comp. of } G\} \leq k-1$, which is a contradiction.

Assume now that $\kappa(G) = 1$. Then there is a cut vertex v in G, and we can write: $V = V_1 \cup V_2$, $|V_1|, |V_2| > 1$, $V_1 \cap V_2 = \{v\}$, and G has no edges between $V_1 \setminus \{v\}$ and $V_2 \setminus \{v\}$. Since G is k-critical, both subgraphs $G[V_1]$ and $G[V_2]$ are $(k - 1)$-colorable. Let c_1 be a $(k - 1)$-coloring of $G[V_1]$ and let c_2 be a $(k-1)$-coloring of $G[V_2]$. By permuting colors if necessary, we can assume that $c_1(v) = c_2(v)$. But then, combining c_1 and c_2 in an obvious way gives a $(k-1)$-coloring of G, which is a contradiction. ■

Remark 2.6. *We have proved that $\kappa(G) \geq 2$ for every k-critical graph, $k \geq 3$. The estimate $\kappa(G) \geq 2$ is tight, as we will see later.*

Theorem 3 (Dirac '53)**.** *Let G be a k-critical graph. Then G is $(k - 1)$-edge-connected.*

Proof. Assume, towards a contradiction, that there is a k-critical graph G with $\kappa'(G) \leq k - 2$. Then there is a subset $S \subseteq V(G)$, $S \neq \emptyset, V(G)$ such that $|[S, \overline{S}]| \leq k-2$. Since G is k-critical, both subgraphs $G[S], G[\overline{S}]$ are $(k-1)$-colorable. Let $(U_1, \ldots, U_{k-1})$ be the color classes of a $(k-1)$-coloring c_1 of $G[S]$, and let $(W_1, \ldots, W_{k-1})$ be the color classes of a $(k-1)$-coloring c_2 of $G[\overline{S}]$. Define an auxiliary bipartite graph Γ with parts A and B, where: $A = [k - 1]$ (color classes in c_1), $B = [k - 1]$ (color classes in c_2). The edges of Γ are defined as follows: $(i, j) \in E(\Gamma)$ if and only if G has no edge between U_i and W_j. If G contains a perfect matching $\mathfrak{f} = \{(i, f(i)) \,|\, i \in [k - 1]\}$, then we have a $(k - 1)$-coloring of G with color classes as follows: $U_1 \cup W_{f(1)}, U_2 \cup W_{f(2)}, \ldots, U_{k-1} \cup W_{f(k-1)}$, contradicting the assumption on G. Now, if $(i, j) \notin E(\Gamma)$ then U_i and W_j are connected by and edge of G. Since $|[S, \overline{S}]| \leq k - 2$ the graph Γ misses at most $k - 2$ edges. So Γ is obtained from the complete bipartite graph $K_{k-1,k-1}$ by omitting $\leq k - 2$ edges.

Claim 2.7. *Let Γ be obtained from $K_{m,m}$ by omitting $\leq m-1$ edges. Then Γ has a perfect matching.*

Proof. Left as an exercise (use Hall/Kőnig). ■

Remark 2.8. *The above claim is tight. That is, $m-1$ cannot be replaced by m.*

Hence Γ has a perfect matching. As we explained before, this implies that $\chi(G) \leq k-1$, which is a contradiction. ■

Remark 2.9. *The statement of the theorem is tight for every k-critical graph G with $\delta(G) = k-1$ (in this case, $\kappa'(G) \leq k-1$).*

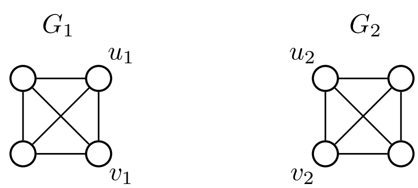
2.4 Hajós Construction

Definition 2.10. *Let G_1, G_2 be graphs with $V_1 \cap V_2 = \emptyset$. Let $e_1 = (u_1, v_1) \in E(G_1)$, $e_2 = (u_2, v_2) \in E(G_2)$. The* Hajós sum *of G_1 and G_2 is defined as:*

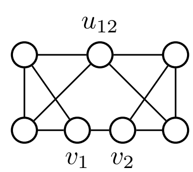
1. *Delete edges e_1, e_2.*
2. *Identify vertices u_1, u_2 to a new vertex u_{12}.*
3. *Add an edge (v_1, v_2).*

Remark 2.11. *By deleting u_{12} and either v_1 or v_2, we disconnect the Hajós sum of G_1 and G_2, implying that the vertex-connectivity of a Hajós sum is at most 2.*

Example:



The Hajós sum of G_1 and G_2 is:



Theorem 4 (Dirac '53)**.** *For every $k \geq 2$, the Hajós sum of two k-critical graphs is k-critical.*

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Lecture 4

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1 Color-Critical Graphs

Definition 1.1. *Given vertex-disjoint graphs G_1, G_2, the* join $G_1 \cup G_2$ *is obtained from G_1, G_2 by adding all edges between $V(G_1)$ and $V(G_2)$.*

Proposition 1.2. *Let G_1, G_2 be k_1, k_2-critical graphs, respectively. Then the join $G_1 \cup G_2$ is $(k_1 + k_2)$-critical.*

Proof. Left as an exercise. ■

1.1 Hajós Construction

Let us recall the definition of the Hajós sum from the previous lecture.

Definition 1.3. *Let G_1, G_2 be graphs with $V_1 \cap V_2 = \emptyset$. Let $e_1 = (u_1, v_1) \in E(G_1)$, $e_2 = (u_2, v_2) \in E(G_2)$. The* Hajós sum *of G_1 and G_2 is defined as:*

1. *Delete edges e_1, e_2.*
2. *Identify vertices u_1, u_2 to a new vertex u_{12}.*
3. *Add an edge (v_1, v_2).*

Theorem 1 (Dirac '53)**.** *Let G_1, G_2 be k-critical graphs ($k \geq 2$). Let $e_1 = (u_1, v_1) \in E(G_1)$, $e_2 = (u_2, v_2) \in E(G_2)$. Then the Hajós sum G of G_1 and G_2 is k-critical.*

Proof.

1. We claim that $\chi(G) \geq k$. Suppose not, and let $c : V(G) \to [k-1]$ be a $(k-1)$-coloring of G. Then c restricted to V_1 gives a $(k-1)$-coloring of $G_1 \setminus \{e_1\}$. Similarly, c restricted to V_2 gives a $(k-1)$-coloring of $G_2 \setminus \{e_2\}$. But then, due to the criticality of G_1, we have $c(u_1) = c(v_1) = c(u_{12})$, and due to the criticality of G_2, we have $c(u_2) = c(v_2) = c(u_{12})$. We derive that $c(v_1) = c(v_2)$, which is a contradiction since $(v_1, v_2) \in E(G)$. We conclude that $\chi(G) \geq k$. Furthermore, it is easy to observe that G is k-colorable.
2. We will show that for every $e \in E(G)$, $\chi(G \setminus \{e\}) \leq k - 1$[1] (in fact, $\chi(G \setminus \{e\}) = k - 1$). If $e = (v_1, v_2)$, we can argue as before. Now, if $e \in E(G_1)$, then, as G_1 is k-critical, there exists a $(k-1)$-coloring c_1 of $G - \{e\}$. Since $e_1 \in G_1 - \{e\}$, we have $c_1(u_1) \neq c_1(v_1)$. Also, since G_2 is k-critical, there exists a $(k-1)$-coloring of $G_2 - \{e_2\}$. We can assume that $c_2(u_2) = c_1(u_1)$ (by permuting colors if necessary). We also have $c_2(u_2) = c_2(v_2)$. We can now "glue" c_1 and c_2 into

[1] This is enough for proving criticality, as both G_1 and G_2, being k-critical graphs, do not have isolated vertices, and thus G does not have them either. Therefore, removal of any vertex from G eliminates at least one edge.

one coloring c in an obvious way (this is possible, as $c_1(u_1) = c_2(u_2)$). Since $c_1(u_1) \neq c_1(v_1)$, but $c_1(u_1) = c_2(u_2) = c_2(v_2)$, we have $c(v_1) \neq c(v_2)$. Hence the edge (v_1, v_2) is also properly colored by c.
The case of $e \in E(G_2)$ is treated similarly.

■

1.2 Universality of Hajós Construction

Definition 1.4. *A graph G is k-*constructible *if one of the following holds:*

1. *$G = K_k$.*
2. *G is the Hajós sum of two k-constructible graphs.*
3. *If G_0 is k-constructible and $(u, v) \notin E(G_0)$ for some $u \neq v \in V(G_0)$, then G is obtained from G_0 by merging u and v.*

Theorem 2 (Hajós '61)**.** *$\chi(G) \geq k$ if and only if G contains a k-constructible subgraph.*

Proof. $(\Longleftarrow)$: Assume that G has a k-constructible subgraph $G_0 \subseteq G$. We need to prove: $\chi(G) \geq k$. As $\chi(K_k) = k$, we shall prove that operations 2 and 3 preserve $\chi(G) \geq k$. Let us start with operation 2 (Hajós sum). Let G_1, G_2 be two k-constructible graphs, and let $e_1 = (u_1, v_1) \in E(G_1), e_2 = (u_2, v_2) \in E(G_2)$. Now, let G be the Hajós sum of G_1 and G_2 (with respect to the edges e_1, e_2). We assume that $\chi(G_1) \geq k$ and $\chi(G_2) \geq k$, and we need to prove that $\chi(G) \geq k$. Suppose not, and let $c : V(G) \to [k-1]$ be a $(k-1)$-coloring of G. Then $c(v_1) \neq c(v_2)$. But then c restricted to $V(G_1)$ is a $(k-1)$-coloring of $G_1 \setminus \{e_1\}$. We then have that $c(u_1) = c(v_1)$ (as otherwise G_1 would be $(k-1)$-colorable). Also, c restricted to $V(G_2)$ is a $(k-1)$-coloring of $G_2 \setminus \{e_2\}$. But then $c(u_2) = c(v_2)$ (as otherwise G_2 would be $(k-1)$-colorable). It then follows that $c(v_1) = c(v_2)$ (as $c(u_1) = c(u_2) = c(u_{12})$), which is contradiction.

We now deal with operation 3. If $\chi(G) \leq k-1$, then let $c : V(G) \to [k-1]$ be a $(k-1)$-coloring of G. We can produce a $(k-1)$-coloring c_0 of G_0 as follows:

$$c_0(w) = \begin{cases} c(w)\ , & w \neq u, v \\ c(uv), & w = u \text{ or } w = v \end{cases} .$$

It is easy to see that c_0 is a proper $(k-1)$-coloring of G_0, implying $\chi(G_0) \leq k-1$, which is a contradiction.
$(\Longrightarrow)$: Assume that $\chi(G) \geq k$. We need to prove: there exists a k-constructible subgraph $G_0 \subseteq G$. Assume not and let G be a maximal[2] counterexample. Then:

1. $\chi(G) \geq k$.
2. G has no k-constructible subgraph.
3. For every $e \notin E(G)$, the graph $G + \{e\}$ contains a k-constructible subgraph G_e.

Claim 1.5. *The non-adjacency relation in G is an equivalence relation.*

[2]Here we do not need to apply Zorn's lemma as the number of vertices of G is finite.

Proof. The reflexivity and the symmetry are obvious, and so it is left to prove transitivity. That is, we need to prove that if $e_1 = (u, v), e_2 = (v, w) \notin G$ then also $(u, w) \notin G$. Assume it is not the case. Then, due to maximality, $G + e_1$ has a k-constructible subgraph G_{e_1}, and similarly $G + e_2$ has a k-constructible subgraph G_{e_2}. Take vertex-disjoint copies of G_{e_1} and G_{e_2} and apply the Hajós sum of G_{e_1}, G_{e_2} with respect to $e_1 = (v, u), e_2 = (v, w)$. We get a graph G_0 containing the edge (u, w). Now, for every vertex $x \neq v$ appearing in both G_{e_1}, G_{e_2}, we merge the two appearances of x in G_0 (i.e., operation 3). The obtained graph G^* is a subgraph of G (as we assumed that $(u, w) \in E(G)$). We have that G^* is k-constructible (obtained from two k-constructible graphs G_{e_1}, G_{e_2} by applying operation 2 once and operation 3 several (≥ 0) times). Since $G^* \subseteq G$, we have a contradiction. ■

Claim 1.6. *If the non-adjacency relation in a graph G is an equivalence relation, then G is a complete r-partite graph for some $r \geq 0$.*

Proof. Left as an exercise. ■

By Claims 1.5 and 1.6 we have that G is a complete r-partite graph, satisfying $\chi(G) \geq k$. It follows that $r \geq k$. But then $K_k \subseteq G$, implying that G has a k-constructible subgraph, which is a contradiction. ■

1.3 Sparse Color-Critical Graphs

Proposition 1.7. *For every $k \geq 4$, there exists a k-critical graph on n vertices if and only if $n = k$ or $n \geq k + 2$.*

Proof. We first prove it for $k = 4$. The join $C_{2n+1} \cup K_1$ is a 4-critical graph on $2n + 2$ vertices. This implies that 4-critical graphs exist for $n = 4, 6, 8$[3]. Now, if G is a 4-critical graph on n vertices, then the Hajós sum of G and K_4 is a 4-critical graph on $n + 3$ vertices. Thus the admissible values for the number of vertices in a 4-critical graph are

$$\{4, 6, 8\} + \{3r \mid r \geq 0\}.$$

Therefore, for every $n = 4$ or $n \geq 6$ there exists a 4-critical graph on n vertices. Since the join of a k-critical graph G on n vertices and K_1 is a $(k + 1)$-critical graph on $n + 1$ vertices, we get that a k-critical graph on n vertices exists if $n = k$ or $n \geq k + 2$.

Now we prove that there is *no* k-critical graph on $k + 1$ vertices. Assume to the contrary that G is a k-critical graph on $k + 1$ vertices. Since $G \neq K_{k+1}$ there are $u \neq v \in V(G)$ such that $(u, v) \notin E(G)$. Now, since G is k-critical, we have that $\delta(G) \geq k - 1$. But then $N(u) = N(v) = V(G) \setminus \{u, v\}$. By criticality, $G - \{u\}$ has a $(k - 1)$-coloring $c : (V(G) - \{u\}) \to [k - 1]$. Since $N(u) = N(v)$, we can extend c to u by assigning $c(u) = c(v)$. This implies that $\chi(G) \leq k - 1$, which is a contradiction. ■

For every $k \geq 4$, and $n = k$ or $n \geq k + 2$ let us define

$$f_k(n) = \min\{|E(G)| : G \text{ is } k\text{-critical}, |V(G)| = n\}.$$

We have that $f_k(k) = \binom{k}{2}$. Also, since $\delta(G) \geq k - 1$ it follows that $f_k(n) \geq \frac{k-1}{2}n$. By Brooks' theorem, $f_k(n) > \frac{k-1}{2}n$ for $n \geq k + 2$. Observe that due to Hajós construction, $f_k(n + k - 1) \leq f_k(n) + \binom{k}{2} - 1$. Then, using (roughly) Fekete's (subadditivity) lemma, it follows that $f_k = \lim_{n\to\infty} \frac{f_k(n)}{n}$ exists.

Conjecture 1.8 (Ore '67). *We have an equality $f_k(n) = f_k(n - k + 1) + \binom{k}{2} - 1$.*

[3] In fact, for every even $n \geq 4$, but this will be enough for our purposes.

This would imply

$$f_k = \frac{1}{2}\left(k - \frac{2}{k-1}\right). \tag{1}$$

Kostochka and Yancey (2014) proved (1) for every $k \geq 4$. In fact, they have proved Ore's conjecture for $k = 4$.

1.4 Dense Color-Critical Graphs

Question (Erdős '49): For fixed $k \geq 4$, does there exist a k-critical graph on n vertices with $\Theta(n^2)$ edges?

Theorem 3 (Dirac '52 for $k \geq 6$, Toft '70 for $k = 4, 5$)**.** *For every fixed $k \geq 4$, there exists a constant $c > 0$ such that for all sufficiently large n, there exists a k-critical graph G on n vertices with at least cn^2 edges.*

Proof. It is easy to see (using Hajós sums, joins, etc.) that it is enough to prove the statement for $k = 4$ and a dense sequence of values of n. We present a construction, as follows. Assume that n is odd. Define a graph G on $4n$ vertices as follows. Take two disjoint sets of vertices A, B, each of size n, and put a complete bipartite graph between them. Then add a (disjoint) cycle C_A on n vertices and put a perfect matching between A and C_A. Similarly, add a (disjoint) cycle C_B on n vertices and put a perfect matching between B and C_B. We observe that:

1. $|V(G)| = 4n$, $|E(G)| > n^2$.

2. $\chi(G) \geq 4$. Indeed, if $c : V(G) \to [3]$, then one of the parts A, B, say A, contains one color only in c. Say this color is 3. But then color 3 is forbidden on every vertex of C_A, and thus C_A is colored in two colors $1, 2$, which is a contradiction.

3. $\chi(G - \{e\}) \leq 3$ for every $e \in E(G)$. We will not prove it formally, but rather present several cases. If $e = (x, y)$ is between A and B, then we can color x, y by color 1, rest of the vertices in A by color 2, and rest of the vertices in B by color 3. One can observe that C_A can be colored by colors $1, 2, 3$. Indeed, as there is a (perfect) matching connecting A and C_A, we can color the neighbor of x in C_A by color 2, and the rest of the vertices of C_A by colors $1, 3$, alternatingly. Similarly, we can color the vertices of C_B by colors $1, 2, 3$; If $e = (x, y)$ is between A and C_A, then we color the vertices in A by color 1, one vertex in B by color 2 and the rest of the vertices in B by color 3. One can observe that C_B can be colored by colors $1, 2, 3$, in the same way as in the previous case. The same is true for C_A, as we can color y by color 1, and the rest of the vertices of C_A by colors $2, 3$, alternatingly.

We conclude that G is 4-critical, on $4n$ vertices, with $> n^2$ edges. ■

1.5 Long Paths (and Cycles) in Color-Critical Graphs

Question: Let G be a k-critical graph on n vertices ($k \geq 4$ fixed, $n \to \infty$). Does G contain a long path (cycle)?

We present a construction of a k-critical graph on n vertices with no paths of length $> C(k) \log n$ (for some constant $C = C(k) > 0$). The construction is essentially due to Gallai '63. We assume that $k \geq 4$. Let T be a tree of maximum degree at most $k - 1$. Let $(H_t)_{t \in V(T)}$ be a family of k-critical graphs, sharing a common vertex x_0 and disjoint otherwise. Start with $\bigcup_{t \in V(T)} H_t$, and for each *ordered* edge $(t, t') \in E(T)$ with $t \neq t'$, choose a vertex $v_{tt'} \in V(H_t)$ adjacent to x_0 such that if $(t, t'), (t, t'') \in E(T)$ then $v_{tt'} \neq v_{tt''}$. This is possible as $\Delta(T) \leq k - 1$, $\delta(H_t) \geq k - 1$ due to k-criticality of H_t. Now, for each *unordered* edge $\{t, t'\} \in E(T)$ we do the following:

1. Delete edge $(x_0, v_{tt'})$.
2. Delete edge $(x_0, v_{t't})$.
3. Add an edge $(v_{tt'}, v_{t't})$.

It is easy to see that the so derived graph G can be obtained from disjoint copies of H_t's using Hajós sums. Hence, G is k-critical. Now, choose:

1. T to be a $(k-1)$-regular tree of depth $h \geq 0$.
2. Choose $H_t = K_k$, for every $t \in V(T)$.

We can observe that the maximum length of a path in $G - \{x_0\}$ is at most $2h \cdot (k-1)$. Altogether, G:

1. is a k-critical graph.
2. has $\approx (k-2)^h(k-1)$ vertices.
3. has longest path of length $\leq 2h \cdot (k-1) \cdot 2 = 4h \cdot (k-1)$.

Hence G is a graph on n vertices without a path of length $\frac{4(k-1)}{\log(k-2)} \log n$ (as $\log n \approx \log((k-2)^h(k-1)) \geq h \log(k-2)$).

1 Color-Critical Graphs

1.1 Long Paths (and Cycles) in Color-Critical Graphs

Theorem 1 (Alon-Krivelevich-Seymour '00)**.** *For every $k \geq 4$, in every k-critical graph G on n vertices there exists a path of length at least $\frac{\log(n-1)}{\log(k-1)}$.*

Proof. We will use the following property of spanning trees in connected graphs.

Claim 1.1. *For every connected graph G and every vertex $v_0 \in V(G)$ there exists a rooted spanning tree T with root v_0 such that for every $e = (u, v) \in E(G) \setminus E(T)$, one of the vertices u, v is an ancestor of the other vertex, with respect to v_0.*

Proof. If we will execute the DFS algorithm on G according to any permutation of the vertices σ, such that v_0 is the first vertex in σ, then the resulting spanning tree will satisfy claim's conditions. ■

Let G be a k-critical graph. Choose $v_0 \in V(G)$ arbitrarily, and obtain a spanning tree T as in Claim 1.1 (we note that as G is k-critical, we have that G is connected). Assume that the depth of T is r. We observe that it suffices to prove that $r \geq \frac{\log(n-1)}{\log(k-1)}$. We will order the vertices of every edge $e = (u, v) \in E(T)$ such that u is closer to v_0 (in T) than v. For each edge $e = (u, v) \in E(T)$, we define *type*(e) to be the distance between e and v_0 (i.e., the distance between u and v_0). We observe that *type*$(e) \in \{0, \dots, r-1\}$. As G is k-critical, for every $e \in E(T)$, there exists a $(k-1)$-coloring c_e of $G - \{e\}$ such that $c_e(v_0) = 1$. Assume now that $e = (u, v)$ is an edge with *type*$(e) = j$, and let $P = (v_0, v_1, \dots, v_j = u)$ be the path in T between v_0 and u. We denote $S_e = (c_e(v_1), \dots, c_e(v_j)) \in [k-1]^j$.

Claim 1.2. *Let $e \neq e' \in E(T)$ be edges of T that satisfy type$(e) =$ type(e'). Then we have $S_e \neq S_{e'}$.*

Proof. Assume, towards a contradiction, that $S_e = S_{e'}$. Denote by w the lowest (i.e., farthest from v_0) common ancestor of u, u' (where $e = (u, v), e' = (u', v')$). In addition, we denote by y the first vertex on the path between w and v (in T). In particular, if $w = u$ then $y = v$. We denote by T_y the subtree of T rooted at y. We define a $(k-1)$-coloring $c : V(G) \to [k-1]$ of G, as follows:

$$c(x) = \begin{cases} c_{e'}(x) \ , & x \in T_y \\ c_e(x) \ , & x \in V(G) \setminus T_y \end{cases} .$$

It is easy to see that c colors properly all edges $e \in E(G)$ that satisfy $e \subseteq V(T_y)$ or $e \subseteq V(G) \setminus V(T_y)$. Thus, it is left to verify that the edges of G which go between $V(T_y)$ and $V(G) \setminus V(T_y)$ are colored properly by c. Let $e \in E(G)$ be such an edge. Due to the property of T, one of the vertices of e belongs to the path between w and v_0 in T. But we have assumed that $S_e = S_{e'}$, and so both colorings $c_e, c_{e'}$ are identical on the vertices of the path between w and v_0. We conclude that c colors the edge e properly. Thus, c is a $(k-1)$-coloring of G, which contradicts the fact that $\chi(G) = k$. ■

We now complete the proof of the theorem. As $S_e \neq S_{e'}$ for every $e \neq e' \in E(T)$ with $type(e) = type(e')$ and $S_e, S_{e'} \in [k-1]^j$, T contains at most $(k-1)^j$ edges of type j. We recall that $|E(T)| = n-1$, and that $type(e) \in \{0, \ldots, r-1\}$. Thus, we get that

$$n - 1 \leq \sum_{j=0}^{r-1} (k-1)^j < (k-1)^r,$$

and so $r \geq \frac{\log(n-1)}{\log(k-1)}$. We conclude that G contains a path of length at least $\frac{\log(n-1)}{\log(k-1)}$. ■

Remark 1.3. *Dirac proved that every* 2*-connected graph* G *that contains a path of length* t*, also contains a cycle of length at least* $2\sqrt{t}$*.*

We will prove a weaker version of this claim, as follows.

