0366.4817 Graph and Hypergraph Coloring

Spring Semester 2022

Homework assignment 1

Due date: Monday, April 4, 2022

Problem 1. Show that every graph G with m edges satisfies: $\chi(G) \leqslant \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$.

Problem 2. Prove: $\chi(G) \leqslant 2^k$ if and only if $E(G)$ is a union of k bipartite graphs.

Problem 3. Prove that $\chi(\bar{G}) = \omega(\bar{G})$ for a bipartite graph G.

Problem 4. Show that a graph G is k-colorable if every vertex of G lies in fewer than $\binom{k}{2}$ odd cycles.

Problem 5. **(a)** Prove that $\chi(G_1 \cup G_2) \leqslant \chi(G_1) \cdot \chi(G_2)$. **(b)** Let D be an orientation of a graph G with $\chi(G) > rs$, for positive integers r, s. Suppose that the vertices of G are assigned distinct real labels. Prove that D contains an increasing path of length r or a decreasing path of length s. **(c)** Deduce the Erdős-Szekeres theorem: every sequence of $rs + 1$ distinct real numbers has a monotone increasing subsequence of length $r + 1$ or a monotone decreasing subsequence of length $s + 1$.

Problem 6. Let G be a graph having no induced path on four vertices. (Such graphs are called cographs.) Prove that for any permutation σ on $V(G)$, the greedy algorithm run on G according to σ produces an optimal coloring of G. [*Hint*: Suppose that the algorithm uses k colors for σ, and let i be the smallest integer such that G has a clique consisting of vertices assigned colors i through k in this coloring. Prove that $i = 1$.]

Problem 7. Let G be a 3-regular K_4-free graph with m edges. Prove that G has a bipartite subgraph with at least $\frac{7m}{9}$ edges.