Problem 1. Let $G \sim G(n, 0.5)$. Prove that whp $\eta(G) = O\left(\frac{n}{\sqrt{\log n}}\right)$, where $\eta(G)$ is the clique contraction number of $G$.

Problem 2. Let $G \sim G(n, 0.5)$. Prove that there exists a constant $c > 0$ such that whp $G$ contains a subdivision of $K_t$, for $t \geq c\sqrt{n}$.

Problem 3. Let $G$ be a triangle-free graph graph on $n$ vertices. Show that $\chi(G) \leq 2\sqrt{n}$.

Problem 4. Let $H$ be a 3-uniform hypergraph on at least five vertices, in which every pair of vertices is contained in the same number $\lambda > 0$ of edges of $H$. Prove that $H$ does not have Property $B$.

Problem 5. A hypergraph $H$ is called 3-vertex-critical if $\chi(H) = 3$ but $\chi(H') \leq 2$ for every proper induced subhypergraph $H' \subseteq H$. Show that a 3-vertex-critical hypergraph $H = (V, E)$ satisfies: $|E| \geq |V|$.

Problem 6. Let $G = (V, E)$ be a graph with $\chi'(G) = k$. Prove: there exists a $k$-edge-coloring of $G$ in which every color class contains $\left\lfloor \frac{|E|}{k} \right\rfloor$ or $\left\lceil \frac{|E|}{k} \right\rceil$ edges.

The exercises below will NOT be graded — but you can try your hand on them:

Exercise 1. (a) Show that for every $k$ there exist a tree $T$ and an order $\sigma$ on the vertices of $T$ such that the greedy algorithm uses more than $k$ colors when run on $T$ according to $\sigma$. (b) Let $G \sim G(n, c/n)$ for a constant $c > 0$. Argue that for every $k > 0$ whp the greedy algorithm uses more than $k$ colors when run on $G$ according to the identity permutation.

Exercise 2. Prove from first principles that for $G \sim G(n, 0.5)$, one has whp: $\eta(G) = \Omega\left(\frac{n}{\sqrt{\log n}}\right)$:
(a) Let $0 < p, p_1, p_2$ satisfy $1 - p = (1 - p_1)(1 - p_2)$. Let $G \sim G(n, p), G_i \sim G(n, p_i), i = 1, 2$. Argue that that $G$ has the same distribution as $G_1 \cup G_2$. (This is called double exposure.) (b) Prove that $G_1 \sim G(n, 1/4)$ whp has disjoint sets $V_1, \ldots, V_t$, each of size $\Theta(\sqrt{\log n})$, covering altogether at least $n/2$ vertices and each spanning a connected graph in $G$. (c) Use $G_2$ to find whp an edge between every pair $V_i, V_j$ as above.

Exercise 3. Show that whp for a constant $c > 0$, a random graph $G \sim G(n, c/n)$ is whp Class 1: (a) Fix a partition $[n] = V_1 \cup V_2$, $|V_i| \geq \lfloor n/2 \rfloor$. Argue that whp there is a vertex $v \in V_1$ having at least $\frac{0.9\ln n}{\ln \ln n}$ neighbors in $V_2$. Conclude that whp $\Delta(G) \geq \frac{0.9\ln n}{\ln \ln n}$. (b) Prove that whp there are no adjacent vertices in $G$ whose degrees exceed $\frac{0.9\ln n}{\ln \ln n}$. Conclude that whp the vertices of maximum degree in $G$ form an independent set.