Claim 1.4. *Let* G *be a* 2*-connected graph that contains a path of length at least* $2s^2$*. Then* G *contains a cycle of length at least* $2s$*.*

Proof. Let P be a path of length $2s^2$ in G that connects u and v. As G is 2-connected, there are 2 internally vertex-disjoint paths between u and v (due to Menger's theorem). If $\text{dist}(u, v) \geq s$ then the union of the above two paths gives a cycle of length $\geq 2s$, as required. Otherwise, there is a path Q between u and v whose length $< s$. Let us write the vertices of $Q \cap P$ in the order of their appearance on Q (where u is the first vertex of Q). As $|Q| < s$ and $|P| \geq 2s^2$, there are two consecutive vertices x, y in the (ordered) list of $Q \cap P$ such that the distance between them along P is at least $\frac{2s^2}{s} = 2s$. We will add to this path between x and y along P the subpath of Q that connects x and y (which is internally vertex-disjoint from the subpath along P), obtaining a cycle of length $> 2s$. ■

Corollary 2. *For every* $k \geq 3$*, every* k*-critical graph* G *that contains a path of length* t*, also contains a cycle of length at least* $2\sqrt{t}$*.*

Proposition 1.5. *Let* G *be a* 2*-connected graph that has an odd cycle (i.e.,* $\chi(G) \geq 3$*), and a cycle of length* t*. Then* G *contains an odd cycle of length at least* $t/2$*.*

Proof. Let C be a cycle of length t in a graph G, and let A be some odd cycle in G. If t is odd the we are done. Thus we can assume that C is an even cycle. Now, if $V(A) \cap V(C) = \emptyset$, then, due to the 2-connectivity of G, there exist two vertex-disjoint paths P_1, P_2 between A and C in G. Then, in $A \cup C$ together with P_1, P_2, we can find an odd cycle of length at least $t/2$. Indeed, we can take the longest path along C between the endpoints of P_1, P_2 on C (which has length at least $t/2$), together with P_1, P_2, and the path along A between the endpoints of P_1, P_2 on A which will close an odd cycle (as A is an odd cycle, one of these paths has an even length and the other has an odd length). The case where $|V(A) \cap V(C)| = 1$ is treated similarly. Thus, we assume that $|V(A) \cap V(C)| \geq 2$. Now, we divide the edges of $A \setminus C$ into intervals between the vertices in $A \cap C$. As $\chi(A \cup C) \geq 3$ and no two (inner) vertices of distinct intervals are adjacent, there exists at least one interval such that the graph consisting of it and C is 3-chromatic. It is then easy to see that C together with the above interval contain an odd cycle whose length is at least $t/2$. ■

Corollary 3. *Let* $k \geq 4$ *be a fixed integer. Then every* k*-critical graph* G *contains an odd cycle of length* $\Omega_k(\sqrt{\log n})$*.*

Theorem 4 (Shapira-Thomas '11)**.** *Let* $k \geq 4$ *be a fixed integer. Then in every* k*-critical graph* G *on* n *vertices there exists an odd cycle of length* $\Omega_k(\log n)$ *(in particular,* G *has circumference* $\Omega_k(\log n)$*).*

2 Coloring Planar Graphs

2.1 Euler's Formula and the 5-Color Theorem

Euler's formula (1758): Let $G = (V, E)$ be a *planar* graph with face-set F, that has r connected components. It holds that

$$|V| - |E| + |F| = r + 1.$$

The most known case of the above formula is when $r = 1$ (i.e., when G is connected).

Corollary 5. *Let $G = (V, E)$ be a simple graph. Then:*

1. $|E| \leq 3|V| - 6$.
2. *If G is triangle-free then* $|E| \leq 2|V| - 4$.

Proof. 1. Without loss of generality (by adding edges), we can assume that G is connected, and every face is a cycle of length at least 3. Let us assume that the sizes of the faces are $f_1, f_2, \ldots$. Then, by double counting the edges, we get that $2|E| = \sum_i f_i \geq 3|F|$. Plugging it into Euler's formula, we get that $|V| - |E| + \frac{2}{3}|E| \geq 2 \Rightarrow |E| \leq 3|V| - 6$.

2. We use the fact that $f_i \geq 4$ for all i, and do a similar counting.

■

Corollary 6. *Every simple planar graph G is 5-degenerate.*

Proof. From item 1 of Corollary 5 it follows that $|E| \leq 3|V| - 6$. Thus, G has a vertex v with $d(v) < 6$. This argument is also valid for every induced subgraph of G (as this is a hereditary property). Therefore, G is 5-degenerate. ■

Corollary 7. *Every simple planar graph G is 6-colorable (as every d-degenerate graph is $d+1$-colorable).*

Theorem 8 (Heawood 1890)**.** *Every simple planar graph G is 5-colorable.*

Proof. The proof proceeds by induction on $n = |V(G)|$.
Base: $n \leq 5$ – the claim is trivial.
Step: Assume that the claim holds for every simple planar graph G with $< n$ vertices. Let G be a simple planar graph on n vertices. We know that G is 5-degenerate, and let v be a vertex with $d(v) \leq 5$. Denote $G' = G \setminus \{v\}$. By induction, G' has a 5-coloring $c' : V(G') \to [5]$. If in c' there is a color $i \in [5]$ that does not appear on the neighbors of v in G, we can extend the coloring c' to G by assigning $c'(v) = i$, obtaining a 5-coloring of G. Thus, we may assume that it is not the case. Then $d_G(v) = 5$, and all five colors appear on the neighborhood $N_G(v)$ of v in G in the coloring c'. Now, consider an embedding of G in the plane, and assume that the neighbors of v, ordered in the clockwise direction, are $v_1, \ldots, v_5$ such that $c'(v_i) = i$ for all $i \in [5]$. (This can be achieved by permuting the colors, as we assumed that all of them appear exactly once.) For $1 \leq i \neq j \leq 5$, we denote by $G_{i,j}$ the subgraph of G that is spanned by the vertices that are colored by i, j in c'. We observe that if we will swap between the colors in every connected component of $G_{i,j}$, we will (still) obtain a proper 5-coloring of G'. Let us now look on $G_{1,3}$. If v_1, v_3 are *not* in the same connected component of $G_{1,3}$, then we will swap the colors in the component of v_1 in $G_{1,3}$, and get a 5-coloring of G' such that two neighbors of v (v_1, v_3) are colored by color 3, and so we can extend c' by assigning $c'(v) = 1$. Thus, we now assume that v_1, v_3 are in the same connected component of $G_{1,3}$. Then, there is a path P_1 in $G_{1,3}$ that connects v_1 and v_3. As

$V(P_1) \subseteq V(G_{1,3})$, all the vertices of P_1 are either colored by color 1 or by color 3 in c'. Similarly, we can assume that $G_{2,4}$ contains a path P_2 that connects v_2 and v_4, such that all the vertices of P_2 are either colored by color 2 or by color 4 in c'. Due to the clockwise ordering of the v_i's and the planarity of G, we have that v_2 resides inside the (simple) closed curve $v - v_1 - P_1 - v_3 - v$, while v_4 resides outside of it. Thus, by the Jordan curve theorem, P_1 and P_2 intersect. As G is planar, this intersection must be at some vertex, which is common to both of them. However, as $V(P_1) \subseteq V(G_{1,3})$ and $V(P_2) \subseteq V(G_{2,4})$, we have that $P_1 \cap P_2 = \emptyset$, which is a contradiction. We conclude that c' can be extended (possibly after permuting the colors in some $G_{i,j}$) to a proper coloring of v, and so $\chi(G) \leq 5$. ■

2.2 The 4-Color Theorem (4CT)

A brief history:

- 1852 – **Francis Guthrie** conjectured that every (simple) planar graph (or map) is 4-colorable.
- 1879 – **Kempe** presented a mistaken proof (which contained many novel and correct ideas, including Kempe's chains).
- 1977 – **Appel** and **Haken** presented a proof that $\chi(G) \leq 4$ for every (simple) planar graph. Their proof used a massive computer checking, after they managed to reduce the problem to checking a finite number of graph configurations (around 2000)[1].
  The 4-color theorem is widely considered to be a major and a fundamental result is graph theory.

Remark 2.1. *From item 2 of Corollary 5 it is immediate that every (simple) planar triangle-free graph is 4-colorable (as it has vertex of degree at most 3, and thus 3-degenerate).*
Grötzsch ('59) showed that every (simple) planar triangle-free graph is 3-colorable.

3 Colorings, Density, Minors and Subdivisions

Definition 3.1. *Let G be a graph and let $e = (u, v) \in E(G)$. The* contraction *of e is the graph operation of replacing u, v by a single vertex uv, such that instead of every edge that connects a vertex $w \neq u, v$ to a vertex in $\{u, v\}$ there will be an edge between w and the new vertex uv.*

Definition 3.2. *Let H, G be graphs. We say that H is a* minor *of G if H can be obtained from G by removing vertices and contracting edges. Notation: $H \prec G$.*

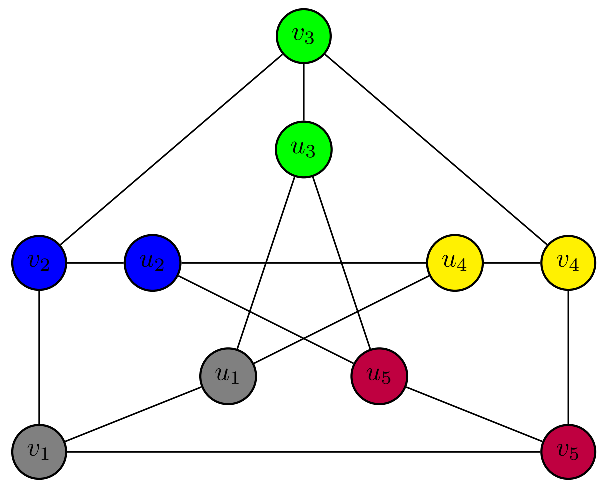
We now give an alternative definition for a minor.

Definition 3.3. *Let $H = (U, F)$, $G = (V, E)$ graphs such that $U = \{u_1, \ldots, u_r\}$. If H is a minor of G then there exist disjoint sets $V_1, \ldots, V_r \subseteq V$ such that:*

1. *$G[V_i]$ is connected for all i.*
2. *If $(u_i, u_j) \in F$, then G contains an edge between V_i and V_j.*

Example: Petersen's graph:

[1] A simpler proof, yet using the same ideas and still relying on computers, was published by Robertson, Sanders, Seymour, and Thomas in 1997.



Using Definition 3.3 with $V_i = \{u_i, v_i\}$ for $i \in [5]$, one can easily observe that the Petersen graph contains a minor of K_5.

Notation: $\eta(G) = \max\{r \mid K_r \prec G\}$ – clique contraction number of G.
It is immediate that $\eta(G) \geq \omega(G)$.
Example: If G is the Petersen graph then $\eta(G) = 5$.

Definition 3.4. *Let G be a graph and let $e = (u, v) \in E(G)$. The* subdivision *of e is the graph obtained from G by adding a new vertex w (of degree two) on (u, v) such that w is connected only to u, v.*

In general, a *subdivision* of G is obtained by a series of edge subdivisions.

Definition 3.5. *Let H, G be graphs. We say that G contains a* subdivision *of H (or that H is a* topological minor *of G) if a subdivision of H is isomorphic to a subgraph of G.*

Question: For which integer values r, Petersen's graph contains a subdivision of K_r?
Answer: $r \leq 4$.

Remark 3.6. *We note that if a graph H is a subdivision of a graph G, then H has vertices of degree $\Delta(G)$.*

We conclude that Petersen's graph does not contain a subdivision of K_5, as its maximum degree is 3.
Question: Why are these notions important?

Theorem 9 (Kuratowski 1930)**.** *A graph G is planar if and only if it does not contain a subdivision of K_5 and of $K_{3,3}$.*

Theorem 10 (Wagner 1937)**.** *A graph G is planar if and only if it does not contain a minor of K_5 and of $K_{3,3}$.*

3.1 Connection of Minors and Subdivisions to Coloring

Question: Is it true that if $\chi(G) \geq k$ then G contains some given subgraph H?

Theorem 11 (Erdős '59)**.** *For every pair of integers k, g there exists a graph G that contains no cycles of length $\leq g$ (i.e.,* girth$(G) > g$*) and satisfies $\chi(G) > k$.*

However, the following two conjectures were raised.

Conjecture 3.7 (Hadwiger '43)**.** *If* $\chi(G) \geq k$ *then* $\eta(G) \geq k$*.*

Conjecture 3.8 (Hajós 1940s)**.** *If* $\chi(G) \geq k$ *then* G *contains a subdivision of* K_k*.*

We note that Hajós' conjecture is stronger, as existence of a subdivision implies existence of a minor.

Hadwiger's conjecture – what is known:

- $k = 1$: trivially true.
- $k = 2$: trivially true.
- $k = 3$: true, as any cycle is a subdivision of K_3.
- $k = 4$: was proved by Hadwiger himself in the same paper where he posed the conjecture. The following stronger result was given by Dirac.

  Theorem 12 (Dirac '52)**.** *Let* G *be a graph with* $\delta(G) = 3$*. Then* G *contains a subdivision of* K_4*.*

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Lecture 6

Instructor: Prof. Michael Krivelevich *Scribe: Yevgeny Levanzov*

1 Colorings, Density, Minors and Subdivisions

1.1 Hadwiger and Hajós Conjectures

Theorem 1 (Dirac '52)**.** *Let G be a graph with $\delta(G) \geq 3$. Then G contains a subdivision of K_4.*

Proof. We will prove a stronger statement: if G has at most one vertex of degree < 3, then G contains a subdivision of K_4. The proof is by induction on $n = |V(G)|$.
Base: $n = 4$ – all degrees are 3, and so $G = K_4$.
Step: Assume that the claim holds for all graphs on $< n$ vertices, and let G be a graph on n vertices. Assume first that G has a unique vertex x with $d(x) < 3$. If $d(x) < 2$, then $G' = G - \{x\}$ has at most one vertex of degree < 3, and so we can apply induction on G', completing the proof. Now, if $d(x) = 2$, then let y, z be the neighbors of x. If $(y, z) \notin E(G)$, then consider $G' = G - \{x\} + (y, z)$. By induction, G' has a subdivision of K_4. If this subdivision uses edge (y, z), replace it by the path (y, x, z) in G, getting a subdivision of K_4. Now, if $(y, z) \in E(G)$, we can assume that $d(y) = d(z) = 3$. (otherwise, we can delete x and apply induction). Assume first that y, z have another common neighbor w. If $d(w) > 3$, then $G' = G - \{x, y, z\}$ has at most one vertex of degree < 3, and so we can apply induction. Otherwise, if $d(w) = 3$, then $G' = G - \{x, y, z, w\}$ has at most one vertex of degree < 3, and so we can again apply induction. The only case left (for $d(x) = 2$) is when y, z do not have another common neighbor (apart from x). In this case, contract x, y, z (i.e., the edges $(x, y), (y, z)$) to a vertex x', and denote by G' the graph obtained after the contraction. As G' has at most one vertex of degree < 3, we can apply induction and get a subdivision of K_4 in G'. We can then turn it into a subdivision of K_4 in G by using vertices x, y, z instead of x' if needed.

The only case left is when $\delta(G) \geq 3$. By deleting edges (if needed) from G we can assume that $\delta(G) = 3$. Let x be a vertex of degree 3, and let u, v, w be its neighbors. If all edges inside $\{u, v, w\}$ are present, then we get (together with x) a copy of K_4. Hence we can assume, without loss of generality, that $(u, v) \notin E(G)$. Then let $G' = G - \{x\} + \{(u, v)\}$. We observe that G' has at most one vertex of degree < 3 (vertex w), and so we can apply induction to G' and get a subdivision of K_4 in G'. Replacing the edge (u, v) in this subdivision (if exists) by the path (u, x, v), we get a subdivision of K_4 in G. ∎

Corollary 2. *Hadwiger's conjecture holds for $k = 4$.*

Proof. If $\chi(G) \geq 4$, then G contains a 4-critical subgraph G'. It holds that $\delta(G') \geq 3$, and so we can apply Theorem 1 to get a subdivision of K_4 and hence a minor of K_4. ∎

Further cases of Hadwiger's conjecture:

- $k = 5$: equivalent to the 4-color theorem (Wagner '37).
- $k = 6$: equivalent to the 4-color theorem (Robertson, Seymour, Thomas '93).
- $k \geq 7$: open.

The latest result regarding Hadwiger's conjecture is the following:

Theorem 3 (Postle '20)**.** *If $K_t \not\preceq G$ then $\chi(G) = O(t(\log\log t)^6)$.*

1.2 Complete Minors and Average Degree

Possible approach: If $\chi(G) \geq k$ then G contains a k-critical subgraph G'. It holds that $\delta(G') \geq k-1$.

Possible statement: Denote $\overline{d}(G) = \frac{|E(G)|}{|V(G)|}$. We would like to argue that if $\overline{d}(G)$ is large, then $\eta(G)$ is large.
Kostochka '82, '84 and independently Thomason '84, proved that if $\overline{d}(G) \geq d$ then $\eta(G) = \Omega\left(\frac{d}{\sqrt{\log d}}\right)$.
The above bound is optimal, as can be seen by looking at the random graph $G(n, 1/2)$.

Theorem 4 (Kostochka '82, '84; Thomason '84)**.** *Let $G = (V, E)$ be a graph satisfying $\frac{|E(G)|}{|V(G)|} \geq d$, where d is a sufficiently large integer. Then G has a complete minor of order $\geq \frac{d}{10\sqrt{\ln d}}$ (and thus $\eta(G) \geq \frac{d}{10\sqrt{\ln d}}$).*

Proof. (Alon-Krivelevich-Sudakov '22) Assume throughout the proof that the parameters $d, n = |V(G)|$ are large enough.

Lemma 1.1. *Let $H = (V, E)$ be a graph on at most n vertices with $\delta(H) \geq n/6$. Denote $t = \frac{n}{\sqrt{\log n}}$ and let $A_1, \ldots, A_t \subseteq V$ with $|A_j| \leq \frac{n}{e^{\sqrt{\ln n}/3}}$ for all $j \in [t]$. Then there exists a subset $B \subseteq V$, $|B| \leq 3.1\sqrt{\ln n}$ such that:*

1. *B dominates all but at most $\frac{n}{e^{\sqrt{\ln n}/3}}$ vertices of H (i.e., the number of vertices outside B not having any neighbor in B is $\leq \frac{n}{e^{\sqrt{\ln n}/3}}$).*
2. *$B \setminus A_j \neq \emptyset$ for every $j \in [t]$.*
3. *$\delta(H[B]) \geq |B|/10$.*

Proof. Denote $s = 3.1\sqrt{\ln n}$. Choose a set B of s vertices at random with repetitions (i.e., choose s times a random vertex in V, repetitions are allowed).
Typical properties of B:

1. Let $v \in V(H)$. We have

$$\Pr[v \text{ is not dominated by } B] = \left(1 - \frac{d(v)}{|V|}\right)^s \underbrace{\leq}_{1-x\leq e^{-x}} e^{-s/6} \leq e^{-\sqrt{\ln n}/2}.$$

   Hence, B is expected *not* to dominate $\leq \frac{|V|}{e^{\sqrt{\ln n}/2}} \leq \frac{n}{e^{\sqrt{\ln n}/2}}$ vertices. Now, by Markov's inequality, with probability tending to 1 we have that B dominates all but $\leq \frac{n}{e^{\sqrt{\ln n}/3}}$ vertices.

2. $\Pr[B \subseteq A_j] = \left(\frac{|A_j|}{|V|}\right)^s \leq \left(\frac{6|A_j|}{n}\right)^s \leq \left(\frac{6}{e^{\sqrt{\ln n}/3}}\right)^s = o\left(\frac{1}{n}\right)$.
   Hence, by a union bound over all subsets A_j, we derive that with probability tending to 1 it holds that $B \setminus A_j \neq \emptyset$ for all $j \in [t]$.

3. Notice that we choose $s = O(\sqrt{\log n})$ vertices with repetition from V, with $|V| \geq n/6$. Hence it is unlikely to repeat a vertex. Let the choices for B be $v_1, \dots, v_s$ (as argued above, we can assume that they are distinct). Now, conditioning on $v_i = v$, the random variable $d(v_i, B)$ (i.e., the degree of v_i into B) is distributed as $\mathrm{Bin}(s-1, d(v)/|V|)$. Hence, noting that $d(v)/|V| \geq 1/6$, by standard concentration statements (e.g., Chernoff) for binomial random variables, we obtain:

$$\Pr\left[d(v_i, B) \leq \frac{s}{10}\right] \ll \frac{1}{s}.$$

Applying the union bound over all v_i's, we get the desired property of B.

■

We now prove the theorem. Let $G = (V, E)$ be a graph on n vertices and denote $d = \frac{|E(G)|}{|V(G)|}$. Let G' be a minor of G such that $\frac{|E(G')|}{|V(G')|} \geq d$ and $|V(G')| + |E(G')|$ is minimal. Let us study the properties of G'. Assume that an edge e of G' belongs to exactly t triangles of G'. By contracting e we get a graph with one vertex and $t+1$ edges less than G'. Since G' is minimal, we derive that $t + 1 > d \Rightarrow t \geq d$. We conclude that every edge of G' belongs to $\geq d$ triangles of G'. Since G' is minimal, we actually have $\frac{|E(G')|}{|V(G')|} = d$. Hence G' has a vertex v of degree $d(v) \leq 2d$. Let H be the subgraph of G' induced by the neighborhood of v. Then:

1. $|V(H)| \leq 2d$.
2. $\delta(H) \geq d$ (as every edge belongs to $\geq d$ triangles).

Remark 1.2. *The above argument, that a graph of average degree $\geq d$ has a dense minor on at most $2d$ vertices with minimum degree at least d, is due to Mader.*

Since the notion of minors is transitive, finding a complete minor of order t in H means finding a complete minor of the same order in G. We claim that H contains a $(d/3)$-connected subgraph H_0 with minimum degree $\delta(H_0) \geq 2d/3$. Indeed, if H is $(d/3)$-connected, then we can take $H_0 = H$. Otherwise, there is a partition $V(H) = A \dot\cup B \dot\cup S$ such that $|S| \leq d/3$, $A, B \neq \emptyset$, and H has no edges between A and B. Assume, without loss of generality, that $|A| \leq |B|$, which implies $|A| \leq d$. Since $\delta(H) \geq d$, every vertex $a \in A$ has $\geq d - |S| \geq 2d/3$ neighbors in A. Therefore, $a_1 \neq a_2 \in A$ have at least $d/3$ common neighbors. Hence the induced subgraph $H_0 := H[A]$ has at most $2d$ vertices, is $\geq (d/3)$-connected, and satisfies $\delta(H_0) \geq d/3$.

