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Allocation in multi-agenda disputes: A set-valued games approach ☆

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ABSTRACT

We study allocation problems when agents negotiate across different agendas. Unlike existing papers on multi-agenda disputes, we consider environments in which resources are constrained and investing (time or effort) in one agenda reduces the ability to invest in other agendas. We introduce a class of cooperative games, referred to as set-valued games (SVG): The value of each coalition is a subset of payoff vectors. Each vector is associated with a distribution of the resources that the coalition may allocate across the agendas. In this environment we introduce and analyze the notion of the core. We show that the core allows for more cooperation opportunities and exchanging favors than existing cooperative multi-agenda models. Proving this relies on a general notion of a comparative advantage. It is shown that the classical core characterization, resorting to duality, does not hold in the current setup.

1. Introduction

Typically a (cooperative) *game* measures the 'worth' of any subgroup of individuals. This value can be interpreted as the productivity of the subgroup under consideration when investing time and effort in a particular enterprise, its social or political power, etc. Often, though, there are multiple enterprises one could invest in whereas the resources, say time, are constrained. In such scenarios there are many different time allocations to each of the different projects, each of which resulting in a different outcome. It seems that if one wishes to model such tradeoffs without pre-committing to a specific time allocation, and thereby to a specific level of production in each project, one needs to consider a more general notion of a game than the classical ones.

In this paper we introduce and study the concept of *set-valued game*: each subgroup of individuals is associated with a *set of real valued vectors*. The set of vectors associated with a subgroup represents *all* production possibilities across the different enterprises. Note that this approach does not take a stand on the aggregation of (or preferences over) payoffs across the different agendas, as in existing work on multi-agenda disputes. We seek to study the primitive form, as opposed to the reduced one, and wish to consider a model that is robust to the aggregation process. As in the classical theory, we address the issue of allocation. The appropriate notion of the core of a set-valued game is defined and analyzed.

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Vector games have been studied ever since Lind (1996). Set-valued games and their cores were studied as non-transferable utility games (see, e.g., Predtetchinski and Herings, 2004), and more generally as in the present paper (see van den Nouweland et al., 1989, Fernandez et al., 2004 and Lehrer and Teper, 2018). The present paper introduces a new model and interpretation to allocation in multi-agenda disputes and studies its relation with and differences from existing economic theories on this topic. We show that our approach to multi-agenda disputes yields interesting results compared to the standard ones in cooperative games.[2] First, the notion of the core presented here provides more opportunities for cooperation, and exchange of "power" across the different agendas, among the game participants. Second, we show that the classical Bondareva-Shapley result on core non-emptiness (Shapley, 1967; Bondareva, 1963) does not hold in our model. The well-known balancedness condition is sufficient for core non-emptiness, but is not necessary.

For the sake of clarity of our results, consider a grand group of n individuals that have to invest in k enterprises. A set-valued game (henceforth SVG) is a function $\mathbf{v}$ that associates a subset $\mathbf{v}(S) \subseteq \mathbb{R}^k_+$ for each subgroup (or coalition) S of individuals out of N. Under the classical notion of a game, $k = 1$ and $\mathbf{v}(S)$ is a singleton. The interpretation behind an SVG $\mathbf{v}$ is the following. Every $x \in \mathbb{R}^k_+$ is a vector of production levels for each of the k projects. A coalition can produce the bundle $x = (x_1, \ldots, x_k)$ if by distributing their limited resources (say one unit of time) across the different projects, its members can accomplish x_ℓ of project ℓ, for every $\ell = 1, \ldots, k$. If $x \in \mathbb{R}^k_+$ is indeed produceable by coalition S (in one unit of time), then $x \in \mathbf{v}(S)$. A member of the *core* of an SVG, a vector of dimension $n \times k$, is a payoff for each player for each of the k enterprises, that is feasible (that is, in $\mathbf{v}(N)$), and such that no coalition can deviate and do better by itself in every one of the k enterprises.[3]

We start by paying specific attention to a special class of SVGs called *multi-game based*. An SVG in this class is a convex span of k (classical) games. Formally, letting v_ℓ be the ℓth game in the base of the SVG $\mathbf{v}$, the value of a coalition S is the set of all vectors for which the ℓth production component is $\alpha_\ell v_\ell(S)$ (where the α_ℓ's are non-negative and sum to 1). This corresponds to production output being linear in effort. For this class of games, it is interesting to see the relation between our solution concept and the cores of the individual games (which are the base of the SVG), and between our notion of a core and existing approaches to allocation in multi-agenda disputes, in which the payoffs across agendas are aggregated uniformly (in particular, see Bloch and de Clippel, 2010 and Gayer and Persitz, 2016).

We establish the following results. First, we show that it is possible that while the cores of all base games, and of the sum of these games, are all empty, and thus the solution concepts in both Bloch and de Clippel and Gayer and Persitz are empty, the core of the SVG itself is not empty. We further show that if the cores of all base games are non-empty, then the convex span of these cores are a subset of the core of the SVG. In addition, this containment could be strict. In particular, we prove that whenever the cores of the base games all consist of a single allocation, then there is always an option of logrolling among several individuals that is not possible when resorting to previous approaches.[4] These results indicate that the approach and notion of a solution presented here are conceptually different than ones found in the literature, and allow for more cooperation.

