0366.4817 Graph and Hypergraph Coloring

Spring Semester 2022

Homework assignment 4

Due date: Monday, June 27, 2022

Problem 1. Let $G \sim G(n, 0.5)$. Prove that **whp** $\eta(G) = O\left(\frac{n}{\sqrt{\log n}}\right)$, where $\eta(G)$ is the clique contraction number of G.

Problem 2. Let $G \sim G(n, 0.5)$. Prove that there exists a constant $c > 0$ such that **whp** G contains a subdivision of K_t, for $t \geqslant c\sqrt{n}$.

Problem 3. Let G be a triangle-free graph graph on n vertices. Show that $\chi(G) \leqslant 2\sqrt{n}$.

Problem 4. Let H be a 3-uniform hypergraph on at least five vertices, in which every pair of vertices is contained in the same number $\lambda > 0$ of edges of H. Prove that H does not have Property B.

Problem 5. A hypergraph H is called 3-vertex-cricical if $\chi(H) = 3$ but $\chi(H') \leqslant 2$ for every proper induced subhypergraph $H' \subsetneq H$. Show that a 3-vertex-critical hypergraph $H = (V, E)$ satisfies: $|E| \geqslant |V|$. [*Hint*: dimension ...]

Problem 6. Let $G = (V, E)$ be a graph with $\chi'(G) = k$. Prove: there exists a k-edge-coloring of G in which every color class contains $\left\lfloor \frac{|E|}{k} \right\rfloor$ or $\left\lceil \frac{|E|}{k} \right\rceil$ edges.

The exercices below will NOT be graded — but you can try your hand on them:

Exercise 1. (a) Show that for every k there exist a tree T and an order σ on the vertices of T such that the greedy algorithm uses more than k colors when run on T according to σ. **(b)** Let $G \sim G(n, c/n)$ for a constant $c > 0$. Argue that for every $k > 0$ **whp** the greedy algorithm uses more than k colors when run on G according to the identity permutation.

Exercise 2. Prove from first principles that for $G \sim G(n, 0.5)$, one has **whp**: $\eta(G) = \Omega\left(\frac{n}{\sqrt{\log n}}\right)$: **(a)** Let $0 < p, p_1, p_2$ satisfy $1 - p = (1 - p_1)(1 - p_2)$. Let $G \sim G(n, p)$, $G_i \sim G(n, p_i)$, $i = 1, 2$. Argue that that G has the same distribution as $G_1 \cup G_2$. (This is called double exposure.) **(b)** Prove that $G_1 \sim G(n, 1/4)$ **whp** has disjoint sets $V_1, \ldots, V_t$, each of size $\Theta(\sqrt{\log n})$, covering altogether at least $n/2$ vertices and each spanning a connected graph in G. **(c)** Use G_2 to find **whp** an edge between every pair V_i, V_j as above.

Exercise 3. Show that **whp** for a constant $c > 0$, a random graph $G \sim G(n, c/n)$ is **whp** Class 1: **(a)** Fix a partition $[n] = V_1 \cup V_2$, $|V_i| \geqslant \lfloor n/2 \rfloor$. Argue that **whp** there is a vertex $v \in V_1$ having at least $\frac{0.9 \ln n}{\ln \ln n}$ neighbors in V_2. Conclude that **whp** $\Delta(G) \geqslant \frac{0.9 \ln n}{\ln \ln n}$. **(b)** Prove that **whp** there are no adjacent vertices in G whose degrees exceed $\frac{0.9 \ln n}{\ln \ln n}$. Conclude that **whp** the vertices of maximum degree in G form an independent set.