Set $i = 0$, and perform the following iteration $t = d/10\sqrt{\ln d}$ times. Let $H_i = (V_i, E_i) \subseteq H_0$, and suppose $A_i, \dots, A_{i-1}$ are subsets of V_i of cardinalities $|A_j| \leq \frac{2d}{e^{\sqrt{\ln(2d)/3}}}$ (where A_j represents the non-neighbors of the previously constructed connected subset B_j in V_i). We assume – and justify it later – that H_i is connected and has $\delta(H_i) > d/3$. Then in every shortest path $P = (v_0, v_1, \dots)$ in H_i the neighborhoods of vertices $v_0, v_3, v_6, \dots$ are pairwise-disjoint. Hence P has at most 15 vertices (since $|V(H_i)|/\delta(H_i) < 6$), implying that the diameter of H_i is at most 14. Applying Lemma 1.1 to H_i (recalling that $|V(H_i)| \leq 2d$), we find a set B_i with $|B_i| \leq 3.1\sqrt{\ln(2d)}$ such that B_i dominates all but $\leq \frac{2d}{e^{\sqrt{\ln(2d)/3}}}$ vertices of V_i, $B_i \backslash A_j \neq \emptyset$ for all $j \in [i-1]$, and $\delta(H_i[B_i]) \geq |B_i|/10$. This implies that $H_i[B_i]$ has at most 9 connected components. Now, turn B_i into a connected set by connecting its components sequentially, altogether adding at most $8 \cdot 13 = O(1)$ vertices to B_i, as we need to add (at most) 8 paths between the (at most) 9 connected components of $H_i[B_i]$, and each such shortest path has at most $15 - 2 = 13$ inner vertices. So now $H_i[B_i]$ is connected and has cardinality $|B_i| \leq (3.1 + o(1))\sqrt{\ln(2d)}$, B_i is connected to every previous B_j (as $B_i \setminus A_j \neq \emptyset$), and B_i dominates all but $\leq \frac{2d}{e^{\sqrt{\ln(2d)/3}}}$ vertices of V_i. Now we update:

1. $V_{i+1} = V_i \setminus B_i$.

2. A_i = vertices of V_{i+1} that are not dominated by B_i, $|A_i| \le \frac{2d}{e^{\sqrt{\ln(2d)/3}}}$.

3. $A_j = A_j \cap V_{i+1}$, for all $j \in [i-1]$.

Altogether at all t iterations we delete $\le t \cdot (3.1 + o(1))\sqrt{\ln(2d)} < d/3$ vertices. Hence indeed each H_i is connected and has $\delta(H_i) > \delta(H_0) - d/3 \ge d/3$, as promised. Therefore eventually we find t disjoint subsets $B_1, \ldots, B_t \subseteq V(H_0)$ such that $H[B_i]$ is connected for every i and H_0 has an edge between B_i, B_j for every $i \ne j$. We conclude that $\eta(H_0) \ge t \Rightarrow \eta(H) \ge t \Rightarrow \eta(G) \ge t$. ■

1.3 Complete Minors and Independence Number

Recall: for any graph G, it holds that $\chi(G) \ge |V(G)|/\alpha(G)$, where $\alpha(G)$ is the independence number of G.

Theorem 5 (Duchet-Meyniel '82)**.** *For any graph G, $\eta(G) \ge \frac{|V(G)|}{2\alpha(G)-1}$.*

Proof. Notice that it is enough to prove it for connected graphs. Indeed, if the connected components of G are $C_1, \ldots, C_t$, and $|C_i| \le \eta(C_i)(2\alpha(C_i) - 1)$, $i = 1, \ldots, t$, then trivially we have $\eta(G) \ge \eta(C_i)$ and $\alpha(G) = \sum_{i=1}^{t} \alpha(C_i)$. Hence we have $|C_i| \le \eta(G)(2\alpha(C_i) - 1)$. Summing up, we obtain:

$$|V(G)| = \sum_{i=1}^{t} |C_i| \le \eta(G) \cdot 2 \sum_{i=1}^{t} \alpha(C_i) - \eta(G) \cdot t = \eta(G) \cdot 2\alpha(G) - \eta(G) \cdot t \le \eta(G)(2\alpha(G) - 1).$$

Assume now that G is connected. We claim that there exists a *connected* dominating set[1] D and an independent set $I \subseteq D$ such that $|D| \le 2|I| - 1$. To prove it, take a largest connected set D containing an independent set I such that $|D| \le 2|I| - 1$ (this is well-defined, as the singleton $D = \{v\}$ satisfies the requirements). We claim that D is dominating. If not, then there is a vertex $u \in V(G) \setminus D$ not dominated by D. Since G is connected, we can take u to be at distance exactly 2 from D. Hence u has no neighbors in D, but has a neighbor v which in turn has a neighbor in D. But then we can add $\{u, v\}$ to D and u to I, in contradiction to the maximality of D. Let now (D, I) be as above, $|I| \le \alpha(G)$, $|D| \le 2\alpha(G) - 1$. Let $G' = G[V(G) \setminus D]$. Since D is connected and dominating, it follows that $\eta(G) \ge \eta(G') + 1$. Also, trivially, $\alpha(G') \le \alpha(G)$. Applying induction to G', we obtain:

$$|V(G) \setminus D| = |V(G')| \le \eta(G')(2\alpha(G') - 1) \le (\eta(G) - 1)(2\alpha(G) - 1).$$

As $|D| \le 2\alpha(G) - 1$, summing up, we obtain:

$$|V(G)| = |V(G) \setminus D| + |D| \le (\eta(G) - 1)(2\alpha(G) - 1) + (2\alpha(G) - 1) = \eta(G)(2\alpha(G) - 1).$$

■

2 Perfect Graphs

Definition 2.1. *A graph G is called* perfect *if $\chi(H) = \omega(H)$ for every* induced *subgraph H of G.*

Definition 2.2. *A property of graphs is called* hereditary *if it is closed under taking induced subgraphs.*

[1] A set $D \subseteq V(G)$ is *dominating* in G if $\forall v \in V(G) \setminus D$, v has a neighbor in D.

We conclude that if $\mathcal{P}$ is a *hereditary* class of graphs, for proving that every $G \in \mathcal{P}$ is perfect it is enough to prove that $\chi(H) = \omega(H)$ for every graph H in $\mathcal{P}$.

Examples of perfect graphs:

1. Complete graphs.
2. Empty graphs.
3. Bipartite graphs: if G is bipartite, then $\chi(G) = 1$ if $E(G) = \emptyset$ – then $\omega(G) = 1$, and $\chi(G) = \omega(G) = 2$ otherwise.
4. Complements of bipartite graphs: need to prove: if G is bipartite, then $\chi(\overline{G}) = \omega((\overline{G})$ (homework 1).

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Lecture 7	
Instructor: Prof. Michael Krivelevich	*Scribe: Yevgeny Levanzov*

1 Perfect Graphs[1]

Examples of perfect graphs:

1. Complete graphs.
2. Empty graphs.
3. Bipartite graphs: if G is bipartite, then $\chi(G) = 1$ if $E(G) = \emptyset$ – then $\omega(G) = 1$, and $\chi(G) = \omega(G) = 2$ otherwise.
4. Complements of bipartite graphs: need to prove: if G is bipartite, then $\chi(\overline{G}) = \omega((\overline{G})$ (homework 1).
5. Interval graphs: vertices – closed finite intervals on the line; edges – $(I_1, I_2) \in E(G) \Leftrightarrow I_1 \cap I_2 \neq \emptyset$. We have shown that if G is an interval graph then $\chi(G) = \omega(G)$.
6. Cographs – graphs without an induced copy of P_4, a path on 4 vertices (homework 1).
7. Comparability graphs (see Section 1.1).
8. Chordal graphs (see Section 1.2).

1.1 Comparability Graphs

Definition 1.1. *Given an undirected graph G, a* transitive orientation *D of G is an orientation of the edges of G satisfying*

$$\overrightarrow{(u,v)}, \overrightarrow{(v,w)} \in E(D) \Rightarrow \overrightarrow{(u,w)} \in E(D).$$

A graph G is called a comparability graph *if it admits a transitive orientation.*

The reason such graphs are called "comparability graphs" is due to the following proposition.

Proposition 1.2. *If P is a partial order of a finite set V, then $G = G(P)$ on vertex set V, defined by $(u, v) \in E(G) \Leftrightarrow u, v$ are comparable in P, is a comparability graph.*

Proof. Follows from the definition of a partially ordered set (poset). ■

Example: Let $G = (A \cup B, E)$ be a bipartite graph. Orienting all edges from A to B gives a transitive orientation. Thus G is a comparability graph.

Proposition 1.3 (Berge '60)**.** *Comparability graphs are perfect.*

[1]The original title was "A Lesson in Perfection".

Proof. Since comparability is hereditary, it suffices to prove that $\chi(G) = \omega(G)$ for any comparability graph G. Let G be a comparability graph, and let D be a transitive orientation of G. Let P be a longest directed path in D. Then the vertices of P induce a clique in G of size $|V(P)|$, implying that $\omega(G) \geq |V(P)|$. By the Gallai–Hasse–Roy–Vitaver theorem (which we saw in the first lecture), $\chi(G) \leq |V(P)|$, and so $\chi(G) \leq \omega(G)$, and thus $\chi(G) = \omega(G)$. This implies that G is perfect. ■

Remarks:

1. If G is an interval graph, then $\overline{G}$ is a comparability graph ($(I_1, I_2) \in E(\overline{G})$ if and only if $I_1 \cap I_2 = \emptyset$, orient $\overrightarrow{(I_1, I_2)}$ if I_1 precedes I_2 on the real line).

2. The fact that $\chi(G) = \omega(G)$ for a comparability graph G follows also from Mirsky's theorem (1971): for any finite partially ordered set P, the minimal number of antichains covering P is equal to the maximum length of a chain in P.

1.2 Chordal Graphs

Definition 1.4. *A graph G is called* chordal *(or* triangulated*) if every cycle in G of length at least* 4 *has a chord.*

1.2.1 Characterization of Chordal Graphs

Definition 1.5. *Let G be a graph. Let $G_1, G_2 \subseteq G$ be induced subgraphs such that:*

1. $G = G_1 \cup G_2$.
2. $G_1 \cap G_2 = S$.

Then we say that G is obtained by pasting G_1 and G_2 along S.

Proposition 1.6. *G is a chordal graph if and only if G can be obtained from complete graphs by a sequence of pastings along a complete graph.*

Proof. $(\Longleftarrow)$: Assume that G_1, G_2 are chordal, G is obtained by pasting G_1, G_2 along a complete graph S. We want to show that G is chordal. Let C be a cycle in G of length ≥ 4. If $C \subseteq G_1$ or $C \subseteq G_2$, then C has a chord as G_1, G_2 are chordal. Otherwise, C has a vertex in $G_1 \setminus G_2$ and a vertex in $G_2 \setminus G_1$. Then C has at least 2 non-consecutive vertices residing in S. Since S is a clique, these vertices are connected by an edge, constituting a chord in C.
$(\Longrightarrow)$: Assume that G is chordal. We show by induction on $|V(G)|$ that G can be obtained as described. If G is complete then the statement is obvious (which includes the base case of $|V(G)| = 1$). Assume then that $(a, b) \notin E(G)$. Let $X \subseteq V(G) \setminus \{a, b\}$ be a minimal (by inclusion) set separating a and b. Let C be a component of $G \setminus X$ containing a, and define $G_1 = G[V(C) \cup X]$, $G_2 = G[V(G) \setminus V(C)]$. Now, as G_1, G_2 are induced subgraphs of G, they are chordal (as chordality is hereditary). Moreover, as G_1, G_2 are proper subgraphs of G, by the induction hypothesis they are constructible as described. Obviously, G is obtained from G_1, G_2 by pasting along $S = G[X]$, and so it suffices to prove that S is a complete graph. Assume not, and let $s, t \in X$ with $(s, t) \notin E(G)$. Since X is a minimal separating set, both s and t have neighbors in C. Let P_1 be a minimal path connecting s and t within C. Arguing similarly, we can find a minimal path P_2 connecting s and t in G_2 with all internal vertices outside of X. Taking $C' = P_1 \cup P_2$, recalling the minimality of P_1, P_2 and the assumption $(s, t) \notin E(G)$, we obtain a chordless cycle of length ≥ 4 – a contradiction. ■

We now present an alternative characterization of chordal graphs, as follows.

Definition 1.7. *A vertex v in a graph G is called* simplicial *if the neighborhood $N(v)$ of v induces a clique in G. Given a graph G on n vertices, a* simplicial elimination ordering *of G is an order $\sigma = (v_1, \ldots, v_n)$ on $V(G)$ such that every vertex v_i, $1 \leq i \leq n$, is a simplicial vertex in the induced subgraph $G[\{v_1, \ldots, v_i\}]$.*

Theorem 1 (Dirac '61)**.** *G is a chordal graph if and only if G has a simplicial elimination ordering.*

Proof. We will prove only the easy direction: assuming that G has a simplicial elimination ordering, we need to show that G is chordal. Let C be a cycle of length at least 4 in G. Let v be the last vertex of C according to the simplicial elimination ordering. Then the neighbors preceding v in the ordering, including its two neighbors u, w on the cycle, form a clique. Hence (u, w) is a chord in C.

The proof of the other direction is more challenging. ■

Theorem 2 (Berge '60)**.** *Chordal graphs are perfect.*

Proof. Using the first characterization of chordal graphs, it is enough to prove that if a graph G is obtained from chordal (and thus, by induction[2], perfect) graphs G_1, G_2 by pasting along a complete graph S, then G is perfect. We have $\chi(G_1) = \omega(G_1)$, $\chi(G_2) = \omega(G_2)$. Since G is obtained through pasting, every clique in G resides entirely in G_1 or entirely in G_2, and so we have $\omega(G) = \max\{\omega(G_1), \omega(G_2)\}$. Hence, since chordality is hereditary, it is enough to prove that

$$\chi(G) \leq \max\{\omega(G_1), \omega(G_2)\} = \max\{\chi(G_1), \chi(G_2)\}.$$

Let c_1 be an optimal coloring of G_1, and let c_2 be an optimal coloring of G_2. Since S is a clique, both c_1 and c_2 color the vertices of S in distinct colors. By renaming colors if necessary, we can ensure that $c_1(v) = c_2(v)$ for every $v \in S$. By "gluing" c_1 and c_2 in a natural way, we obtain a coloring c of G in $\max\{\chi(G_1), \chi(G_2)\} = \max\{\omega(G_1), \omega(G_2)\} = \omega(G)$ colors. Then $\chi(G) = \omega(G)$, as required. ■

1.3 Weak Perfect Graph Theorem

We have seen many examples of perfect graphs G where the complement graph $\overline{G}$ is also perfect (e.g., complete and empty graphs, bipartite graphs and their complements, interval and comparability graphs). This is a part of a general phenomenon, formalized by the *Weak Perfect Graph Theorem* of Lovász[3].

Theorem 3 (Weak Perfect Graph Theorem, Lovász '72)**.** *A graph G is perfect if and only if its complement $\overline{G}$ is perfect.*

Proof.

Lemma 1.8. *Let G be a graph. The following are equivalent:*

1. *G is perfect.*

2. *For every induced subgraph H of G and every vertex $v \in V(H)$, there is an independent set I in H such that $v \in I$, and I intersects every maximum clique in H[4].*

[2]Since complete graphs are perfect, the base of the induction holds.

[3]Remarkably, the theorem was proven by Lovász when he was only 22 years old.

[4]In other words, I satisfies $\omega(H \setminus I) = \omega(H) - 1$.

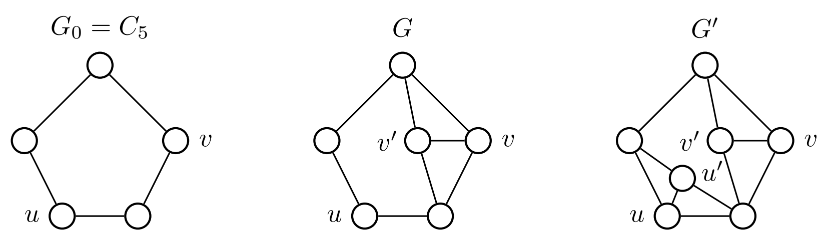
3. *For every induced subgraph H of G, there is an independent set I in H, intersecting every maximum clique in H.*

Proof. (1) $\implies$ (2): Let $\chi(H) = k$, and let $(V_1, \ldots, V_k)$ be an optimal coloring of H. Assume, without loss of generality, that $v \in V_1$. The graph $H \setminus V_1$ has chromatic number $k-1$, and hence, due to perfection of G, $\omega(H \setminus V_1) = k - 1$. As such, V_1 intersects every maximum clique in H.
(2) $\implies$ (3): Trivial.
(3) $\implies$ (1): Assume that $\omega(G) = k$. Color the graph by iteratively finding an independent set I, intersecting every maximum clique (as in (3)), coloring I in a fresh color and discarding it. Since the clique number (of the remaining subgraph) decreases at every iteration due to (3), this procedure colors G in k colors, implying $\chi(G) \leq k = \omega(G)$. Hence $\chi(G) = \omega(G)$. The same argument can be applied for any induced subgraph of G. It follows that G is perfect. ∎

Definition 1.9. *Let $G = (V, E)$ be a graph, and let $v \in V$.* Replicating *v means:*

1. *Adding a new vertex v' to G.*
2. *Connecting v' to v.*
3. *Connecting v' to every neighbor of v.*

Example: Start with $G_0 = C_5$ (a non-perfect graph). Let G be obtained by replicating a vertex of C_5 (vertex v in the figure). If we now replicate any vertex of degree 2 in G (vertex u in the figure), we obtain a graph G' with $\chi(G') = 4 > \omega(G') = 3$ (note that $\chi(G) = \omega(G) = 3$).



We conclude that the property $\chi(H) = \omega(H)$ is *not* preserved by replication (of vertices of H). Hence the following lemma might appear surprising.

Lemma 1.10 (Replication Lemma)**.** *Replicating any vertex of a perfect graph G produces a perfect graph.*

Proof. Let G be a perfect graph and let G' be obtained by replicating a vertex v in G. We will use condition (3) of Lemma 1.8. Let H be an induced subgraph of G'. If H contains at most one vertex from $\{v, v'\}$, then H is isomorphic to an induced subgraph of G, and so as G is perfect, by condition (3) of Lemma 1.8, H has an independent set I, intersecting every maximum clique in H. We now assume that $v, v' \in V(H)$. Denote $H' = H \setminus \{v'\}$. Then $H' \subseteq G$ is an induced subgraph, and thus, by condition (2) of Lemma 1.8, it has an independent set $I \subseteq V(H')$ with $v \in I$, intersecting every maximum clique in H'. It is easy to see that I also intersects every maximum clique in H, as every maximum clique in H containing v' contains v as well. We conclude, using condition (3) of Lemma 1.8, that G' is perfect, as required. ∎

We now prove the Weak Perfect Graph Theorem. Let G be a perfect graph, and we want to prove that $\overline{G}$ is perfect. By condition (3) of Lemma 1.8, it is enough to prove that G has a clique that intersects every maximum independent set (as perfection is hereditary, this argument also holds for every induced subgraph of G). We denote $\alpha(G) = \alpha$, and also denote by k the number of independent sets of size α in G. Now, for every vertex $v \in V(G)$, we denote by α_v the number of independent sets of size α that contain v. We note that $\sum_{v \in V(G)} \alpha_v = k \cdot \alpha$. We now create a new graph G' by replicating every vertex $v \in V(G)$ exactly $\alpha_v - 1$ times (where if $\alpha_v = 0$ – we delete v; if $\alpha_v = 1$ – we do not replicate v). We observe that $|V(G')| = \sum_{v \in V(G)} \alpha_v = k \cdot \alpha$, and that $\alpha(G') = \alpha(G) = \alpha$. (Indeed, replication does not increase the independence number, as for every vertex v, at most one of the vertices v, v' (a replication of v) can belong to an independent set. Moreover, as every vertex u that is not contained in a maximum independent set I must have a neighbor in I, so does its replication u' (and so u' cannot be added to I). Finally, it is clear that every independent set of size α in G survives replication.) We have that $\chi(G') \geq |V(G')|/\alpha(G') = k$. In fact, we claim that $\chi(G') = k$. Indeed, we can color G' by going over all independent sets I of size $|I| = \alpha$ in G, and color one of the replicated vertices (including the original ones) of every $v \in I$ (that was not yet colored) in the same fresh color. This produces a proper k-coloring[5] of G', as copies of distinct vertices from an independent set in G are not adjacent (i.e., form an independent set), and as G has exactly k independent sets of size α. By Lemma 1.10, G' is perfect, and thus $\omega(G') = k$. Let K' be a clique of size k in G'. Now, as $|K' \cap I| \leq 1$ for every independent set I in G', we have that K' intersects every color class in the above described k-coloring of G'. Now, as every color class in the above described k-coloring of G' is a "replica" of a maximum independent set in G, by projecting K' back to the graph G, we obtain a clique K in G, intersecting every maximum independent set in G. The same argument holds also for every induced subgraph of G. Thus, the complement of G satisfies condition (3) of Lemma 1.8, and is thus perfect. ■

Remark: Lovász (1972) also proved the following statement.

Theorem 4 (Lovász '72)**.** *A graph G is perfect if and only if $|V(H)| \leq \alpha(G) \cdot \omega(G)$ for every induced subgraph H of G.*

Clearly, Theorem 4 implies the Weak Perfect Graph Theorem due to the symmetry of the clique and the independence numbers between a graph and its complement.

1.4 Strong Perfect Graph Theorem

Observe that if a graph G is perfect then G does not contain an induced odd cycle of length at least 5. By the Weak Perfect Graph Theorem, G also does not contain a complement of an induced odd cycle of length at least 5. Berge conjectured in 1960 that these conditions are sufficient as well. This has become a theorem.

Theorem 5 (Strong Perfect Graph Theorem, Chudnovsky, Robertson, Seymour, Thomas '2006[6])**.** *A graph G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd cycle of length ≥ 5.*

[5]Moreover, this coloring is equitable, with all color classes having the same size α.

[6]The proof was announced in 2002.

Graph and Hypergraph Coloring, Tel Aviv Univ., Spring 2022 May 2, 2022

Lecture 8

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1 List Coloring

1.1 Basic Definitions and Results

Definition 1.1. *Given a graph $G = (V, E)$, a* list assignment *L is an assignment of color lists $L(v) \subseteq \mathbb{Z}_+$, $v \in V$ to the vertices of G.*

Definition 1.2. *Given a list assignment L for a graph G, an L-*coloring *is a function f from the colors $\bigcup_{v\in V} L(v)$ to $\mathbb{Z}_+$ such that:*

1. *$f(v) \in L(v)$ for every $v \in V(G)$.*
2. *$f(u) \neq f(v)$ for every $e = (u, v) \in E(G)$.*

*If such a choice exists, then G is said to be L-*colorable *(or L-*choosable*).*

Definition 1.3. *A graph G is k-*choosable *(or k-*list-colorable*) if G is L-choosable for* every *list assignment $L = (L(v))_{v\in V(G)}$ such that $|L(v)| = k$ for every $v \in V(G)$.*

Definition 1.4. *The* choice number *(or* list chromatic number*) of G, denoted by $\text{ch}(G)$ or by $\chi_\ell(G)$, is the minimum k for which G is k-choosable.*

This notion was introduced by Vizing in 1976, and independently by Erdős, Rubin and Taylor (ERT) in 1979.

Question: Does it happen that $\text{ch}(G) > \chi(G)$?
Answer: Yes (but very rarely).

Proposition 1.5. *For every graph G, $\text{ch}(G) \geq \chi(G)$.*

Proof. If G is k-choosable then G is L-colorable for the list assignment $L = \{L(v) = \{1, \ldots, k\}\}_{v\in V(G)}$. Thus G is k-colorable and so $\chi(G) \leq \text{ch}(G)$. ∎

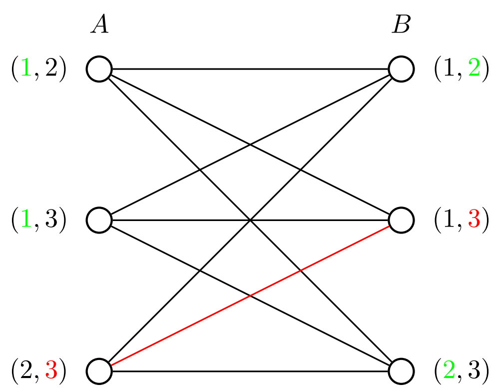
Example: $\text{ch}(C_{2n}) = 2$.

