0366.4817 Graph and Hypergraph Coloring

Spring Semester 2022

Homework assignment 2

Due date: Monday, May 2, 2022

Problem 1. Prove: If G is a k-critical graph and $S \subset V(G)$ is a cutset of G then S does not span a clique in G.

Problem 2. Let G be a k-critical graph, $k \geqslant 3$, and let $e \neq f \in E(G)$.
(a) Prove that there is an independent set I in G such that $I \cap e = \emptyset$, $I \cap f \neq \emptyset$ and $\chi(G - e - I) = k - 2$.
(b) Prove that there is a $(k-1)$-critical subgraph of G containing e but not f.

Problem 3. Show that for an integer $k \geqslant 2$, a graph G having no cycles of length 1 modulo k is k-colorable. [*Hint*: Use the property of the DFS tree used in the class to argue about long paths in critical graphs.]

Problem 4. Prove that the maximum number of edges in an n-vertex graph without a K_4-subdivision is $2n - 3$.

Problem 5. Let $m = \frac{k(k+1)}{2}$. Prove that $K_{m,m}$ contains a subdivision of K_{2k} while $K_{m,m-1}$ does not.

Problem 6. Prove that there exists an absolute constant $c > 0$ such that every graph G of average degree at least d contains a minor of every graph H with at most cd vertices and edges. [*Hint*: Use some of the approaches and the tricks used in the class to prove the Kostochka-Thomason bound.]