Proof. Let $L = (L(v))_{v\in V(C_{2n})}$ be a list assignment satisfying $|L(v)| = 2$ for every $v \in V(C_{2n})$. If the lists in L are identical, then we deal with the usual coloring and G is L-choosable (as $\chi(C_{2n}) = 2$). Otherwise, since $G = C_{2n}$ is connected, we can find two adjacent vertices $x, y \in V(C_{2n})$ such that $c_0 \in L(x) \setminus L(y)$. Now, color $f(x) = c_0$, and then color the vertices of C_{2n} moving from x to y. Each time, when arriving to color $z \neq y$, only one neighbor of z has been colored, and thus the list $L(z)$ contains a color c different from that of the neighbor of z already colored. We can assign $f(z) = c$. Finally when arriving at y, since $f(x) = c_0$ and $c_0 \notin L(y)$, we have at most one color from $L(y)$ used already for its neighbors. We can thus assign the other color to y. ∎

1.2 The Choice Number of Complete Bipartite Graphs

Proposition 1.6. $\mathrm{ch}(K_{3,3}) > 2 = \chi(K_{3,3})$.

Proof.



Let $V(K_{3,3}) = A \cup B$, and let $L = (L(v))_{v \in A\cup B}$ be the list assignment as in the figure above. Let now $f(v) \in L(v)$, $v \in A \cup B$. Then it is easy to see that $|f[A]| = |\{f(a) \,|\, a \in A\}| \geq 2$, $|f[B]| = |\{f(b) \,|\, b \in B\}| \geq 2$ (as no element appears in all three 2-subsets of $[3]$). Since $f[A], f[B] \subseteq [3]$, we have that $f[A] \cap f[B] \neq \emptyset$. Hence there are $a \in A$ and $b \in B$ such that $f(a) = f(b)$, thus witnessing a monochromatic edge $e = (a, b)$ under f. ∎

We can generalize this simple example as follows.

Proposition 1.7 (ERT '79)**.** *Let* $n = \binom{2s-1}{s}$ *for a positive integer* s*. Then* $\mathrm{ch}(K_{n,n}) > s$.

Proof. Let the sides of $K_{n,n}$ be A, B, $|A| = |B| = n = \binom{2s-1}{s}$. We assign to the vertices $u \in A$ all subsets of cardinality s from $[2s-1]$ as color lists. We do the same for the side B. Let now f be such that $f(v) \in L(v)$ for all $v \in A \cup B$. Then the set $f[A] = \{f(a) \,|\, a \in A\}$ has cardinality at least s (as otherwise there is a set $S \subseteq [2s-1]$, $|S| = s$ such that $S \cap f[A] = \emptyset$ — a contradiction). Similarly, the set $f[B] = \{f(b) \,|\, b \in B\}$ has cardinality $|f[B]| \geq s$. But then $f[A] \cap f[B] \neq \emptyset$ (since $f[A], f[B] \subseteq [2s-1]$). This implies that there are vertices $a \in A, b \in B$ for which $f(a) = f(b)$, thus creating a monochromatic edge $e = (a, b)$ under f. ∎

Corollary 1. $\mathrm{ch}(K_{n,n}) \geq (1/2 + o(1)) \log_2 n$[1].

We conclude that we *cannot* bound $\mathrm{ch}(G)$ by *any* function of $\chi(G)$.

Theorem 2 (ERT '79)**.** $\mathrm{ch}(K_{n,n}) \leq (1 + o(1)) \log_2 n$.

Proof. We will prove: if integers n, d satisfy $2^d > 2n$ then $\mathrm{ch}(K_{n,n}) \leq d$. Let the sides of $K_{n,n}$ be A, B, $|A| = |B| = n$, and assume that we are given color lists $L = \{L(v) \,|\, v \in A \cup B\}$ such that $|L(v)| = d$ for every v. Denote $C = \bigcup_{v \in A\cup B} L(v)$. Partition $C = C_A \cup C_B$ at random, by putting every color $c \in C$ into C_A independently and with probability $1/2$. Our goal is to use colors from C_A to color A, and

[1] Follows from the approximation $\binom{2k}{k} = \Theta(2^{2k}/\sqrt{k})$.

colors from C_B to color B. If $L(a) \cap C_A \neq \emptyset$ for all $a \in A$ and $L(b) \cap C_B \neq \emptyset$ for all $b \in B$, then we can color A, B using separate colors. We have

$$\Pr[L(a) \cap C_A = \emptyset] = \Pr[L(b) \cap C_B = \emptyset] = \left(\frac{1}{2}\right)^d.$$

Then by the union bound:

$$\Pr[(\exists a \in A.L(a) \cap C_A = \emptyset) \vee (\exists b \in B.L(b) \cap C_B = \emptyset)] \leq 2n \cdot \left(\frac{1}{2}\right)^d < 1.$$

We conclude that with a positive probability such a split $C = C_A \cup C_B$ exists, and so $\text{ch}(K_{n,n}) \leq d$. ■

To summarize,

$$(1/2 + o(1)) \log_2 n \leq \text{ch}(K_{n,n}) \leq (1 + o(1)) \log_2 n.$$

1.2.1 Connection of ch($K_{n,n}$) to Property B

Definition 1.8. *A* hypergraph $H = (V, E)$ *is a collection of subsets of* V*. If* $|e| = s$, $\forall e \in E$, *then* H *is called* s-uniform.

Definition 1.9. *A hypergraph* $H = (V, E)$ *is* 2-colorable*, or has* Property B[2]*, if there is a partition* $V = A \dot{\cup} B$ *such that for every* $e \in E$, $e \cap A \neq \emptyset$, $e \cap B \neq \emptyset$.

Question: Determine the value of

$$m(d) = \min\{|E(H)| : H = (V, E) \text{ is a } d\text{-uniform hypergraph having } \textit{no} \text{ Property } B\}.$$

Example: $m(2) = 3$.

Proposition 1.10 (ERT '79)**.** *For every* $d \geq 2$, $\text{ch}(K_{m(d),m(d)}) > d$.

Proof. Let $H = (V, E)$ be a d-uniform hypergraph with $m(d)$ edges and without Property B (which exists by the definition of $m(d)$). Take a complete bipartite graph $K_{m(d),m(d)}$ with sides A, B satisfying $|A| = |B| = m(d)$. Now, assign color lists $L(v)$ to all $v \in V(K_{m(d),m(d)})$ as follows: assign distinct edges of H to distinct vertices of side A as color lists, and do the same for side B. If f is a choice of colors for all vertices of $K_{m(d),m(d)}$ without a monochromatic edge, then define $f[A] = \{f(a) \,|\, a \in A\}$, $f[B] = \{f(b) \,|\, b \in B\}$. Then, since f does not produce a monochromatic edge, we have $f[A] \cap f[B] = \emptyset$. We observe that the sets $f[A]$ and $f[B]$ both form a *cover* of H, namely, intersect every edge of H (and these covers are disjoint). But then, by extending $f[A]$ and $f[B]$ arbitrarily to all $V(H)$, that is, adding each of the remaining vertices in $V(H)$ either to $f[A]$ or to $f[B]$, arbitrarily, we get a valid 2-coloring of H – a contradiction, as H is assumed *not* to have Property B. Thus $\text{ch}(K_{m(d),m(d)}) > d$. ■

It is known (and will be discussed later) that

$$\Omega\left(2^d \cdot \sqrt{\frac{d}{\log d}}\right) \leq m(d) \leq O(2^d d^2).$$

This implies that $\text{ch}(K_{n,n}) = (1 + o(1)) \log_2 n$.

[2]Named after Felix Bernstein, who researched this property in 1908.

1.3 Degrees and Choosability – Upper Bound

Recall: For a positive integer d, a graph is called *d-degenerate* if there is an ordering $\sigma = (v_1, \ldots, v_n)$ of $V(G)$ (with $|V(G)| = n$) such that for every $1 \leq i \leq n$, vertex v_i has at most d neighbors preceding it (such an ordering is called *d-degenerate*). The *degeneracy* of G, denoted by $\text{degen}(G)$, is the smallest value d such that G is d-degenerate. Clearly, $\text{degen}(G) \leq \Delta(G)$.
We have proven: $\chi(G) \leq 1 + \text{degen}(G) \leq 1 + \Delta(G)$.

Proposition 1.11. *For every graph G, $\text{ch}(G) \leq 1 + \text{degen}(G)$.*

Proof. Let $d = \text{degen}(G)$, and let $\sigma = (v_1, \ldots, v_n)$ be a d-degenerate order of G. Let $L = \{L(v) \mid v \in V(G)\}$ be a list assignment satisfying $|L(v)| = d+1$ for every $v \in V(G)$. We color $V(G)$ from the family L of lists a vertex by a vertex according to σ. Once arriving at coloring v_i, $1 \leq i \leq n$, out of $d+1$ colors in $L(v_i)$ at most d have been used on the neighbors of v_i preceding it in σ. We can then assign $f(v_i)$ to be a color from $L(v_i)$ not used already on the neighbors of v_i. This implies that $\text{ch}(G) \leq d + 1$. ■

Conclusion: $\text{ch}(G) \leq 1 + \Delta(G)$.

1.4 Choosability Version of Brooks' Theorem

Recall: Let $G = (V, E)$ be a connected graph of maximum degree Δ. Then $\chi(G) \leq \Delta$ unless G is a clique or an odd cycle ($\chi(K_{\Delta+1}) = \Delta + 1, \Delta(K_{\Delta+1}) = \Delta$; $\chi(C_{2n+1}) = 3, \Delta(C_{2n+1}) = 2$).
Our goal now is to prove the choosability version of Brooks' theorem: If G is a connected graph with maximum degree Δ then $\text{ch}(G) \leq \Delta$, unless $G = K_{\Delta+1}$ or $G = C_{2n+1}$.

Lemma 1.12 (Vizing '76)**.** *Let $G = (V, E)$ be a connected graph, and let $L = \{L(v) \mid v \in V\}$ be a list assignment for G satisfying $|L(v)| \geq d(v)$ for all $v \in V$.*

1. *If there is $y \in V$ such that $|L(y)| > d(y)$, then G is L-choosable.*
2. *If G is 2-connected and the lists in L are not identical, then G is L-choosable.*

Proof.
1. Since G is connected there is an order $\sigma = (v_1, \ldots, v_n)$ on V such that:
   (a) $v_n = y$.
   (b) $\forall 1 \leq i \leq n-1$, v_i has at least one neighbor following it in σ.

   (Such an ordering can be obtained, for example, by executing BFS on G starting from y, and then putting the vertices into σ in the reverse order of their discovery.)
   Now, color G vertex by vertex according to σ. For every $1 \leq i \leq n-1$, we have $|L(v_i)| \geq d(v_i)$ and $\leq d(v_i) - 1$ colors have been used on the neighbors of v_i preceding it. Thus there exists a color for v_i. When arriving to color $v_n = y$, we recall that $|L(y)| \geq d(y) + 1$, and hence there is a color available for y as well.

2. We assume that G is 2-connected and not all lists are identical. We can find two vertices $x, y \in V$ such that $(x, y) \in E$ and $c \in L(x) \setminus L(y)$. Define now $G' = G - \{x\}$. We also update the lists:

$$L'(v) = \begin{cases} L(v) & , \ (v, x) \notin E \\ L(v) \setminus \{c\}, & (v, x) \in E \end{cases} .$$

Since $\kappa(G) \geq 2$ and $|V(G')| = |V(G)| - 1$ we derive that G' is connected. We have $|L'(v)| \geq d_{G'}(v)$ for every $v \in V(G')$, $|L'(y)| \geq |L(y)| \geq d_{G'}(y)+1$. Now, by Item 1 we have that G' is L'-choosable. We can extend a legitimate choice for G' by assigning color c to x.

■

Definition 1.13. *A graph G is* degree-choosable *if G is L-choosable for every list assignment $L = \{L(v) \mid v \in V(G)\}$ satisfying $|L(v)| = d(v)$ for all $v \in V(G)$.*

Lemma 1.14. *Let G be a connected graph. If G contains a non-empty induced subgraph H which is degree-choosable, then G is degree-choosable.*

Proof. We are given a list assignment $L = \{L(v) \mid v \in V(G)\}$, $|L(v)| = d(v)$ for all $v \in V(G)$. Let $G' = G \setminus H$. In every connected component C of G' we have $|L(v)| \geq d_C(v)$ for every $v \in C$ and in addition there is a vertex $y \in C$ connected to H (due to connectedness of G), for which $|L(y)| > d_C(y)$. Then C is L-choosable by Lemma 1.12, part 1. Let f be a (proper) choice function for G'. Now, we update the lists $L(v)$ for $v \in V(H)$ in the following natural way:

$$L'(v) = L(v) \setminus \{f(u) \mid u \notin H, (u, v) \in E(G)\}.$$

Then, $|L'(v)| \geq d_G(v) - d_G(v, V(G) \setminus V(H)) = d_H(v)$. Since we assumed that H is degree-choosable, we can complete f by choosing colors for the vertices $v \in V(H)$ from their lists $L'(v)$. ■

Lemma 1.15 (ERT '79)**.** *Let G be a 2-connected graph which is not complete and is not an odd cycle. Then G contains an even cycle with at most one chord.*

Proof. Since G is 2-connected it necessarily contains a cycle. Assume first that G has a triangle. Let Q be a largest clique in G (by our assumption, $|Q| \geq 3$). Since G is 2-connected and not a clique, there is a path connecting two vertices of Q and passing outside of Q (i.e., all its edges are outside of Q). Let P be a shortest path joining two vertices of Q with all its edges being outside of Q. Denote by $x, y \in Q$ the endpoints of P. If P has only two edges, then since Q is maximal, there is a vertex $w \in Q$ not connected to the vertex $z \in P \setminus Q$. Then the 4-cycle $xzywx$ is a cycle of length 4 with one chord. If P has at least 3 edges, then its internal vertices have no neighbors in Q (as otherwise we could shorten P). In this case, we can create an even cycle with at most one chord either by adding the edge (x, y) to P or by adding the path xwy with $w \in Q$ to Q (such w exists as $|Q| \geq 3$).

We now assume that G has no K_3. Let C be a shortest cycle in G. Then C has no chords, and thus we can assume that C is odd (as otherwise we are done). Then, since G is 2-connected, $G \neq C$, we can connect two vertices of C by a path outside of C. Then, by combining P with the appropriate part of C, we can find an even cycle in G. Let C_0 be a shortest even cycle in G. If C_0 has at most one chord then we are done. Otherwise, C_0 has at least two chords. Then, recalling that G has no K_3, and observing that each chord in C_0 creates two odd cycles with the parts of C_0, by performing case analysis, we can show that G contains an even cycle that is shorter than C_0, which is a contradiction. ■

Theorem 3 (Borodin '77; ERT '79)**.** *Let G be a connected graph. Assume that G is* not *degree-choosable. Then all blocks*[3] *in G are either complete graphs or odd cycles.*

Proof. By Lemma 1.14, it suffices to prove that if some block B of G is not complete and not an odd cycle, then B is degree-choosable. Since an edge is a complete graph, we can assume that B is 2-connected. Then, by Lemma 1.15, B contains an even cycle with a most one chord. Thus, by Lemma

[3] A *block* of a graph is a maximal subgraph that has no cut-vertex.

1.14, it is enough to require: an even cycle with at most one chord is degree-choosable. Let H be an even cycle with at most one chord. Let L be a list assignment for $v \in V(H)$ with $|L(v)| \geq d_H(v)$ for all $v \in V(H)$. If the lists in L are not identical, then by Lemma 1.12, part 2, we can color H from the lists. Otherwise, the lists are identical: if H is an even cycle with no chord, then since $\chi(H) = 2$ and $|L(v)| = 2$ for all $v \in V(H)$, we can color H from L; if H is an even cycle with a chord, then since $\Delta(H) = 3$ and all lists are identical, we have that $|L(v)| = 3$ for all $v \in V(H)$, and so as $\chi(H) \leq 3$ we get that H is colorable from the lists in L. ■

A corollary from Theorem 3 is the choosability version of Brooks' theorem.

Corollary 4 (Vizing '76; ERT '79)**.** *If G is a connected graph of maximum degree Δ, then* $\mathrm{ch}(G) \leq \Delta$, *unless $G = K_{\Delta+1}$ or $G = C_{2n+1}$.*

We will see the proof of this corollary in the next lecture.

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Lecture 9

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1 List Coloring

1.1 Choosability Version of Brooks' Theorem

In the last lecture we proved the following theorem.

Theorem 1 (Borodin '77; ERT '79). *Let G be a connected graph. Assume that G is* not *degree-choosable. Then all blocks*[1] *in G are either complete graphs or odd cycles.*

A corollary from Theorem 1 is the choosability version of Brooks' theorem.

Corollary 2 (Vizing '76; ERT '79). *If G is a connected graph of maximum degree Δ, then $\mathrm{ch}(G) \leq \Delta$, unless $G = K_{\Delta+1}$ or $G = C_{2n+1}$ $(\Delta = 2)$*[2].

Proof. If G is $(\Delta - 1)$-degenerate, then G is Δ-choosable. Thus we may assume that G contains an induced subgraph H with $\delta(H) \geq \Delta$. But then $H = G$ and G is a Δ-regular. Assume, towards a contradiction, that G is *not* Δ-choosable. Then by Theorem 1 we have that every block of G is either a complete graph or an odd cycle (in particular, a regular subgraph). Let us look on the block structure of G (which is a tree). If G contains more than one block, then the cut-vertex v of a leaf block B_0 in the block (tree) structure of G has a larger degree than the rest of the vertices of the block – a contradiction. We conclude that G contains a single block, which is either a $K_{\Delta+1}$ or an odd cycle $(\Delta = 2)$. ■

We now present a very recent, short and self-contained proof of Corollary 2.

Proof. (Krivelevich[3] '22) The proof borrows its main idea from the nice argument of Zając for the classical Brooks' theorem, which we saw at the beginning of the course. Let G be a connected graph of maximum degree $\Delta \geq 3$ such that $G \neq K_{\Delta+1}$[4], and denote $n = |V(G)|$. We proceed by induction on n. When $n \leq \Delta$, we can choose distinct colors for the vertices from any Δ-uniform list assignment.

For the induction step, suppose $n > \Delta$ and consider a Δ-uniform list assignment L on G. We seek an L-coloring f. If G has a vertex v with $d(v) < \Delta$, then we can apply the induction hypothesis to obtain an L-coloring of each component of $G - \{v\}$. These colorings together define f on $G - \{v\}$ and use at most $\Delta - 1$ colors on the neighbors of v. Thus a color remains available in $L(v)$ to extend f to an L-coloring of G. Hence we may assume that G is Δ-regular.

Claim 1.1. *If (u, v, w) is an induced 3-vertex path such that a color $f(u)$ has been chosen from $L(u)$, then it is possible to choose $f(w)$ from $L(w)$ so that at most one color from $\{f(u), f(w)\}$ appears in $L(v)$.*

[1] A *block* of a graph is a maximal subgraph that has no cut-vertex.

[2] Note that this implies Brooks' theorem, namely, if G is not a $K_{\Delta+1}$ and not an odd cycle, then $\chi(G) \leq \Delta$.

[3] The proof was discovered by M. Krivelevich several days after the current lecture. It was not shown in class and formally was not part of the course.

[4] We saw in the last lecture that a (connected) graph of maximum degree 2 is 2-choosable if and only if it is not an odd cycle.

Proof. Set $f(w) = f(u)$ if $f(u) \in L(w)$, choose $f(w) \in L(w)$ arbitrarily if $f(u) \notin L(v)$, and choose $f(w) \in L(w) \setminus L(v)$ if $f(u) \in L(v) \setminus L(w)$. We can do this in the last case because $L(v)$ and $L(w)$ are different but have the same size. ∎

We first apply Claim 1.1 in the special case where G has a non-spanning cycle (i.e., not Hamiltonian) C such that some vertex of C has no neighbor outside C. Since G is connected, we can then find two consecutive vertices w and v along C such that all neighbors of w lie on C but v has a neighbor u outside C. By the induction hypothesis, $G \setminus V(C)$ has an L-coloring f. We will extend f to $V(C)$ to obtain an L-coloring of G. Using Claim 1.1, we choose $f(w) \in L(w)$ so that at most one color from $\{f(u), f(w)\}$ appears in $L(v)$.

Index the vertices along C in order as $x_1, \ldots, x_r$ with $x_1 = w$ and $x_r = v$. Having chosen $f(w)$ as above, we next choose colors for $x_2, \ldots, x_r$ in order. For $i \leq r-1$, when we reach x_i at least one of its Δ neighbors is uncolored, and hence a color in its list is available to use as $f(x_i)$. When we reach v, we have used at most one color from $L(v)$ on its neighbors u and w and hence at most $\Delta - 1$ colors on its Δ neighbors, so again a color in $L(v)$ remains available to use as $f(v)$.

Now consider the general case for G. Since G is connected and not complete, G has P_3 as an induced subgraph, say with vertices u, v, w in order. Let P be a longest path in G starting with u, v, w, say $P = (x_1, x_2, x_3, x_4, \ldots, x_\ell)$, where $(x_1, x_2, x_3) = (u, v, w)$. By the choice of P, all neighbors of x_ℓ lie on P. Let x_i be the neighbor of x_ℓ with smallest index. The cycle $(x_i, \ldots, x_\ell, x_i)$ contains all neighbors of x_ℓ; if it does not include all the vertices of G, then the preceding special case applies.

Hence we may assume $\ell = n$ and $i = 1$. Now, since $\Delta \geq 3$, vertex v has a neighbor x_j outside $\{u, w\}$. With $x_3 = w$, we choose colors for the vertices in the order $u, x_3, \ldots, x_{j-1}, x_n, \ldots, x_j, v$. Every vertex in this order has a later neighbor, except for v. However, after choosing any $f(u) \in L(u)$, Claim 1.1 allows us to pick $f(w)$ from $L(w)$ so that at most one color from $\{f(u), f(w)\}$ appears in $L(v)$. Hence for each subsequent vertex, when we reach it we have used at most $\Delta - 1$ colors from its list on its neighbors, so a color remains available for it. ∎

1.2 Degrees and Choosability – Lower Bound

Alon proved in 2000 that every graph G with minimum degree[5] d satisfies $\mathrm{ch}(G) = \Omega(\log d)$.

Remarks:

1. The equivalent of the above claim for regular coloring is (strongly) false. In particular, the chromatic number cannot be bounded from below by *any* function of minimum/average degree. For example, if $G = K_{n,n}$ then $\delta(G) = n$ but $\chi(G) = 2$.

2. The above result allows for a simple algorithm to *estimate* (very crudely) the choice number of a graph G, as follows:

   (a) Find $d := \mathrm{degen}(G)$ (by checking whether G is i-degenerate for $i = 0, 1 \ldots$). We then know that $\mathrm{ch}(G) \leq 1 + d$.

   (b) By the definition of $\mathrm{degen}(G)$, there is a subgraph H of G with $\delta(H) \geq d$. This implies that $\mathrm{ch}(G) \geq \mathrm{ch}(H) = \Omega(\log d)$.

We will prove Alon's result for the regular case.

[5] The statement remains true if we consider graphs of average degree d instead, due to the well-known argument that every graph of average degree d contains a subgraph whose minimum degree is at least $d/2$.

Theorem 3. *Let G be a d-regular graph. It holds that* $\mathrm{ch}(G) = \Omega(\log d)$.

The proof uses the method/concept of *graph containers*, introduced by Saxton and Thomason '2015, and independently by Balogh, Morris and Samotij '2015. For the case of regular graphs, the basic approach was discovered earlier by several researchers, most notably by Sapozhenko '2001.

Lemma 1.2. *Let G be a d-regular graph on n vertices, and let $\varepsilon > 0$. Then there is a collection of sets $\mathcal{C} = \{C_i\}_{i\in I}$ with $C_i \subseteq V(G)$ for all i, such that:*

1. $|\mathcal{C}| \leq \sum_{i\leq \frac{n}{\varepsilon d}} \binom{n}{i}$.
2. *For every $C_i \in \mathcal{C}$, $|C_i| \leq \frac{n}{\varepsilon d} + \frac{n}{2-\varepsilon}$.*
3. *For every independent set $I \subseteq V(G)$ there exists $C \in \mathcal{C}$ such that $I \subseteq C$.*
4. *For every $C \in \mathcal{C}$ it holds that $\Delta(G[C]) \leq \varepsilon d$.*

Remarks:

1. For G as above it holds that $\alpha(G) \leq n/2$, and this is tight (e.g., consider a d-regular bipartite graph with $n/2$ vertices on each side).
2. If G is a (d-regular) bipartite graph with $n/2$ vertices on each side (in particular, $G = K_{d,d}$) then it contains at least $2^{n/2}$ independent sets.

Proof of Lemma 1.2. Let $S \subseteq V(G)$ be an independent set. Set $T = \emptyset$ and perform the following process: as long as there is a vertex $v \in S \setminus T$ such that $|N(v) \setminus N(T)| \geq \varepsilon d$, we update $T := T + \{v\}$. Clearly, this process terminates after at most $\frac{n}{\varepsilon d}$ steps with $|T| \leq \frac{n}{\varepsilon d}$. In addition, we have:

1. $T \subseteq S$.
2. Since S is an independent set, $S \cap N(T) = \emptyset$.
3. Every vertex $v \in S \setminus T$ has at least $(1-\varepsilon)d$ neighbors in $N(T)$.

(We call such a set T a *fingerprint*.)
We now define

$$B = B(T) = \{v \in V(G) \setminus (T \cup N(T)) \,|\, d(v, N(T)) \geq (1-\varepsilon)d\}.$$

Note that B is determined by T alone. Now we define $C = T \cup B$ – a container in the collection. Since $|T| \leq \frac{n}{\varepsilon d}$, we will have at most $\sum_{i\leq \frac{n}{\varepsilon d}} \binom{n}{i}$ sets/containers C. By the definition of B and since for every $v \in S \setminus T$, $d(v, N(T)) \geq (1-\varepsilon)d$, it follows that $v \in B$. This implies that $S \subseteq (T(S) \cup B(T)) = C$. In addition, for every $v \in T$, $d(v, C) = 0$, and for every $v \in (C \setminus T) = B$, $d(v, C) \leq \varepsilon d$, and so $\Delta(G[C]) \leq \varepsilon d$. It is left to estimate from above $|C|$, merely, $|B|$. We have $B \subseteq V(G) \setminus N(T)$, and so

$$|B| \leq n - |N(T)|. \tag{1}$$

Also, every vertex in B sends $\geq (1-\varepsilon)d$ edges to $N(T)$. Since G is d-regular, we have $|B| \cdot (1-\varepsilon)d \leq |N(T)| \cdot d$, and so

$$|B| \leq \frac{|N(T)|}{1-\varepsilon}. \tag{2}$$

Now, multiplying inequality (1) by $\frac{1}{2-\varepsilon}$, inequality (2) by $\frac{1-\varepsilon}{2-\varepsilon}$, and adding them up we get

$$|B| \leq \frac{n}{2-\varepsilon}.$$

Thus we have $|C| = |T| + |B| \leq \frac{n}{\varepsilon d} + \frac{n}{2-\varepsilon}$, completing the proof. ■

Theorem 4. *Let $d \geq k \geq 2$ be integers satisfying:*

$$\left(\frac{ed}{\ln d}\right)^{\frac{\ln d}{d}\cdot k^2} < e^{\left[1-\left(\frac{1}{2}+\frac{1}{\ln d}\right)\frac{k}{k-1}\right]^k}. \tag{3}$$

Let G be a d-regular graph. Then $\mathrm{ch}(G) > k$.

We observe that (3) is satisfied if $d > C2^k k^4$ for some large enough constant $C > 0$.

Proof. Let us fix a palette $\mathcal{K} = \{1, \ldots, k^2\}$. Now, choose lists $\{L(v) \mid v \in V(G)\}$ uniformly at random, where $L(v)$ is a random k-subset of $\mathcal{K}$. We need to prove: with positive probability the choice of lists creates a list assignment $L = \{L(v)\}_{v\in V(G)}$ such that G is not L-choosable. Given a family of lists, let f be a choice function. We denote $S_i = f^{-1}(i) = \{v \in V(G) \mid f(v) = i\}$. We have k^2 disjoint independent sets such that for every $v \in V(G)$ there exists $1 \leq i \leq k^2$ such that $v \in S_i$, $i \in L(v)$. We will use containers in place of independent sets (to make smaller the number of choices in the union bound), as follows. Set $\varepsilon = 1/\ln d$, and apply Lemma 1.2 to get a family $\mathcal{C}$ of containers of size at most

$$|\mathcal{C}| \leq \sum_{i\leq \frac{n}{\varepsilon d}} \binom{n}{i} \leq \sum_{i\leq \frac{n\ln d}{d}} \binom{n}{i} \underbrace{\leq}_{\sum_{i=0}^k \binom{n}{i} \leq \left(\frac{en}{k}\right)^k} \left(\frac{ed}{\ln d}\right)^{\frac{n\ln d}{d}}.$$

It suffices to show that with positive probability for any choice of k^2 containers $C_1, \ldots, C_{k^2} \in \mathcal{C}$ there is a vertex v such that $v \notin \bigcup_{i\in L(v)} C_i$. We start by fixing a choice of k^2 containers. By the above estimate, there number of choices for k^2 containers is at most $\left(\frac{ed}{\ln d}\right)^{\frac{n\ln d}{d}\cdot k^2}$. Given a family of containers $\{C_i\}_{i\in[k^2]}$, define, for $v \in V(G)$, $k_v = |\{i \mid v \in C_i\}|$. Then $\sum_{v\in V(G)} k_v = \sum_{i=1}^{k^2} |C_i|$. Recall that for every i,

$$|C_i| \leq \frac{n}{\varepsilon d} + \frac{n}{2-\varepsilon} = \frac{n\ln d}{d} + \frac{n}{2-1/\ln d} \leq \frac{n}{2}(1 + 1/\ln d).$$

So $\sum_{v\in V(G)} k_v \leq \frac{nk^2}{2}(1 + 1/\ln d)$. Denote $\overline{k} = \frac{1}{n}\sum_v k_v \leq \frac{k^2}{2}(1 + 1/\ln d)$. Now, for a given $v \in V(G)$ the probability that the list $L(v)$ of v satisfies $v \notin \bigcup_{i\in L(v)} C_i$ is

$$\frac{\binom{k^2-k_v}{k}}{\binom{k^2}{k}} \geq g(k_v),$$

where the function $g(z)$ is defined by

$$g(z) = \left(\frac{k^2-k-z}{k^2-k}\right)^k = \left(1 - \frac{z}{k^2-k}\right)^k,$$

for $0 \leq z \leq k^2 - k$, and $g(z) = 0$ otherwise. Hence, due to independent choices of the lists $L(v)$, the probability that the choice of colors does *not* fail for a given choice of containers is at most

$$\prod_{v\in V(G)} (1 - g(k_v)) \leq e^{-\sum_v g(k_v)}.$$

Now, since $g(z)$ is a convex function, by Jensen's inequality we get that

$$e^{-\sum_v g(k_v)} \leq e^{-ng(\overline{k})}.$$

In addition, since $g(z)$ is a non-increasing function and $\overline{k} \le k^2\left(\frac{1}{2} + \frac{1}{\ln d}\right)$, it follows that

$$g(\overline{k}) \ge \left(1 - \frac{k^2\left(\frac{1}{2}+\frac{1}{\ln d}\right)}{k^2-k}\right)^k = \left(1 - \left(\frac{1}{2} + \frac{1}{\ln d}\right)\frac{k}{k-1}\right)^k.$$

Hence, the above probability is at most $e^{-n\left(1-\left(\frac{1}{2}+\frac{1}{\ln d}\right)\frac{k}{k-1}\right)^k}$. By a union bound, if

$$\left(\frac{ed}{\ln d}\right)^{\frac{n\ln d}{d}\cdot k^2} \cdot e^{-n\left(1-\left(\frac{1}{2}+\frac{1}{\ln d}\right)\frac{k}{k-1}\right)^k} < 1,$$

then there is a choice of lists $L = \{L(v)\}_{v \in V(G)}$ failing *all* choices of containers. It follows that if (3) (i.e., the assumption of the theorem) holds then $\mathrm{ch}(G) > k$. In order to solve (3) (directly) for k, we need to solve

$$\exp\left[(1+o(1))\frac{\ln^2 d}{d}\cdot k^2\right] < \exp\left[2^{-k}(1+o(1))\right] \Longrightarrow (1+o(1))\frac{\ln^2 d}{d}\cdot k^2 < 2^{-k}(1+o(1)) \Longrightarrow 2^k k^2 < \frac{d}{\ln^2 d}.$$

So, if $d > C2^k k^4$ for some large enough constant $C > 0$ then the above holds, and $\mathrm{ch}(G) > k$. ■

Corollary 5. *If G is* any *d-regular graph then* $\mathrm{ch}(G) \ge (1 - o(1))\log_2 d$.

Finally, Theorem 3 follows directly from Corollary 5.

1.3 Choosability in Planar Graphs

Theorem 6 (Thomassen '94)**.** *Every planar graph is 5-choosable.*

Proof. Clearly, we can add edges to a planar graph G (while preserving planarity) so that:

1. The outer face of G is bounded by a cycle.
2. Every inner face of G is a triangle.

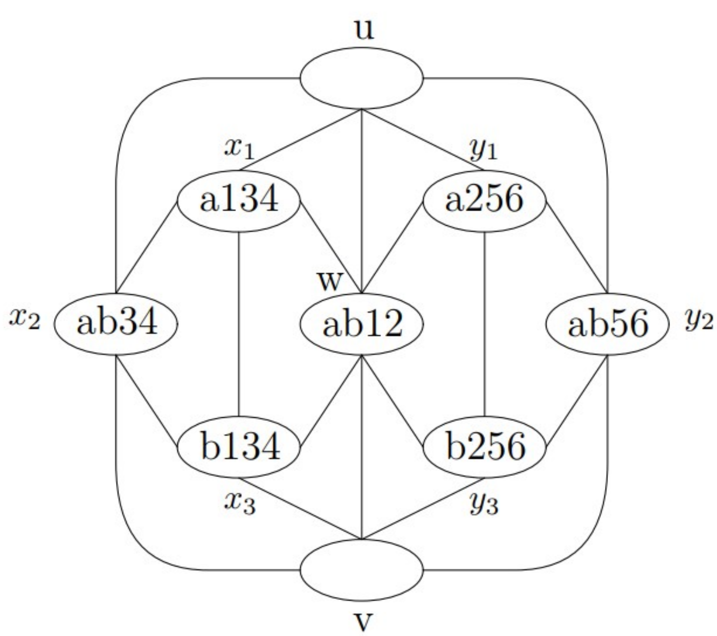
We will prove a stronger statement. Let G be a planar graph as above, and let $L = \{L(v)\}_{v \in V(G)}$ be a list assignment such that:

1. For some two consecutive vertices u, v along the enclosing cycle of the boundary, $|L(u)| = |L(v)| = 1$ and $L(u) \ne L(v)$.
2. $|L(w)| = 3$ for other (from u, v) vertices on the enclosing cycle.
3. $|L(x)| = 5$ for all vertices inside the enclosing cycle.

Then G is L-choosable.
The proof is by induction on $|V(G)|$.
Base: $|V(G)| = 3$ (and so the graph is K_3 with vertices u, v, w). In this case we can write: $L(u) = \{c_1\}, L(v) = \{c_2\}, c_1 \ne c_2$, $|L(w)| = 3$. By choosing $c \in L(w) \setminus \{c_1, c_2\}$ we get a proper L-choosing.
Step: Assume that the external cycle is $(v_1, \ldots, v_p)$. In this case $|L(v_1)| = 1, |L(v_p)| = 1, L(v_1) \ne L(v_p)$, $|L(v_i)| = 3$ for $i \ne 1, p$.
Case 1: G has a chord with respect to the external cycle: $(v_i, v_j) \in E(G)$, $1 \le i \le j-2 \le p-2$. First, color the subgraph of G bounded (geometrically) by the chord (v_i, v_j) and the part of the external cycle

containing v_1, v_p. This can be done by induction. Let f be the corresponding choice function. We now update: $L(v_i) = \{f(v_i)\}, L(v_j) = \{f(v_j)\}, L(v_i) \neq L(v_j)$. Now, apply induction on to the second half of the graph.
Case 2: The external cycle in G has no chord. Let us look at a neighbor of v_2 other than v_1, v_3. Since the external cycle in G does not have chords, all neighbors $u_1, \ldots, u_m$ of v_2 are inside the external cycle. Since every internal face is a triangle, we have a path $u_1 u_2 \ldots u_m$ such that $(v_1, u_1), (v_3, u_m) \in E(G)$. Let $L(v_1) = \{c_1\}$. Since $v_2 \neq v_1, v_p$, we have $|L(v_2)| = 3$. Choose colors $c_2, c_3 \in L(v_2) \setminus \{c_1\}$. Now, update $L(u_i) := L(u_i) \setminus \{c_2, c_3\}$ for every $i \in [m]$. Then, $|L(u_i)| \geq 3$ for every $i \in [m]$. Consider now the graph $G - \{v_2\}$. This is a planar graph with an external cycle $C' = (v_1, u_1, \ldots, u_m, v_3, \ldots, v_p, v_1)$, along which two vertices v_1 and v_p satisfy $|L(v_1)| = |L(v_p)| = 1, L(v_1) \neq L(v_p)$, and $|L(w)| \geq 3$ for the remaining external vertices. Then, by induction, $G - \{v_2\}$ is choosable from the lists. Denote by c' the color chosen for v_3. Now we complete this choice by choosing a color $c \in \{c_2, c_3\} \setminus \{c'\}$ for v_2. We conclude that G is choosable from the lists. ■

We note that Theorem 6 is tight: Voigt ('93) gave an example of a non-4-choosable planar graph. The best (smallest) construction of a non-4-choosable planar graph is due to Maryam Mirzakhani[6] ('96) and has 63 vertices. Here we present a construction of a non-4-choosable planar graph due to Shai Gutner[7] ('96). Let W be the following planar graph:



Now, take 12 vertex-disjoint copies of W, identify vertices u in each of the 12 copies, do the same for v, and finally add edge (u, v). The obtained graph G is clearly still planar and has $7 \cdot 12 + 2 = 86$ vertices.

Now, take the set $S = \{7, 8, 9, 10\}$ and label each of the 12 copies of W by a distinct pair (a, b), where $a \neq b \in S$. Put $S = \{7, 8, 9, 10\}$ for the lists of vertices u, v (which are shared by all the copies). For the copy labeled by (a, b), use a, b where they appear in the figure of W above, to form the lists of the vertices of this copy.

[6] She came up with this construction at the age of 19, later becoming a renowned mathematician, and the first woman to win the prestigious Fields Medal. Sadly, she died of cancer at the young age of 40.

[7] He came up with this construction while still being a high school student.

We claim that G is not choosable from the lists defined above. Suppose the contrary, and let f be a choice of colors from the lists of $V(G)$. Assume $f(u) = a \neq b = f(v)$, and consider the copy of W labeled by (a, b). Now, the colors left for vertex w in this copy are just $\{1, 2\}$. If $f(w) = 1$ then the colors left for the vertices of the triangle $\{x_1, x_2, x_3\}$ are just $\{3, 4\}$, and thus f creates a monochromatic edge inside $\{x_1, x_2, x_3\}$, which is a contradiction. Now, If $f(w) = 2$ then the colors left for the vertices of the triangle $\{y_1, y_2, y_3\}$ are just $\{5, 6\}$, and thus f creates a monochromatic edge inside $\{y_1, y_2, y_3\}$, which is again a contradiction. We conclude that G is not choosable[8] from the lists, and so $\mathrm{ch}(G) > 4$.

[8] Note that G is 3-colorable.

1 Coloring Random Graphs

1.1 Basic Definitions and Notation

Binomial random graph – $G(n,p)$**:**
The vertex set of a graph $G \sim G(n,p)$ is $V(G) = \{1,\dots,n\} = [n]$, and for all $1 \le i < j \le n$, $\Pr[(i,j) \in E(G)] = p = p(n)$, independently.
Equivalently, given $G = ([n], E)$, $\Pr_{G(n,p)}[G] = p^{|E|}(1-p)^{\binom{n}{2}-|E|}$.
We note that $G(n,p)$ is a product probability space.
Case $p = \frac{1}{2}$**:** For all $G = ([n], E)$, $\Pr_{G' \sim G(n,p)}[G = G'] = 2^{-\binom{n}{2}}$ (i.e., uniform distribution).

Erdős–Rényi random graph[1] **–** $G(n,m)$**:**
The sample space of the distribution $G(n,m)$ is

$$\Omega = \{G = ([n], E) \,|\, |E| = m\}.$$

For every $G \in \Omega$, $\Pr[G] = \frac{1}{|\Omega|} = \frac{1}{\binom{\binom{n}{2}}{m}}$ (i.e., uniform distribution).
We have $G(n,p) \approx G(n,m)$ if $m \approx \binom{n}{2}p$.

Remark: The above is not exactly true. For example, if $A =$ "G has exactly m edges", then

$$\Pr_{G \sim G(n,m)}[A] = 1,$$

but

$$\Pr_{G \sim G(n,p)}[A] = \Pr\left[\mathrm{Bin}\left(\binom{n}{2}, p\right) = m\right] = \Theta\left(\frac{1}{\sqrt{\binom{n}{2}p}}\right).$$ [2]

Asymptotic assumptions and notation:

1. The number of vertices $n \to \infty$.
2. A *graph property* P_n is a collection of graphs with vertex-set $[n]$. Then $P = \{P_n\}_{n \in \mathbb{N}}$ is a sequence of graph properties.

[1] This random graph model was introduced in 1959 by Paul Erdős and Alfréd Rényi in their seminal paper. The binomial random graph model was introduced contemporaneously and independently by Edgar Gilbert. Historically, both of these models are called (somewhat inaccurately) the Erdős–Rényi model.

[2] That is, roughly one over the standard deviation.

Definition 1.1. *A graph property* $P = \{P_n\}_{n\in\mathbb{N}}$ *holds* with high probability[3] (**whp**) *in* $G(n,p)$ *if*

$$\lim_{n\to\infty} \Pr[G \sim G(n,p) \text{ has } P_n] = 1.$$

Example: $G \sim G(n, \frac{1}{2111})$ is Hamiltonian **whp**.

Estimates on binomial coefficients:

1. For every integer $1 \le k \le n$,

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \sum_{i=0}^{k} \binom{n}{i} \le \left(\frac{en}{k}\right)^k.$$

2. $\binom{2n}{n} = \Theta(4^n/\sqrt{n})$ (follows from Stirling's approximation).

Markov's inequality:
Let X be a non-negative random variable such that $\mathbb{E}[X]$ exists. Then for every $a > 0$,

$$\Pr[X \ge a] \le \frac{\mathbb{E}[X]}{a}.$$

Therefore if $a \gg \mathbb{E}[X]$ then $\Pr[X \ge a] = o(1)$.

Questions: Given $G \sim G(n, 1/2)$ (or $G(n, p = p(n))$ in general), $\chi(G)$ is a random variable.

1. What is its typical behavior?
2. Algorithmic aspects — developing an efficient (i.e., polynomial-time) algorithm for coloring G with as few colors as possible (i.e., the number of colors should be close to a **whp** upper bound on $\chi(G)$).

1.2 Lower Bounding the Chromatic Number of a Random Graph

Recall: For every graph G, $\chi(G) \ge |V(G)|/\alpha(G)$.
Therefore, it is enough to get a typical upper bound on $\alpha(G)$ for $G \sim G(n, 1/2)$. We define for integer $1 \le k \le n$,

$$f(k) = f(k,n) = \binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}}.$$

Explanation: Let X be the random variable that counts the number of independent sets of size k in $G \sim G(n, 1/2)$. Decompose $X = \sum_{S\subseteq[n],\,|S|=k} X_S$, where X_S is the indicator random variable defined by

$$X_S = \begin{cases} 1, & G[S] \text{ is independent} \\ 0, & \text{otherwise} \end{cases}.$$

[3] Another common term is *asymptotically almost surely (a.a.s.)*.

By the linearity of expectation,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{|S|=k} X_S\right] = \sum_{|S|=k} \mathbb{E}[X_S] = \sum_{|S|=k} \Pr[G[S] \text{ is independent}] = \binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} = f(k,n).$$

Therefore, $f(n,k) =$ the expected number of independent sets of cardinality k.
Define $k_0 = \max\{k \mid f(k,n) \geq 1\}$ (note that k_0 is well-defined). We now estimate k_0:

$$f(n,k) \approx 1 \Longrightarrow \binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} \approx 1 \Longrightarrow \left(\frac{cn}{k}\right)^k \left(\frac{1}{2}\right)^{k(k-1)/2} \approx 1 \Longrightarrow \frac{cn}{k}\left(\frac{1}{2}\right)^{(k-1)/2} \approx 1.$$

Solving the above equation gives

$$k_0 = 2\log_2 n - 2\log_2\log_2 n + O(1) = (2 - o(1))\log_2 n.$$

Thus we expect $\alpha(G) \approx k_0$. We have

$$\frac{f(k-1)}{f(k)} = \frac{\binom{n}{k-1}2^{-\binom{k-1}{2}}}{\binom{n}{k}2^{-\binom{k}{2}}} = \frac{k}{n-k+1} \cdot 2^{k-1} \underbrace{\approx}_{k\approx k_0} \frac{k}{n} \cdot n^{2+o(1)} = k \cdot n^{1+o(1)}. \tag{1}$$

Now, if $k = k_0 + 1$ then, by definition, $f(k) < 1$, and thus for $k = k_0 + 2$ by (1) we have that $f(k) = O(n^{-1})$. Therefore, by Markov's inequality, it holds that for $G \sim G(n, 1/2)$ **whp** $\alpha(G) < k_0 + 2 \Longrightarrow \alpha(G) \leq k_0 + 1$. Therefore **whp**

$$\chi(G) \geq \frac{n}{k_0 + 1} = \frac{n}{(2 - o(1))\log_2 n}.$$

1.3 Upper Bounding the Chromatic Number of a Random Graph

1.3.1 Coloring by Excavation

Lemma 1.2. *Let $n \geq m \geq k$ be integers, and let G be a graph on n vertices satisfying: for every $V_0 \subseteq V(G), |V_0| = m$ it holds that V_0 spans an independent set of size k. Then $\chi(G) \leq n/k + m$.*

Proof. Start with $G_0 := G$, and for as long as $|V(G_0)| \geq m$, find an independent set I of cardinality $|I| = k$, color it by a fresh color, and discard: $G_0 = G_0 - I$. Once $|V(G_0)| < m$ — color the remaining vertices in fresh separate colors.
The total number of colors needed: stage 1 – repeated $\leq n/k$ times and so uses $\leq n/k$ colors; stage 2 – $|V(G_0)| < m$ and so uses $< m$ further colors. Therefore, the total number of colors is $\leq n/k + m$. ■

For a fixed $V_0 \subseteq [n], |V_0| = m$, we have that $G[V_0] \sim G(m, 1/2)$.
Goal: to find $m = m(n), k = k(n)$ such that

$$\Pr[\alpha(G(m, 1/2)) < k] \ll \frac{1}{\binom{n}{m}}.$$

Then, by the union bound,

$$\Pr[\exists V_0 \subseteq [n].\, (|V_0| = m) \wedge (\alpha(G[V_0]) < k)] \leq \binom{n}{m} \cdot \Pr[\alpha(G(m, 1/2)) < k] = o(1).$$

Hence our goal is to find $m = m(n), k = k(n)$ such that for $G \sim G(m, 1/2)$ we will have an exponentially small bound on $\Pr[\alpha(G) < k]$.

1.3.2 Janson's Inequality[4]

General setting:
Ω – a finite ground set, $\{A_i\}_{i\in I}$ – subsets of Ω. Generate a random subset $R \subseteq \Omega$ by $\Pr[r \in R] = p_r$, for all $r \in \Omega$, independently.
Goal: to estimate from above

$$\Pr[\textit{none} \text{ of } A_i \text{ falls into } R].$$

Example: $\Omega = \binom{[n]}{2} = E(K_n)$, $\Pr[e \in R] = p$ for all $e \in E(K_n)$. Now, for every $S \subseteq [n], |S| = 3$, $A_S =$ the set of three edges inside S. We consider the collection $\{A_S\}_{|S|=3}$. The event "No A_S is fully inside R" is equivalent to the event "$K_3 \not\subseteq G$".

Returning back to the general setting, for every $i \in I$, we define the indicator

$$X_i = \begin{cases} 1, & A_i \subseteq R \\ 0, & \text{otherwise} \end{cases}.$$

Let $X = \sum_{i\in I} X_i$ – number of subsets A_i, $i \in I$, fully inside R. It holds trivially that

$$\Pr[\textit{none} \text{ of } A_i \text{ falls into } R] = \Pr[X = 0].$$

We would like to compute $\mathbb{E}[X]$. Denote

$$\mu = \mathbb{E}[X] = \mathbb{E}\left[\sum_{i\in I} X_i\right] = \sum_{i\in I}\prod_{r\in A_i} p_r.$$

Poisson paradigm:
Under certain assumptions[5] we would (like to) have that

$$\Pr[X = 0] = e^{-\mu}.$$

If all A_i, $i \in I$, are disjoint:

$$\Pr[X = 0] = \Pr[\textit{none} \text{ of } A_i \text{ is in } R] = \prod_{i\in I}(1 - \textstyle\prod_{r\in A_i} p_r).$$

Frequently we have $\prod_{r\in A_i} p_r = o(1)$, and so

$$1 - \prod_{r\in A_i} p_r \approx e^{-\prod_{r\in A_i} p_r},$$

and so

$$\Pr[X = 0] \approx \exp\left(-\sum_{i\in I}\prod_{r\in A_i} p_r\right) = e^{-\mu}.$$

Taking correlation into account:
For $i \neq j \in I$, write $i \sim j$ if $A_i \cap A_j \neq \emptyset$. Now define

$$\Delta := \sum_{\substack{i\sim j \\ \text{ord. pairs}}} \Pr[(A_i \subseteq R) \wedge (A_j \subseteq R)] = \sum_{\substack{i\sim j \\ \text{ord. pairs}}} \Pr[X_i = X_j = 1].$$

[4] Appeared in Janson, Łuczak, Ruciński '90.
[5] These assumptions would imply that X is distributed approximately according to $\mathrm{Pois}(\mu)$.

Janson's inequality states that under the above assumptions

$$\Pr[X=0] \le e^{-\mu+\Delta/2}.$$

Generalized/extended Janson's inequality:
Assuming in addition that $\mu \le \Delta$, we have

$$\Pr[X=0] \le e^{-\mu^2/(2\Delta)}.$$

Going back to our setup:
For $k \le m \le n$, $G \sim G(m,1/2)$, we are interested in estimating $\Pr[\alpha(G) < k]$. Let $\Omega = \binom{[m]}{2}$ – set of all pairs of vertices. For every $e \subseteq [m]$, $\Pr[e \in R] = 1/2$, where R is the set of *non-edges* of Ω. Define a family $\{A_S\}, A_S \subseteq \Omega$ as follows: for every $S \subseteq [m], |S| = k$, A_S = set of pairs inside S. We have

$$\Pr[\textit{no } A_S \text{ falls inside } R] = \Pr[\alpha(G) < k].$$

Define now random variables as follows: for every $S \subseteq [m], |S| = k$, define indicator X_S by

$$X_S = \begin{cases} 1, & S \text{ is an indep. set} \\ 0, & \text{otherwise} \end{cases}.$$

Let $X = \sum_{\substack{S\subseteq[m]\\|S|=k}} X_S$ be a random variable counting the number of independent sets of cardinality k in $G(m,1/2)$. Our goal is to estimate $\Pr[X=0]$. Using previous notation,

$$\mu = \mathbb{E}[X] = \binom{m}{k} 2^{-\binom{k}{2}} = f(k,m).$$

Now, we have that

$$\begin{aligned}
\Delta &= \sum_{\substack{S,S'\subseteq[m]\\|S|=|S'|=k\\2\le|S\cap S'|\le k-1}} \Pr[X_S = X_{S'} = 1] \\
&= \underbrace{\binom{m}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}}}_{\substack{\text{choosing } S,\\ \text{requiring } X_S=1}} \cdot \sum_{j=2}^{k-1} \underbrace{\binom{k}{j}}_{S\cap S'} \underbrace{\binom{m-k}{k-j}}_{S'-S} \left(\frac{1}{2}\right)^{\binom{k}{2}-\binom{j}{2}} \\
&= \underbrace{\binom{m}{k}^2 \left(\frac{1}{2}\right)^{2\binom{k}{2}}}_{=\mu^2} \cdot \sum_{j=2}^{k-1} \underbrace{\frac{\binom{k}{j}\binom{m-k}{k-j}}{\binom{m}{k}} \cdot 2^{\binom{j}{2}}}_{:=u_j} \\
&= \mu^2 \cdot \sum_{j=2}^{k-1} u_j,
\end{aligned}$$

where $u_j = \frac{\binom{k}{j}\binom{m-k}{k-j}}{\binom{m}{k}} \cdot 2^{\binom{j}{2}}$. We need to estimate from above the sum $\sum_{j=2}^{k-1} u_j$.

Our goal is to show that for $G \sim G(n,1/2)$ **whp** $\chi(G) \le \frac{n}{2\log_2 n} \cdot (1+o(1))$. We now set our parameters. Let $m = \left\lceil \frac{n}{\log_2^2 n} \right\rceil$. Recall that $k_0 = \max\{k \mid f(k,m) \ge 1\}$. Now, set $k = k_0 - 3$. Recalling the ratio $\frac{f(k-1,m)}{f(k,m)}$ around k_0, we have $f(k_0,m) \geq 1 \Longrightarrow f(k,m) \geq m^{3-o(1)}$. Thus, $\mu \geq m^{3+o(1)}$. Let us now estimate u_2 and u_{k-1}.

$$u_2 = \frac{\binom{k}{2}\binom{m-k}{k-2}}{\binom{m}{k}} \cdot 2^1 \leq \frac{\binom{k}{2}\binom{m}{k-2}}{\binom{m}{k}} \cdot 2 \underbrace{=}_{k \ll m} O\left(\frac{k^2 \cdot k^2}{m^2}\right) = O(k^4/m^2),$$

and

$$\begin{aligned} u_{k-1} &= \frac{\binom{k}{k-1}\binom{m-k}{1}}{\binom{m}{k}} \cdot 2^{\binom{k-1}{2}} \leq \frac{km}{\binom{m}{k} 2^{-\binom{k}{2}+k-1}} \\ &= \frac{km}{\mu \cdot 2^{k-1}} \underbrace{=}_{k \approx 2\log_2 m \cdot (1-o(1))} \frac{km}{\mu \cdot m^{2-o(1)}} = \frac{k}{\mu \cdot m^{1-o(1)}} \underbrace{=}_{\mu \geq m^{3+o(1)}} o(1/m^2). \end{aligned}$$

Tedious but standard computation (do at home!) shows that in fact

$$\sum_{j=2}^{k-1} u_j = O(\max\{u_2, u_{k-1}\}).$$

We have shown that $u_2 = O(k^4/m^2)$, $u_{k-1} = o(1/m^2)$. It follows that $\Delta = \mu^2 \cdot O(u_2) = \mu^2 \cdot O(k^4/m^2)$. By extended[6] Janson's inequality, $\Pr[X=0] = e^{-\mu^2/(2\Delta)}$. Plugging in $m = \lceil \frac{n}{\log_2^2 n} \rceil$, $k = k_0(m) - 3 = 2\log_2 m \cdot (1-o(1)) = 2\log_2 n \cdot (1-o(1))$, we get that

$$\Pr[X=0] \leq e^{-cm^2/k^4}$$

for some constant $c > 0$. Hence,

$$\Pr[\exists V_0 \subseteq [n].\, (|V_0| = m) \wedge (\alpha(G[V_0]) < k)] \leq \binom{n}{m} \cdot e^{-cm^2/k^4} \leq 2^n \cdot e^{-cn^2/\log^8 n} = o(1).$$

Hence **whp** in $G \sim G(n,1/2)$ the assumptions of Lemma 1.2 hold with $m = \lceil \frac{n}{\log_2^2 n} \rceil$, $k = k_0 - 3 = 2\log_2 n \cdot (1-o(1))$, and we conclude that **whp** in $G \sim G(n,1/2)$ it holds that

$$\chi(G) \leq n/k + m = \frac{(1+o(1))n}{2\log_2 n} + O\left(\frac{n}{\log^2 n}\right) = (1+o(1)) \cdot \frac{n}{2\log_2 n}.$$

We have thus proven:

Theorem 1 (Bollobás '88)**.** *For $G \sim G(n,1/2)$ **whp** it holds that $\chi(G) = (1+o(1)) \cdot \frac{n}{2\log_2 n}$.*

1.4 The Choice Number of a Random Graph

Questions: Given $G \sim G(n,1/2)$ (or $G(n,p=p(n))$ in general), what is the typical behavior (or value) of $\mathrm{ch}(G)$?

We know that always $\mathrm{ch}(G) \geq \chi(G)$, and thus **whp** in $G \sim G(n,1/2)$ it holds that

$$\mathrm{ch}(G) \geq (1+o(1)) \cdot \frac{n}{2\log_2 n}.$$

We show that this bound is typically tight, by proving a lemma analogous to Lemma 1.2 (excavation lemma).

[6] As $\mu \geq m^{3-o(1)}$, and a more careful (yet simple) calculation shows that in fact $\Delta = \mu^2 \cdot \Theta(k^4/m^2)$, it follows that $\mu \leq \Delta$ and so indeed we can apply the extended inequality.

Lemma 1.3. *Let $n \geq m \geq k$ be integers and let G be a graph on n vertices satisfying: for every $V_0 \subseteq V(G), |V_0| = m$ it holds that $\alpha(G[V_0]) \geq k$. Then $\mathrm{ch}(G) \leq n/k + m$.*

The main argument in the proof is due to Jeff Kahn (appears in Alon '93).

Proof. Assume we are given a list assignment $L = (L(v))_{v \in V(G)}$ satisfying $|L(v)| \geq n/k + m$ for every v. We need to prove: there is a choice $f(v) \in L(v), v \in V(G)$ such that $f(u) \neq f(v)$ for every edge $e \in E(G)$. We proceed as follows. Set $G_0 = G$. Assume that there exists a color c appearing in at least m lists $L(v), v \in V(G_0)$. Let $V_0 = \{v \in V(G_0) \,|\, c \in L(v)\}$. By the assumption, $|V_0| \geq m$. By the lemma's assumption, V_0 contains an independent set $I = I_c$ of cardinality k. We then color all vertices $v \in I$ by color c, discard I by updating $G_0 = G_0 - I$, and delete color c from all lists of the remaining vertices. Now, when this process stops:

1. Every color c appears in lists of $< m$ vertices of G_0.

2. We have repeated the deletion phase $\leq n/k$ times, each time deleting ≤ 1 color from every list.

It follows that $|L(v)| \geq n/k + m - n/k = m$ for every v. We now define an auxiliary bipartite graph $\Gamma = (A \cup B, E)$, as follows. We let $A = V(G_0), B = \bigcup_{v \in V(G_0)} L(v)$, and $(v, c) \in E$ if and only if $c \in L(v)$ (where the $L(v)$'s are the updated lists). We are looking for a matching M in Γ saturating side A. The existence of such M means that we can choose colors $c \in L(v)$ for any $v \in V(G_0)$ (according to M) such that:

1. Each vertex gets a color.

2. Each color is chosen at most once.

Since $d_\Gamma(v) \geq m$ for every $v \in A$ and $d_\Gamma(c) \leq m$ for every $c \in B$, the graph Γ is easily seen[7] to have a required matching. This completes the proof of the lemma. ■

Recall that we have proven that for: $m = \lceil \frac{n}{\log_2^2 n} \rceil$, $k = k_0(m) - 3 = 2 \log_2 n \cdot (1 - o(1))$, **whp** every m vertices of $G \sim G(n, 1/2)$ span an independent set of cardinality k. Hence by the Lemma 1.3, **whp** $\mathrm{ch}(G) \leq n/k + m = (1 + o(1)) \cdot \frac{n}{2 \log_2 n}$. Since we have also argued that **whp**

$$\chi(G) \geq \frac{n}{k_0(n) + 1} = (1 + o(1)) \cdot \frac{n}{2 \log_2 n},$$

we conclude that:

Theorem 2 (Kahn)**.** *For $G \sim G(n, 1/2)$ **whp** it holds that $\mathrm{ch}(G) = (1 + o(1))\chi(G) = (1 + o(1)) \cdot \frac{n}{2 \log_2 n}$.*

[7]Taking $\emptyset \neq S \subseteq A$, we must have $|S| \leq |N(S)|$, as otherwise there would be a vertex $c \in N(S)$ of degree $> m$. The claim then follows from Hall's theorem.

1 Coloring Random Graphs

In the last lecture we have proved:

Theorem 1 (Bollobás '88)**.** *For $G \sim G(n, 1/2)$* ***whp*** *it holds that $\chi(G) = (1 + o(1)) \cdot \frac{n}{2\log_2 n}$.*

The proof of the upper bound used the notion of *coloring by excavation*: each time *find* a large independent set, color it in a fresh color, discard it, and proceed.

Question: Is there an efficient algorithm for coloring $G \sim G(n, 1/2)$ typically in relatively few colors (i.e., close to the upper bound on $\chi(G)$)?

1.1 An Algorithm for Coloring a Random Graph

Greedy coloring:
Recall: Given a graph $G = (V, E), |V| = n$, and a permutation $\sigma = (v_1, \ldots, v_n)$ on V, we color the vertices in order of σ, by choosing a color for v_i to be the smallest color available, namely, the smallest color not taken by its previously colored neighbors.

Setup: Given $G \sim G(n, 1/2)$, fix $\sigma = e$ to be the identity permutation on $[n]$. Define

$$\chi_g(G) := \text{number of colors used by the greedy algorithm when running according to } \sigma.$$

Note that $\chi_g(G)$ is a random variable. We would like to analyze its behavior.

Theorem 2 (Grimmett-McDiarmid '75)**.** *For $G \sim G(n, 1/2)$* ***whp*** *it holds that $\chi_g(G) \le (1+o(1)) \cdot \frac{n}{\log_2 n}$.*

Recalling that **whp** $\chi(G) = (1 + o(1)) \cdot \frac{n}{2\log_2 n}$, it follows from Theorem 2 that **whp** $\frac{\chi_g(G)}{\chi(G)} \le 2 + o(1)$.

Proof. Set $k = \left\lceil \frac{n}{\log_2 n - 3\log_2\log_2 n} \right\rceil$. We will prove: for $G \sim G(n, 1/2)$ **whp** $\chi_g(G) \le k$. Let f be the coloring produced by the greedy algorithm. If f uses $> k$ colors: denote by A_i, $1 \le i \le n$, the event "Vertex i is the first to get color $k + 1$". We have

$$\Pr[\chi_g(G) > k] = \Pr\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} \Pr[A_i].$$

We will estimate for every $i \in [n]$, $\Pr[A_i] = o(1/n)$, and then the theorem will follow. If A_i happens, then by the time we arrive to color vertex i, all k colors have been used in $[i-1]$ and vertex i has some neighbor in every color class. Fix a non-trivial partition P of $[i-1]$ into k parts: $P = (C_1, \ldots, C_k)$. Denote by B_P the event: "f produces the partition P on $[i-1]$". Then:

$$\Pr[A_i \mid B_P] = \prod_{j=1}^{k}\left(1 - \left(\frac{1}{2}\right)^{|C_j|}\right) \underbrace{\le}_{\text{convexity}} \left(1 - \left(\frac{1}{2}\right)^{\frac{i-1}{k}}\right)^{k}$$

$$\leq \left(1-\left(\frac{1}{2}\right)^{\frac{n}{k}}\right)^k \underbrace{\leq}_{1-x\leq e^{-x}} e^{-\left(\frac{1}{2}\right)^{\frac{n}{k}}\cdot k}$$
$$= \exp\left\{-\left(\frac{1}{2}\right)^{\log_2 n - 3\log_2\log_2 n}\cdot\frac{n}{\log_2 n}\cdot(1+o(1))\right\}$$
$$= e^{-(1+o(1))\cdot\log_2^2 n} = o(1/n).$$

Therefore, by the law of total probability, we have:

$$\Pr[A_i] = \sum_{\substack{P\,=\,\text{non-trivial}\\ \text{partition of } [i-1]}} \Pr[B_P]\cdot\Pr[A_i \mid B_P] = o(1/n)\cdot\sum_P \Pr[B_P] = o(1/n).$$

This completes the proof of the theorem. ■

We now show a complementary lower bound for $\chi_g(G)$.

Theorem 3 (Grimmett-McDiarmid '75)**.** *For $G \sim G(n, 1/2)$* ***whp*** *it holds that every color class produced by the greedy algorithm has size $\leq \log_2 n + 2(\log_2 n)^{1/2}$.*

Proof. If some color class C_i, $i \in [n]$, produced by the greedy algorithm, has cardinality $\geq \log_2 n + 2(\log_2 n)^{1/2}$, then there are sets $A, B \subseteq C_i$, $|A| = t_1, |B| = t_2$, where $t_1 = \log_2 n$ and $t_2 = 2(\log_2 n)^{1/2}$, $A \cap B = \emptyset$, and $A < B$ (namely, every vertex from A precedes every vertex from B). Denote by M_A^i the event: "A is the set of first t_1 vertices colored by color i". Then, for $B \subseteq [n]$, $|B| = t_2$, $A < B$, we have

$$\Pr[B \subseteq C_i \mid M_A^i] \leq \prod_{j=0}^{t_2-1} \underbrace{\left(\frac{1}{2}\right)^{t_1}}_{\substack{\text{next vertex}\\ \text{in } B \text{ has}\\ \text{no edges to } A}} \cdot \underbrace{\left(\frac{1}{2}\right)^{j}}_{\substack{\text{next vertex}\\ \text{in } B \text{ has no}\\ \text{edges to prior}\\ \text{vertices from } B}} = \left(\frac{1}{2}\right)^{t_1\cdot t_2+\binom{t_2}{2}}.$$

Hence,

$$\Pr[\text{exists such } B \mid M_A^i] \leq \binom{n}{t_2}\left(\frac{1}{2}\right)^{t_1\cdot t_2+\binom{t_2}{2}} \leq n^{t_2}\left(\frac{1}{2}\right)^{t_1\cdot t_2}\left(\frac{1}{2}\right)^{\binom{t_2}{2}}$$
$$= n^{t_2}\cdot\frac{1}{n^{t_2}}\cdot\left(\frac{1}{2}\right)^{\binom{t_2}{2}} = \left(\frac{1}{2}\right)^{\binom{t_2}{2}}$$
$$= \left(\frac{1}{2}\right)^{(1-o(1))\cdot 2\log_2 n} = n^{-2+o(1)}.$$

Therefore,

$$\Pr[|C_i| \geq t_1+t_2] = \sum_{\substack{A\subseteq[n],\\ |A|=t_1}} \Pr[M_A^i]\cdot\sum_{\substack{B\subseteq[n],\\ |B|=t_2,\\ A<B}} \Pr[B\subseteq C_i \mid M_A^i] = o(1/n)\cdot\sum_A \Pr[M_A^i] = o(1/n).$$

Hence, by the union bound over all $\leq n$ color classes, we have that

$$\Pr[\text{exists color class } C_i,\ |C_i| \geq t_1+t_2] \leq n\cdot o(1/n) = o(1).$$

■

Conclusion: For $G \sim G(n, 1/2)$ **whp** it holds that $\chi_g(G) = (1 + o(1)) \cdot \frac{n}{\log_2 n}$ and $\chi(G) = (1 + o(1)) \cdot \frac{n}{2 \log_2 n}$. Thus **whp** $\frac{\chi_g(G)}{\chi(G)} = 2 + o(1)$.

Remark 1.1. *The greedy algorithm is also robust:* ***whp*** *in $G(n, 1/2)$, for* any *permutation σ on $[n]$ the greedy algorithm uses at most $(1 + o(1)) \cdot \frac{n}{\log_2 n}$ colors.*

Open problem: Given $G \sim G(n, 1/2)$, does there exist a polynomial-time algorithm finding typically an independent set of cardinality $(1 + \varepsilon) \cdot \log_2 n$ for some constant $\varepsilon > 0$?

1.2 Hadwiger's and Hajós' Conjecture for Random Graphs

Reminder: A graph $H = (\{u_1, \ldots, u_t\}, F)$ is contained in a graph $G = (V, E)$ as a *minor* if there exist disjoint sets $V_1, \ldots, V_t \subseteq V$ such that

1. $G[V_i]$ is connected for every $i \in [t]$.
2. $(u_i, u_j) \in F \Longrightarrow G$ contains an edge between V_i and V_j.

Notation: $H \prec G$.

Let $\eta(G) = \max\{t \,|\, K_t \prec G\}$.

Hadwiger's conjecture '43: For every graph G, $\eta(G) \geq \chi(G)$.

Question: Given $G \sim G(n, 1/2)$, what are the typical values of $\eta(G), \chi(G)$?

Recall: **whp** $\chi(G) = \Theta\left(\frac{n}{\log n}\right)$. Also, Kostochka and independently Thomason ('80s) showed that if G is a graph that has average degree d then $\eta(G) = \Omega\left(\frac{d}{\sqrt{\log d}}\right)$.
For $G \sim G(n, 1/2)$: $|E(G)| \sim \text{Bin}\left(\binom{n}{2}, 1/2\right)$, and thus[1] **whp** $|E(G)| = (1 + o(1)) \cdot \frac{n^2}{4}$. It follows from Kostochka-Thomason that **whp** $\eta(G) = \Omega\left(\frac{n}{\sqrt{\log n}}\right)$. Therefore **whp** in $G(n, 1/2)$ we have $\eta(G) \gg \chi(G)$.

Reminder: A graph $G = (V, E)$ has a *subdivision* of K_k if there is a vertex set $V_0 \subseteq V, |V_0| = k$, and a collection of $\binom{k}{2}$ internally disjoint paths (of length ≥ 1) connecting all pairs from V_0.

Hajós' conjecture 1940s: For every graph G, if $\chi(G) = k$ then G contains a subdivision of K_k.

Proposition 1.2. *Let G be a graph on n vertices containing a subdivision of K_k. Then there is a set $V_0 \subseteq V(G), |V_0| = k$ such that $e(V_0) \geq \binom{k}{2} - (n - k)$.*

Proof. Let V_0 be the branch vertices of a subdivision of K_k in G. All $\binom{k}{2}$ pairs of vertices (from V_0) are connected by internally disjoint paths in G. Out of these $\binom{k}{2}$ paths, at most $n - k$ have a vertex outside V_0. The rest should be connected by paths lying entirely in V_0, each having at least one edge (in fact, exactly one edge) inside V_0. Thus, $e(V_0) \geq \binom{k}{2} - (n - k)$. ■

Let now $G \sim G(n, 1/2)$, and set $k = \lceil 3\sqrt{n} \rceil$. For a given subset $V_0 \subseteq V(G), |V_0| = k$, we have $e_G(V_0) \sim \text{Bin}\left(\binom{k}{2}, 1/2\right)$ and thus $\mathbb{E}[e_G(V_0)] = (1 + o(1)) \cdot \frac{k^2}{4} = (1 + o(1)) \cdot \frac{9n}{4}$. By applying Chernoff-type bounds on the tails of the binomial random variables, we derive: for $G \sim G(n, 1/2)$ **whp** it holds that for every $V_0 \subseteq V(G), |V_0| = k$, we have $e(V_0) = (1+o(1)) \cdot \frac{9n}{4}$. Also, $\binom{k}{2} - (n-k) = (1+o(1)) \cdot \frac{9n}{2} - n = (1+o(1)) \cdot \frac{7n}{2}$.

[1] Follows from Chernoff-Hoeffding bounds.

Hence, using Proposition 1.2, we conclude:

Theorem 4 (Erdős-Fajtlowicz '77)**.** *For $G \sim G(n, 1/2)$* ***whp*** *it holds that G has* no *subdivision of size $\geq 3\sqrt{n}$.*

Recalling that **whp** in $G \sim G(n, 1/2)$ we have $\chi(G) = \Theta\left(\frac{n}{\log_2 n}\right)$, it follows that Hajós' conjecture fails miserably for a typical random graph.

1.3 Coloring Locally-Sparse Graphs

Question: Let $\mathcal{H} = \{H_1, \ldots, H_m\}$ be a family of graphs. Suppose a graph G has *no* copy of any $H_i \in \mathcal{H}$. What can be said about $\chi(G)$?

Proposition 1.3. *Let T be a tree with k edges, and let G be a graph satisfying $\chi(G) > k$. Then G contains a copy of T.*

The proposition follows from the fact that G contains a $(k+1)$-critical subgraph G_0, and it holds that $\delta(G_0) \geq k$. By a well-known argument, every graph with minimum degree $\geq k$ contains a copy of every tree with k edges.

Definition 1.4. *Given a graph G, the* girth *of G, denoted by* $\mathrm{girth}(G)$*, is the length of a shortest cycle in G. If G is a forest, i.e., has no cycles, then we define* $\mathrm{girth}(G) = \infty$.

Theorem 5 (Erdős '59)**.** *Let k, ℓ be positive integers. Then there exists a graph G with:* $\mathrm{girth}(G) > \ell$ *and $\chi(G) > k$.*

Proof. Consider $G \sim G(n, p)$, where $p = p(n)$ is chosen as follows: set a constant $0 < \alpha < 1/\ell$ and then set $p(n) = n^{-1+\alpha}$. Now, let X be a random variable counting the number of cycles of length $\leq \ell$ in G. Decompose: $X = X_3 + \cdots + X_\ell$, where X_i is the number of cycles of length i in G. We have, due to the linearity of expectation,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=3}^{\ell} X_i\right] = \sum_{i=3}^{\ell} \mathbb{E}[X_i].$$

For a given i,

$$\mathbb{E}[X_i] = \underbrace{\frac{n(n-1)\cdots(n-i+1)}{2i}}_{\substack{\text{number of } i\text{-cycles in } K_n \\ (2i \text{ – choice of start. vertex and dir.})}} \cdot p^i < (np)^i.$$

Thus

$$\mathbb{E}[X] < \sum_{i=3}^{\ell} (np)^i = \sum_{i=3}^{\ell} n^{\alpha i} = O(n^{\alpha \ell}) = o(n).$$

By using Markov's inequality, we derive:

$$\Pr[X > n/2] = o_n(1) \Longrightarrow \Pr[X \leq n/2] = 1 - o_n(1).$$

Let $t = \lceil \frac{3 \ln n}{p} \rceil$. Then

$$\Pr[\alpha(G) \geq t] \leq \underbrace{\binom{n}{t}}_{\substack{\text{choose a set} \\ V_0, |V_0| = t}} \cdot \underbrace{(1-p)^{\binom{t}{2}}}_{\text{require } e(V_0) = 0} < n^t e^{-p\binom{t}{2}}$$

$$= \left[ne^{-p(t-1)/2}\right]^t = \left[ne^{-(1-o(1))\frac{3}{2}\ln n}\right]^t$$
$$= \left[n^{-1/2+o(1)}\right]^t = o(1).$$

It follows that **whp** $\alpha(G) < t$. Hence, for every large enough n, there is a graph G on n vertices such that:

1. $X(G) \leq n/2$ (i.e., the number of cycles of length $\leq \ell$ is at most $n/2$).
2. $\alpha(G) \leq t$.

Take G as above, and delete one arbitrary vertex from every cycle of length $\leq \ell$. We get a graph G_0, satisfying:

1. $|V(G_0)| \geq n/2$.
2. $\text{girth}(G_0) > \ell$.
3. $\alpha(G_0) \leq \alpha(G) \leq t$.

We conclude that
$$\chi(G_0) \geq \frac{|V(G_0)|}{t} \geq \frac{n/2}{\lceil 3n^{1-\alpha} \ln n \rceil} \geq n^{\alpha/2} > k.$$

■

1.3.1 Explicit Constructions of Graphs with High Chromatic Number and without Short Cycles

Zykov's construction (1949):
We want to construct a sequence $\{G_k\}$ of K_3-free graphs such that $\chi(G_k) = k$. Assume that $G_1 = K_1$ and that we have already constructed $\{G_1, \ldots, G_k\}$. Our goal is to construct G_{k+1}. Take an independent set X, $|X| = \prod_{i=1}^{k} |V(G_i)|$. Each $x \in X$ is associated with, and connected to, a k-tuple $(v_1, \ldots, v_k)$, $v_i \in V(G_i)$.

Theorem 6 (Zykov '49)**.** *G_k is k-chromatic and K_3-free.*

Proof. The proof proceeds by induction on k. The induction base is trivial. For the induction step, we first observe that $\chi(G_{k+1}) \leq 1 + \chi(G_k)$, as we just add an independent set X. Thus, by induction hypothesis, we have $\chi(G_{k+1}) \leq k+1$. We will now show that $\chi(G_{k+1}) \geq k+1$. Assume the contrary, and let f be a proper k-coloring of G_{k+1}. Since $\chi(G_i) = i$ (by induction hypothesis), f uses at least i distinct colors on $V(G_i)$, $1 \leq i \leq k$. Then there is a k-tuple $(v_1, \ldots, v_k)$, $v_i \in V(G_i)$, such that f uses distinct colors on $v_1, \ldots, v_k$. Let $x \in X$ be connected to $v_1, \ldots, v_k$. Then $f(x) = f(v_i)$ for some $1 \leq i \leq k$ — a contradiction.

We now show that G_{k+1} is K_3-free. If $\{x, y, z\}$ is a triangle, then since X is an independent set, we have $|X \cap \{x, y, z\}| \leq 1$. Also, every $x \in X$ has exactly one neighbor in each G_i. Since $G_1, \ldots, G_k$ are not connected by edges, there is $1 \leq i \leq k$ such that $x, y, z \in V(G_i)$, but then $K_3 \subseteq G_i$ — a contradiction. ■

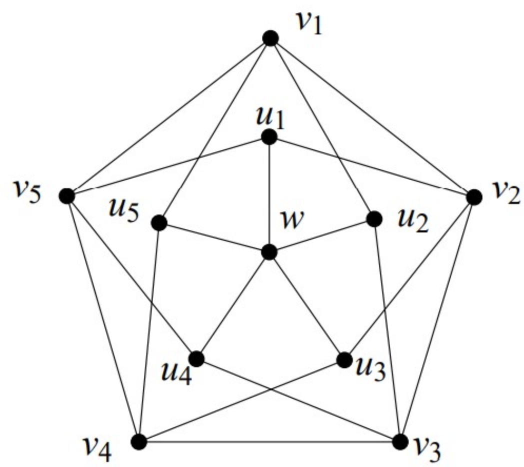
Mycielski's construction (1955):
Given a graph G with $V(G) = \{v_1, \ldots, v_n\}$, define the *Mycielskian* $M(G)$ of G as follows:

1. Add an independent set $U = \{u_1, \ldots, u_n\}$.

2. For every $i \in [n]$, connect u_i to all neighbors of v_i in G.

3. Add another vertex w, and connect w to all of U.

Example: Grötzsch graph $= M(C_5)$:



Theorem 7 (Mycielski '55)**.** *Let G be a k-chromatic K_3-free graph. Then $M(G)$ is $(k+1)$-chromatic and K_3-free.*

Proof. We first show that $\chi(M(G)) \leq 1 + \chi(G)$. If $g : V(G) \to [k]$ is a k-coloring of G, then the following is clearly a $(k+1)$-coloring of $M(G)$:

$$g(v_i) = f(v_i),\ 1 \leq i \leq n;$$
$$g(u_i) = f(v_i),\ 1 \leq i \leq n;$$
$$g(w) = k + 1.$$

We now show that $\chi(M(G)) \geq k + 1$. Assume the contrary, and let $f : V(M(G)) \to [k]$ be a proper k-coloring of $M(G)$. We can assume, without loss of generality, that $f(w) = k$. It follows that $f(u_i) \neq k$ for every i. Let $A = \{v_i \in V(G) \mid f(v_i) = k\}$. We can assume that $A \neq \emptyset$ as otherwise f colors G in $k - 1$ colors — contradicting $\chi(G) = k$. Also, we can clearly assume that A is an independent set. Define a new coloring $g : V(G) \to [k-1]$ as follows:

$$g(v_i) = \begin{cases} f(u_i), & v_i \in A \\ f(v_i), & v \in V(G) \setminus A \end{cases} .$$

It is easy to see that g is a proper $(k-1)$-coloring of G — a contradiction.

We now show that $M(G)$ is K_3-free. Assume the contrary, and let $\{x, y, z\}$ be a $K_3 \subseteq M(G)$. Then $\{x, y, z\} \cap \{w\} = \emptyset$, and $|\{x, y, z\} \cap U| \leq 1$. As G is K_3-free, we may assume, without loss of generality, that $x = u_i$ for some $i \in [n]$. Then $y, z \in V(G)$ and so $(v_i, y), (v_i, z), (y, z) \in E(G)$, which implies that $K_3 \subseteq G$ — a contradiction. ■

Remark 1.5. *If G has n vertices and m edges, then $M(G)$ has $2n + 1$ vertices and $3m + n$ edges.*

Blanche Descartes[2] **(1947, 1955):**
We start with an odd cycle G_3. In order to construct the graph G_{k+1} from G_k we proceed as follows. Take an independent set X, $|X| = k(|V(G_k)| - 1) + 1$. Then take $\binom{|X|}{|V(G_k)|}$ disjoint copies of G_k. For every choice of a subset $Y \subseteq X$ of cardinality $|Y| = |V(G_k)|$, connect Y by a perfect matching to its own copy of G_k.

Theorem 8 (Blanche Descartes '47, '55)**.** *If G_3 has length ≥ 7, then G_k is a k-chromatic graph with* girth$(G_k) \geq 6$.

Proof. The proof proceeds by induction on k. The induction base is trivial. For the induction step, we first observe that $\chi(G_{k+1}) \leq 1 + \chi(G_k)$, as we just add an independent set X. Thus, by induction hypothesis, we have $\chi(G_{k+1}) \leq k + 1$. We will now show that $\chi(G_{k+1}) \geq k + 1$. Assume the contrary, and let f be a proper k-coloring of G_{k+1}. By the pigeonhole principle, there is a monochromatic subset $Y \subseteq X$ of size $|V(G_k)|$. Assume that $f(v) = j$ for every $v \in Y$. Now, denote by G_k^Y the copy of G_k associated to Y, i.e., the copy of G_k to which Y is connected by a matching. It follows that no vertex of G_k^Y is colored by color j, and thus $\chi(G_k^Y) \leq k - 1$ — contradicting $\chi(G_k) = k$.

We now show that girth$(G_{k+1}) \geq 6$. Assume the contrary, and let C be a cycle of length ≤ 5 in G_{k+1}. We must have a vertex $x \in C \cap X$ (as by induction hypothesis, girth$(G_k) \geq 6$, and there are no edges between the copies of G_k). Let y, z be the neighbors of x along C. By the construction, y, z belong to different copies of G_k. Assume, without loss of generality, that $y \in G_k^1, z \in G_k^2$. Since y, z have only one neighbor in X (follows from the construction) and there are no edges between G_k^1 and G_k^2, there exist $y' \neq y \in C \cap G_k^1$, $z' \neq z \in C \cap G_k^2$. As $(y', z') \notin E(G_{k+1})$, we get that $C \setminus \{x, y, z, y', z'\} \neq \emptyset$, and so $|C| \geq 6$. ■

[2] A collaborative pseudonym that was used by the British mathematicians R. Leonard Brooks, Arthur Harold Stone, Cedric Smith, and William T. Tutte. The four met in 1935 as undergraduate students at Trinity College, Cambridge. The pseudonym originated by combining the initials of their names to form BLAC. It was extended to BLAnChe, while the surname Descartes was chosen as a play on the common phrase "carte blanche".
The construction shown here is attributed to Tutte.

Graph and Hypergraph Coloring, Tel Aviv Univ., Spring 2022 May 30, 2022

Lecture 12

Instructor: Prof. Michael Krivelevich *Scribe: Yevgeny Levanzov*

1 Hypergraph Coloring

Definition 1.1. *A* hypergraph *H is an ordered pair $H = (V, E)$, where V is a set of* vertices *and E is a family of subsets of V; the elements of E are called* edges*. If $|e| = r$ for every $e \in E$ then H is called* r-uniform*. If $|e| \leq r$ for every $e \in E$ then we say that H has* rank r*.*

Example: A 2-uniform hypergraph is a graph.

Definition 1.2. *Let $H = (V, E)$ be a hypergraph. A function $f : V \to [k]$ is called a* k-coloring *of H if no edge $e \in E$ of H is monochromatic under f, that is, there are $u, v \in e$ such that $f(u) \neq f(v)$.*
A hypergraph is k-colorable *if it admits a k-coloring.*

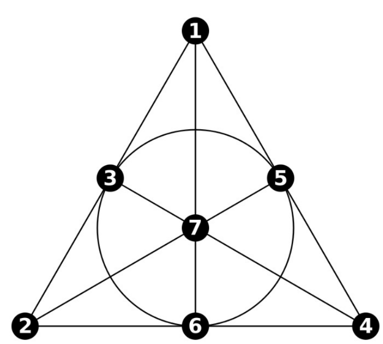
1.1 Property B

Definition 1.3. *A hypergraph H is said to have* Property B[1] *if H is 2-colorable.*

Extremal problem: Define

$$m(n) = \min\{|E(H)| : H = (V, E) \text{ is a } n\text{-uniform hypergraph having } no \text{ Property } B\}.$$

Example: $m(2) = 3$ — $H = K_3$.
It holds that $m(3) \leq 7$, as can be seen by the *Fano plane* (which is a *projective plane*[2] of order 2), where the lines correspond to hyperedges:



The Fano plane is also a *Steiner triple system* on 7 vertices. It can be easily verified that the above hypergraph is not 2-colorable (as the only way for a set of size 3 to intersect all edges is to contain the 3 vertices of some line/edge). In fact, it holds that $m(3) = 7$.
Our goal is to understand the asymptotic behavior of $m(n)$ when $n \to \infty$.

[1] Named after Felix Bernstein, who researched this property in 1908.

[2] A *projective plane* consists of lines and points such that through every pair of points there is exactly one line, and every two lines intersect in exactly one point. A projective plane has *order* n if it has $n^2 + n + 1$ lines, $n^2 + n + 1$ points, $n + 1$ points on each line, and $n + 1$ lines through each point.

1.2 A Lower Bound for $m(n)$

Proposition 1.4 (Erdős '63)**.** *It holds that $m(n) \geq 2^{n-1}$.*

Proof. Let $H = (V, E)$ be an n-uniform hypergraph with $< 2^{n-1}$ edges. We need to prove that H is 2-colorable. Color V in red/blue at random: for every $v \in V$, $\Pr[v \text{ is red}] = \Pr[v \text{ is blue}] = 1/2$, independently. For every $e \in E$, denote by A_e the event "e is monochromatic under the random coloring". It holds that

$$\Pr[A_e] = \underbrace{2}_{\substack{\text{choose}\\\text{color}}} \cdot \underbrace{2^{-n}}_{\substack{\text{require all } n\\\text{vertices of } e \text{ are}\\\text{of chosen color}}} = 2^{-n+1}.$$

Then, by the union bound,

$$\Pr\left[\bigcup_{e\in E} A_e\right] \leq \sum_{e\in E} \Pr[A_e] = |E| \cdot 2^{-n+1} < 2^{n-1} \cdot 2^{-n+1} = 1.$$

We conclude that there is a coloring in red/blue without any monochromatic edge. ∎

Theorem 1 (Cherkashin, Kozik[3] '15)**.** *If there exists $p \in [0, 1]$ such that*

$$k(1-p)^n + k^2 p < 1, \tag{1}$$

then $m(n) > 2^{n-1}k$.

Proof. Let $H = (V, E)$ be an n-uniform hypergraph with $m = 2^{n-1}k$ edges. We need to prove, assuming (1), that H is 2-colorable. For each vertex $v \in V$, assign a label x_v, uniformly chosen from $[0, 1]$. With probability 1, all x_v's are distinct. Given the labels $\{x_v\}_{v\in V}$, we order the vertices in V in the increasing order of their labels, thus creating a (random) permutation σ of V. Given σ, we (try to) 2-color V as follows. Go vertex-by-vertex according to σ, color v in blue unless forced otherwise, namely, unless v is the last (by σ) vertex of some edge $e \in E$, all of whose other vertices have been colored blue. In this case, v is colored red. We will prove, assuming (1), that the above algorithm succeeds with positive probability. Observe that the algorithm cannot color an entire edge $e \in E$ in blue, but might color some edge entirely in red. If an edge $f \in E$ gets colored red eventually, then every vertex $v \in f$, including the first vertex by σ, is the last vertex of some other edge $e \in E$. So in particular there are edges $e \neq f \in E$ such that $e \cap f = \{v\}$ is the last vertex of e and the first vertex of f. Let us call such an ordered pair (e, f) as above a *conflicting* pair. We conclude that if there are *no* conflicting pairs then the algorithm succeeds. We now estimate the probability of getting a conflicting pair. We divide $[0, 1] = L \cup M \cup R$ (left, middle, right), where $L = [0, (1-p)/2), M = [(1-p)/2, (1+p)/2), R = [(1+p)/2, 1]$. Let us first estimate the probability of having a conflicting pair (e, f) such that $e \cap f = \{v\}$ and $x_v \in L \cup R$. If $x_v \in L$, then the entire edge e should fall into L (i.e., the labels of all its vertices are in L); if $x_v \in R$, then the entire edge f should fall into R. For a given $e \in E$, $\Pr[\forall u \in e. x_u \in L] = \left(\frac{1-p}{2}\right)^n$. Similarly, for a given $f \in E$, $\Pr[\forall u \in f. x_u \in R] = \left(\frac{1-p}{2}\right)^n$. Hence, the probability of having a conflicting pair (e, f) with $e \cap f = \{v\}$, and $x_v \in L$ or $x_v \in R$ is at most

$$2m \left(\frac{1-p}{2}\right)^n \leq 2^{n-1}k \cdot \frac{(1-p)^n}{2^{n-1}} = k(1-p)^n.$$

[3] They provided a much simpler proof to a previous result of Radhakrishnan-Srinivasan '00.

Now we estimate the probability of having a conflicting pair (e, f) with $e \cap f = \{v\}$ and $x_v \in M$. We have $\Pr[x_v \in M] = |M| = p$. Now, given that v has label $x_v \in M$, $\Pr[v \text{ is the last vertex of } e \,|\, x_v] = x_v^{n-1}$ and $\Pr[v \text{ is the first vertex of } f \,|\, x_v] = (1 - x_v)^{n-1}$. Therefore,

$$\Pr[(e, f) \text{ is a conflicting pair} \,|\, x_v] = x_v^{n-1} \cdot (1 - x_v)^{n-1} = (x_v(1 - x_v))^{n-1} \leq \left(\frac{1}{4}\right)^{n-1}.$$

Hence,

$$\Pr[(e, f) \text{ is a conflicting pair with } e \cap f = \{v\} \text{ and } x_v \in M] \leq p\left(\frac{1}{4}\right)^{n-1}.$$

The number of ordered pairs (e, f) with $|e \cap f| = 1$ is at most m^2, hence

$$\begin{aligned}\Pr[\text{exists a conflicting pair of this type}] &\leq m^2 \cdot p\left(\frac{1}{2}\right)^{2n-2} \\ &= (2^{n-1}k)^2 \cdot p\left(\frac{1}{2}\right)^{2n-2} \\ &= k^2 p.\end{aligned}$$

Hence, $\Pr[\text{exists a conflicting pair}] \leq k(1-p)^n + k^2 p$. We assumed (in (1)) that $k(1-p)^n + k^2 p < 1$ and thus $\Pr[\textit{no} \text{ conflicting pair}] > 0$. We conclude that with positive probability the algorithm succeeds in 2-coloring of H. ■

Corollary 2. *It holds that* $m(n) = \Omega\left(2^n \cdot \sqrt{\frac{n}{\log n}}\right)$.

Proof. We use the inequality $1 - x \leq e^{-x}$ and get that $k(1 - p)^n + k^2 p \leq ke^{-pn} + k^2 p$. Let $f(p) = ke^{-pn} + k^2 p$. Optimizing, we set $p^* = \frac{\ln(n/k)}{n}$. Then $f(p^*) = \frac{k^2}{n}(1 + \ln(k/n))$. Now, choosing $k = c\sqrt{\frac{n}{\ln n}}$, where $c > 0$ is some constant, we get that $f(p^*) < 1$, and so (1) is satisfied (for large enough n). Hence $m(n) = \Omega\left(2^n \cdot \sqrt{\frac{n}{\log n}}\right)$. ■

1.3 An Upper Bound for $m(n)$

Theorem 3 (Erdős '64)**.** *It holds that* $m(n) = O(2^n n^2)$.

Proof. Let V be a set of vertices of size $|V| = v$, where the value of $v = v(n)$ will be chosen later. Generate m subsets $e_1, \ldots, e_m \subseteq V$, each of size n, uniformly at random from V, with repetitions. Define now $H = (V, E)$, where $E = \{e_1, \ldots, e_m\}$ (possibly with multiple edges). We will prove: for a "good" choice of $m = m(n)$, the random hypergraph H is non-2-colorable with positive probability. This would imply that $m(n) \leq m$. Let us fix a coloring χ of V, namely, a partition $V = A \cup B, |A| = a, |B| = b, a + b = v$. We estimate the probability that χ is a 2-coloring of H. For a random n-subset $e_i \subseteq V$,

$$\Pr[e_i \text{ is monochromatic under } \chi] = \frac{\binom{a}{n} + \binom{b}{n}}{\binom{v}{n}} \underbrace{\geq}_{\text{convexity}} \frac{2\binom{v/2}{n}}{\binom{v}{n}} =: p.$$

Hence,

$$\Pr[\text{none of } e_i,\ i \in [m],\ \text{is monochromatic under } \chi] = (1 - p)^m.$$

The number of potential 2-colorings of V is $\leq 2^v$. Hence, by the union bound, if $2^v \cdot (1-p)^m < 1$, then with positive probability the hypergraph $H = (V, \{e_1, \dots, e_m\})$ is not 2-colorable. Observe that $2^v \cdot (1-p)^m \leq 2^v e^{-pm}$, and so we need to solve $2^v e^{-pm} < 1$. Choose $m = \lceil \frac{v \ln 2}{p} \rceil$, then there is an n-uniform hypergraph with m edges which is not 2-colorable. We have

$$p = \frac{2\binom{v/2}{n}}{\binom{v}{n}} = 2^{1-n} \cdot \prod_{i=0}^{n-1} \frac{v-2i}{v-i}.$$

If we assume that $v \gg n^{3/2}$, then we can estimate:

$$\begin{aligned} 2^{1-n} \cdot \prod_{i=0}^{n-1} \frac{v-2i}{v-i} &= 2^{1-n} \cdot \prod_{i=0}^{n-1} (1 - i/v + O(i^2/v^2)) \\ &\leq 2^{1-n} \cdot \prod_{i=0}^{n-1} e^{-i/v + O(i^2/v^2)} \\ &= 2^{1-n} e^{-n^2/2v} (1 + o(1)). \end{aligned}$$

Choose now $v = cn^2$, where $c > 0$ is some constant. Then we get that there exists an n-uniform hypergraph H with $\leq c2^n n^2$ edges which is not 2-colorable. Hence $m(n) = O(2^n n^2)$. ∎

Remark 1.5. *Recalling the lower bound argument, we observe that if $v = \Theta(n^2)$, and $e, f \subseteq V$ are random n-subsets of V, then $\Pr[|e \cap f| = 1]$ is bounded away from 0 (hence the expected number of such pairs is $\Theta(m^2)$).*

1.4 Applying the Lovász Local Lemma[4]

Theorem 4 (Lovász Local Lemma (LLL), symmetric case; Erdős, Lovász '75)**.** *Let $A_1, \dots, A_m$ be events in an arbitrary probability space. Suppose that each event A_i is (mutually[5]) independent of all other events A_j but at most d of them. Suppose further that $\Pr[A_i] \leq p$, $1 \leq i \leq m$. If $ep(d+1) \leq 1$, then*

$$\Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > 0.$$

Proof. See books on the probabilistic method (e.g., Alon-Spencer, *The Probabilistic Method*). ∎

Corollary 5 (Erdős, Lovász '75)**.** *Let $H = (V, E)$ be a hypergraph. Assume that every edge $e \in E$ has at least k vertices and intersects at most d other edges. If $e \cdot \frac{1}{2^{k-1}}(d+1) \leq 1$, then H is 2-colorable.*

Proof. Color V at random: $\Pr[v \text{ is red}] = \Pr[v \text{ is blue}] = 1/2$, independently. For every $f \in E$, let A_f be the event "f is monochromatic". We have $\Pr[A_f] = 2 \cdot 2^{-|f|} \leq 2^{1-k}$. In addition, each A_f is (mutually) independent of $\{A_{f'} \mid f \cap f' = \emptyset\}$, and the number of edges outside of the above set is $\leq d$ by our assumption. Hence by the LLL, if $e \cdot \frac{1}{2^{k-1}}(d+1) \leq 1$, then $\Pr\left[\bigwedge_{f \in E} \overline{A_f}\right] > 0$. This implies that H is 2-colorable. ∎

[4] Appeared in Erdős, Lovász '75.

[5] A set of events $A_1, \dots, A_n$ is *mutually independent* if every event is independent of any intersection of the other events and their complements. In particular, for every $I \subseteq [n]$, $\Pr[\bigwedge_{i \in I} A_i] = \prod_{i \in I} \Pr[A_i]$.

Corollary 6. *Let $H = (V, E)$ be an n-uniform hypergraph of maximum degree Δ. If $e \cdot \frac{1}{2^{n-1}} \cdot n\Delta \leq 1$, then H is 2-colorable.*

Proof. Apply Corollary 5 with $k = n$ and $d = n\Delta$ (as each edge intersects at most $n\Delta$ other edges). ■

2 Edge Coloring

2.1 Basic Definitions

Definition 2.1. *Let $G = (V, E)$ be a multigraph. A function $f : E \to [k]$ is called a k-*edge-coloring *of G if $f(e) \neq f(e')$ for every $e, e' \in E$ such that $e \cap e' \neq \emptyset$.*

Definition 2.2. *A multigraph G is said to be k-*edge-colorable *if it has a k-edge-coloring.*

Definition 2.3. *The* chromatic index *of a multigraph G, denoted by $\chi'(G)$, is the smallest k such that G is k-edge-colorable.*

Remark 2.4.

1. *If $f : E(G) \to [k]$ is a k-edge-coloring of a graph G, then each color class $f^{-1}(i) = \{e \in E(G) \mid f(e) = i\}$ is a* matching*. Thus, a k-edge-coloring of G is a partition of $E(G)$ into k matchings.*

2. *Observe that a k-edge-coloring of a graph G is a k-vertex-coloring of the* line graph[6] *$L(G)$. Thus, $\chi'(G) = \chi(L(G))$.*

2.2 Trivial Bounds on the Chromatic Index

Claim 2.5. *For every graph G it holds that $\chi'(G) \geq \Delta(G)$.*

The above claim is analogous to the claim that $\chi(L(G)) \geq \omega(L(G))$ (as $\omega(L(G)) \geq \Delta(G)$).

Proof. If f is a legal edge-coloring of G, then for every $v \in V(G)$, the $d(v)$ edges containing v are all colored in distinct colors by f, and thus f uses at least $d(v)$ colors. ■

Claim 2.6. *For every graph G it holds that $\chi'(G) \geq \frac{|E(G)|}{\nu(G)}$, where $\nu(G)$ is the* matching number *of G (i.e., the size of a maximum matching in G).*

The analogous claim for vertex-coloring is that $\chi(G) \geq |V(G)|/\alpha(G)$.

Proof. Let $f : E(G) \to [k]$ be a valid k-edge-coloring of G. Then each color class $E_i = f^{-1}(i) = \{e \in E(G) \mid f(e) = i\}$ forms a matching in G, implying $|E_i| \leq \nu(G)$. Since $E(G) = E_1 \dot{\cup} \ldots \dot{\cup} E_k$, we obtain $|E(G)| = \sum_{i=1}^{k} |E_i| \leq k\nu(G)$, and so $k \geq |E(G)|/\nu(G)$. ■

Claim 2.7. *For every graph G it holds that $\chi'(G) \leq 2\Delta(G) - 1$.*

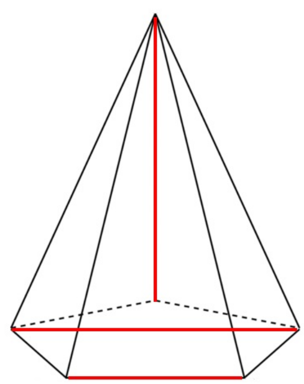
Proof. Let us look at the line graph $L(G)$. For an edge $e = (u, v) \in E(G)$, we have that e intersects $\leq (d(u) - 1) + (d(v) - 1)$ other edges. Thus $\Delta(L(G)) \leq 2(\Delta(G) - 1) = 2\Delta(G) - 2$. By the trivial form of Brooks' theorem, we have that $\chi(L(G)) \leq \Delta(L(G)) + 1 \leq 2\Delta(G) - 1$, and thus $\chi'(G) = \chi(L(G)) \leq 2\Delta(G) - 1$. ■

[6] The *line graph* of a graph G is the graph whose vertices are the edges of G, and two vertices are adjacent if their corresponding edges in G intersect.

Conclusion: $\Delta(G) \le \chi'(G) \le 2\Delta(G) - 1$.
In particular, unlike in vertex-coloring, here we have an easy 2-approximation for the chromatic index.

Example: We want to compute $\chi'(K_n)$.

Case 1: $\chi'(K_{2n}) = 2n - 1$.
Proof: We have $\chi'(K_{2n}) \ge \Delta(K_{2n}) = 2n-1$. For the upper bound, we can think of an edge-coloring of K_{2n} as a representation of the pairings of a $2n$-player round-robin tournament (i.e., where every player plays against all others). Then, every round (in some paring system) is a matching of size n and there are $2n - 1$ rounds.
Geometric proof of the upper bound:
Put a regular $(2n - 1)$-gon on the plane, plus an apex vertex above its center. Now, form a matching by taking any edge e of the base $(2n-1)$-gon, plus $n-2$ diagonals of the polygon parallel to it. Finally, add the edge connecting the apex to the unique vertex of the polygon not covered by the previously chosen edges (see figure below for illustration). Rotating the picture $2n - 1$ times, we get a cover of $E(K_{2n})$ by $2n - 1$ matchings.



Case 2: $\chi'(K_{2n-1}) = 2n - 1$.
Proof: K_{2n-1} has $\binom{2n-1}{2}$ edges and satisfies $\nu(K_{2n-1}) = n - 1$. Thus $\chi'(K_{2n-1}) \ge \frac{\binom{2n-1}{2}}{n-1} = \frac{(2n-1)(2n-2)}{2(n-1)} = 2n - 1$. For the upper bound, taking the geometric argument for $\chi'(K_{2n}) \le 2n - 1$ and chopping off the apex provides an edge coloring of K_{2n-1} in $2n - 1$ colors. Alternatively, considering the argument with a round-robin tournament, having an odd number of players implies that in each round one player is free, and thus we will have an extra round, that is, $2n - 2 + 1 = 2n - 1$ rounds in total, each of which is a matching.

2.3 Edge Coloring of Bipartite Graphs

Theorem 7 (Frobenius marriage theorem)**.** *Let $G = (A \cup B, E)$ be a d-regular bipartite multigraph, $d > 0$. Then G has a perfect matching.*

This a simple consequence of Hall's theorem.

Theorem 8 (Kőnig '1916)**.** *If G is a bipartite graph, then $\chi'(G) = \Delta(G)$.*

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Lecture 13

Instructor: Prof. Michael Krivelevich — *Scribe: Yevgeny Levanzov*

1 Edge Coloring

1.1 Edge Coloring of Bipartite Graphs

Theorem 1 (Frobenius marriage theorem)**.** *Let $G = (A \cup B, E)$ be a d-regular bipartite multigraph, $d > 0$. Then G has a perfect matching.*

This a simple consequence of Hall's theorem.

Theorem 2 (Kőnig '1916)**.** *If $G = (A \cup B, E)$ is a bipartite multigraph, then $\chi'(G) = \Delta(G)$.*

Proof. First, make a given bipartite graph into a bipartite graph with equally-sized parts by adding isolated vertices if necessary. Let now $G = (A \cup B, E)$ be a bipartite multigraph of $\Delta(G) = \Delta$ and with $|A| = |B|$. For as long as there are vertices $a \in A, b \in B$ such that $d(a), d(b) < \Delta$, add an edge (a, b) (possibly in addition to already existing edges between a and b). We end up with a Δ-regular multigraph $G' \supseteq G$. We can obviously assume $\Delta > 0$. By Theorem 1, G' contains a perfect matching M_1. We color M_1 by color 1, update $G' := G' - M_1$, and repeat. After Δ steps the graph G' empties, meaning $\bigcup_{i=1}^{\Delta} M_i = E(G')$. Hence $\chi'(G') \leq \Delta$ and so $\chi'(G) \leq \chi'(G') \leq \Delta$. Since trivially $\chi'(G) \geq \Delta$, we conclude that $\chi'(G) = \Delta$. ■

1.2 Vizing's Theorem and its Extensions

Vizing '64, '65 and Gupta '66 proved: for a (loopless) multigraph G it holds that $\chi'(G) \leq \Delta(G) + \mu(G)$, where $\mu(x, y)$ is the multiplicity of an edge (x, y) in G, and $\mu(G) = \max_{x \neq y \in V(G)} \mu(x, y)$. For simple graphs, we have $\mu(G) \leq 1$, implying

$$\chi'(G) \leq \Delta(G) + 1.$$

Recalling the trivial lower bound $\chi'(G) \geq \Delta(G)$, we derive that for *every* simple graph G, $\chi'(G) \in \{\Delta(G), \Delta(G)+1\}$[1]. Simple graphs G with $\chi'(G) = \Delta(G)$ are *Class 1* and[2] those with $\chi'(G) = \Delta(G)+1$ are *Class 2*.

Example: $G = K_{2n} \Longrightarrow \chi'(G) = 2n-1 = \Delta(G)$ — Class 1, $G = K_{2n-1} \Longrightarrow \chi'(G) = 2n-1 = \Delta(G)+1$ — Class 2.

Ore '68 proved: $\chi'(G) \leq \max_{v \in V(G)} d(v) + \mu(v) \leq \Delta(G) + \mu(G)$, where $\mu(v) = \max_{u \in N_G(v)} \mu(u, v)$[3]. Ore '67 also proved:

$$\chi'(G) \leq \max\left\{\Delta(G), \max_{(x,y,z) \in \mathcal{P}} \frac{d(x) + d(y) + d(z)}{2}\right\},$$

[1] This stands in striking contrast to the chromatic number of a graph, for which we do not have a good approximation, and where bounds on $\chi(G)$ are not directly related to the degrees of G.

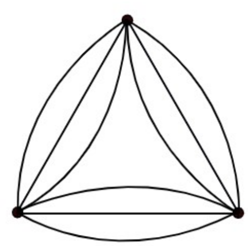
[2] Given a graph G, it is NP-hard to decide whether it belongs to Class 1 of Class 2.

[3] Obviously, we can assume that G does not contain isolated vertices, and thus $\mu(v)$ is well-defined.

where $\mathcal{P}$ is the set of ordered triples of distinct vertices forming a path of length 2 in G.
We can derive from this immediately:

Theorem 3 (Shannon '49)**.** *Let G be a multigraph of maximum degree Δ. Then $\chi'(G) \le \frac{3\Delta}{2}$.*

Shannon's bound is tight, as can be seen in the following example:
Example: "Fat" triangle: a multigraph G with vertices u, v, w, where each pair is connected by exactly k edges (see figure below). We have $|E(G)| = 3k$, and every pair of edges is intersecting, implying that $\chi'(G) = 3k$. Finally, observe that G is $2k$-regular, and thus $\chi'(G) = 3k = \frac{3}{2}\Delta(G)$.



We will prove the following theorem, generalizing[4] both theorems of Ore:

Theorem 4 (Andersen '77; Goldberg '77, '84)**.** *Let G be a loopless multigraph. Then*

$$\chi'(G) \le \max\left\{\Delta(G), \max_{(x,y,z)\in\mathcal{P}} \left\lfloor \frac{d(x)+\mu(x,y)+\mu(y,z)+d(z)}{2} \right\rfloor\right\}, \quad (1)$$

where $\mathcal{P}$ is the set of ordered triples of distinct vertices forming a path of length 2 in G.

Theorem 4 will follow in turn from the following theorem:

Theorem 5. *Let q be an integer, and let G be a loopless multigraph with an edge (y, w). Assume that $\chi'(G - (y, w)) \le q$. If $d(y), d(w) \le q$, and $d(x) + \mu(x, y) + \mu(y, z) + d(z) \le 2q + 1$ for every pair of distinct neighbors x, z of y, then $\chi'(G) \le q$.*

We start by showing that Theorem 5 implies Theorem 4.

Proof (Theorem 5 $\Rightarrow$ Theorem 4). The proof is by induction on $|E(G)|$.
Base: $|E(G)| = 0$ – trivial.
Step: Let $(y, w) \in E(G)$. Denote $G' = G - (y, w)$. Denote by q the right-hand side of (1), namely:

$$q = \max\left\{\Delta(G), \max_{(x,y,z)\in\mathcal{P}} \left\lfloor \frac{d(x)+\mu(x,y)+\mu(y,z)+d(z)}{2} \right\rfloor\right\}.$$

This bound applies trivially also to G'. Hence, by induction, $\chi'(G') \le q$. Also, $d(y), d(w) \le \Delta(G) \le q$. If x, z are distinct neighbors of y, then by the definition of q,

$$\left\lfloor \frac{d(x)+\mu(x,y)+\mu(y,z)+d(z)}{2} \right\rfloor \le q.$$

Thus $d(x) + \mu(x, y) + \mu(y, z) + d(z) \le 2q + 1$. Hence Theorem 5 applies, and we derive that $\chi'(G) \le q$, as required. ■

[4] Indeed, as $d(x) + \mu(x, y) \le d(x) + \mu(x)$ and $d(y) + \mu(y, z) \le d(y) + \mu(y)$, the bound in (1) is at most the one in the first theorem of Ore. In addition, as $\mu(x, y) + \mu(y, z) \le d(y)$, we also get that the bound in (1) is at most the one in the second theorem of Ore.

Proof of Theorem 5. (Kostochka '14) Let $G' = G - (y, w)$, where (y, w) is one edge out of possibly several edges connecting y and w. Let φ be a q-edge-coloring of G'. For every $v \in V(G)$, define the set $O(v) \subseteq [q]$ to be the set of colors missing at the edges incident to v at coloring φ. If w is the only neighbor of y then $O(w) \subseteq O(y)$. Since we have deleted (y, w), we have at most $q - 1$ edges touching w (as $d_G(w) \leq q$, by the assumption of the theorem), implying $O(w) \neq \emptyset$. Let then $c \in O(w) \cap O(y)$. Then we can extend φ to G by coloring (y, w) in color c, thus obtaining a q-edge-coloring of G. Hence we may assume that $|N_G(y)| \geq 2$. If for some $v \in N_G(y)$ we have $\varphi((y, v)) \in O(w)$, then we can obtain a new q-coloring of $G - (y, v)$ by shifting color $\varphi((y, v))$ to (y, w). More generally, a *color fan* with respect to φ is a sequence $(v_0, v_1, \ldots, v_k)$ of neighbors of y such that

1. $v_0 = w$.
2. $\varphi((y, v_i)) \in O(v_{i-1})$, $i = 1, \ldots, k$.

For every color fan we can shift colors as before an get a q-coloring of $G - (y, v_k)$. If then there is a color $c \in O(y) \cap O(v_k)$, then, after the shift, we can use c to color (y, v_k), thus completing a q-coloring of G. Hence we can assume that $O(z) \cap O(y) = \emptyset$ for every neighbor z of y reachable by some color fan. Let X be the set of neighbors of y reachable by color fans. Trivially, $w \in X$, and we can assume[5] that $X - \{w\} \neq \emptyset$, implying $|X| \geq 2$. We now define an auxiliary directed graph H as follows:

1. $V(H) = X$.
2. For every edge $(y, v) \in E(G')$ such that $\varphi((y, v)) \in O(u)$, where $u, v \in X$, we put a directed edge $\overrightarrow{(u, v)}$ in H.

Thus, a color fan in G with respect to φ is a directed path from w in H. Suppose that $\chi'(G) > q$.

Claim 1.1. *The sets $O(v)$ are disjoint for $v \in X \cup \{y\}$.*

Let now derive Theorem 5 from Claim 1.1. If the sets $O(v), v \in X$ are disjoint then for $(y, v) \in E(G')$, the color $\varphi((y, v))$ belongs to at most one list $O(u)$. Thus every edge (y, v) constitutes at most one edge entering v in H. Counting the edges of H by their heads, we get

$$|E(H)| = \sum_{v \in X} d^-_H(v) \leq \sum_{v \in X} \mu(y, v) - 1,$$

where we subtract 1 from the sum because the edge (y, w) is not in G'. We now count by tails. For every $v \in X$, every color in $O(v)$ appears as a color of some edge (y, z) at y (otherwise, by shifting along a color fan, we can use this color to color edge (y, v), thus obtaining a q-coloring of G). Hence every such color causes a directed edge in H leaving v. Hence $d^+_H(v) = |O(v)| = q - d_{G'}(v)$. Adjusting by 1 again due to (y, w) being deleted, we have

$$|E(H)| = \sum_{v \in X} d^+_H(v) \geq 1 + \sum_{v \in X} (q - d_G(v)).$$

Comparing the two estimates on $|E(H)|$, we derive

$$-1 + \sum_{v \in X} \mu(y, v) \geq 1 + \sum_{v \in X} (q - d_G(v)),$$

[5] Indeed, as there is a color $c \in O(w)$, if c does not appear on any edge (y, v), then we could color (y, w) by c. Hence, $\varphi((y, v)) = c$ for some v, implying that v is reachable by a color fan, and so $v \in X$.

or

$$\sum_{v\in X}(d_G(v)+\mu(y,v)) \geq q|X|+2. \tag{2}$$

By the theorem's assumption, the largest two terms in the above sum are $\leq 2q+1$ together, and so every other term is $\leq q$. It follows that the left-hand side of (2) is $\leq 2q+1+(|X|-2)q = q|X|+1$, thus getting a contradiction. This yields $\chi'(G) \leq q$.

It is left to prove Claim 1.1.

Proof of Claim 1.1. We have already argued that $O(y)\cap O(z) = \emptyset$ for every $z \in X$. Let $z \in X$. If $O(z) = \emptyset$, there is nothing to prove about it. Let then $\beta \in O(z)$, $\alpha \in O(y)$ ($O(y) \neq \emptyset$ as $d_{G'}(y) < q$). We have that $\alpha \neq \beta$. Then α is present at z. Let now P be the longest[6] α/β-path starting from z. Let u be the last vertex of P. We claim that $u = y$. If not, let Q be the directed path from w to z in H (exists as $z \in X$ and thus there is a color fan starting at w and reaching z). We can flip colors α and β along P, getting a (new) q-coloring of G'. If $u \notin V(Q)$, then by flipping colors along P, we release color α for z, and then use it to color (y,z); If $u \in V(Q)$, then $O(u)\cap O(y) = \emptyset$, and we can release color α for u, and then color edge (y,u) in α, thus completing a q-coloring of G. Let now $z, z' \in X$ with $O(z)\cap O(z') \neq \emptyset$. Take $\beta \in O(z)\cap O(z')$, then $\alpha \neq \beta$. Let now P, P' be α/β-paths leading from z, z', respectively, to y, as described/argued before. But then at the first meeting vertex[7] of paths P and P' there are two edges of the same color — a contradiction, as φ is a proper edge-coloring. ■

This completes the proof of the theorem. ■

Let us now switch to simple graphs.

Definition 1.2. *A (simple) graph G is called* critical *if*

1. $\chi'(G) = \Delta(G)+1$.
2. $\chi'(G') \leq \Delta(G)$ *for every* $G' \subsetneq G$.

Clearly, every graph G from Class 2 contains a critical subgraph.

Corollary 6. *Let G be a critical graph, and let $y \in V(G)$. Then y has at least two neighbors x, z such that $d(x) = d(z) = \Delta(G)$.*

Proof. Since G is critical, $\chi'(G-(y,w)) = \Delta(G)$ for any edge $(y,w) \in E(G)$. But $\chi'(G) = \Delta(G)+1$, meaning Theorem 5 does not apply with $q = \Delta(G)$. Hence, as $d(y), d(w) \leq q = \Delta(G)$, we have that $d(x)+\mu(x,y)+\mu(y,z)+d(z) > 2q+1$ for some two distinct neighbors x, z of y. Hence, as $\mu(x,y) = \mu(y,z) = 1$ (as G is simple), we have $d(x)+d(z) \geq 2q+2-2 = 2q$. This implies that $d(x) = d(z) = \Delta(G)$. ■

Corollary 7. *Let G be a simple graph of maximum degree Δ. If the set V_Δ of vertices of degree Δ spans only a matching (possibly empty), then $\chi'(G) = \Delta$.*

Proof. Assume the contrary. Let G' be a critical subgraph of G. Take y to be a vertex of degree Δ in G' (there exists such a vertex, as otherwise $\chi'(G') \leq \Delta$). Then by Corollary 6, y has two neighbors $x, z \in V_\Delta$ — a contradiction. ■

Corollary 8. *Let G be a simple graph with a unique vertex of maximum degree. Then G is Class* 1*, i.e., $\chi'(G) = \Delta(G)$.*

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[6] Note that this is well-defined.

[7] If the first meeting vertex is y, the claim follows from the fact that $\alpha \in O(y)$.