We then prove a non Bondareva-Shapley result: an appropriate definition of balancedness, which serves as a characterization of core non-emptiness in the classical cooperative games setup,[5] provides a sufficient condition for the core of an SVG to be non-empty, but it is not necessary. An example is provided in which an SVG has a non-empty core while it is not balanced. We point to the fact that a characterization in the current setup cannot rely on straightforward convexity and duality considerations as in the Bondareva-Shapley result. Finally, we generalize the Bondareva-Shapley condition in a proper way, and show that it 1) coincides with the classical notion of balancedness in case of classical cooperative games, and 2) is necessary and sufficient for core non-emptiness in a large class of SVGs.

The paper is organized as follows. Section 2 introduces the notion of set-valued games and related notions that are important for the analysis of such games. Section 3 presents the notion of a core. Section 4 discusses the relation of the core of an SVG to solution concepts in other approaches to multi-agenda disputes. Section 5 points to the fact that a straightforward generalization of the Bondareva-Shapley balancedness condition cannot serve as a characterization of the core non-emptiness in the context of SVGs. Section 6 provides a proper generalization of the notion of balancedness, showing that it coincides with the classical one for standard cooperative games. Then, Section 7 shows that the proposed generalization characterizes the core non-emptiness for SVGs that satisfy some convexity conditions. Lastly, all proofs that do not appear in the main text are given in the Appendix.

2. Set-valued games

2.1. Cooperative games

We begin by recalling a few classical definitions in cooperative games. Let $N = \{1, \ldots, n\}$ be a finite set of players. A *coalition* is any subset of N, $S \subseteq N$. According to the classical definition, a *cooperative game* v associates a nonnegative

[2] Work on the topic of multi-dimensional bargaining exists also in the literature of noncooperative games. See Bloch and de Clippel (2010) for a review.

[3] Note that this is consistent with the approach that no particular aggregation of the payoffs across projects is considered.

[4] Logrolling is the trading of favors such as vote trading by legislative members.

[5] The characterization also holds in many other setups, such as those discussed in Gayer and Persitz (2016) and Assa et al. (2016).

number $v(S)$ to each coalition $S \subseteq N$. One of the leading notions of solution for games is the *core*: the core of a game v is defined as

$$\mathrm{Core(v)} = \left\{ (\mathrm{x^i})_{\mathrm{i=1,\dots,n}} : \sum_{\mathrm{i \in S}} \mathrm{x^i} \geq \mathrm{v(S)}, \forall \mathrm{S} \subseteq \mathrm{N}, \ \sum_{\mathrm{i \in N}} \mathrm{x^i} = \mathrm{v(N)} \right\}. \tag{1}$$

The *cover* of a game v at the grand coalition N is defined as,[6]

$$\hat{v}(N) = \max \left\{ \sum_{j=1}^{\ell} \alpha_j v(A_j); \ \sum_{j=1}^{\ell} \alpha_j \mathbf{1}_{A_j} = \mathbf{1}_N, \ \alpha_j \geq 0, \ A_j \subseteq N, \ j = 1, \dots, \ell \right\}.$$

Note that $v(N) \leq \hat{v}(N)$ for every game v. Bondareva (1963) and Shapley (1967) provided a necessary and sufficient condition for core non-emptiness: the core of a game v is not empty if and only if v is *balanced*, that is $v(N) = \hat{v}(N)$.

2.2. The definition

A *set-valued game* is a generalization of the notion of a game. For some $k \in \mathbb{N}$, a set-valued game associates to each coalition a *subset of vectors in* $\mathbb{R}^k_+$.

Definition 1. A *set-valued game (SVG)* over a collection N of players, is a function $\mathbf{v} : 2^N \to 2^{\mathbb{R}^k_+}$ defined on all the subsets of N, which satisfies:

1. $\mathbf{v}(\emptyset) = \{0\}$
2. *Closedness*. For every $S \subseteq N$, $\mathbf{v}(S)$ is a non-empty closed set
3. *Comprehensiveness*. If $x \in \mathbf{v}(S)$ and $y \in \mathbb{R}^k_+$ is such that[7] $y \leq x$, then $y \in \mathbf{v}(S)$.

We will explicitly refer to a classical cooperative game as *a game*, as opposed to the more general set-valued game that will be referred to as SVG. In the remainder of this section we discuss various properties of SVGs and their relation to games. The role of the definitions and examples we introduce is to motivate the notion of an SVG, and to familiarize the reader with this notion and related definitions pertinent to the analysis in subsequent sections.

Note that the definition of an SVG relies on transferable utilities within each game and non-transferable utilities across games. One can generalize the definition, and extend the analysis below appropriately, relying solely on non-transferable utilities: an environment in which each game is a non-transferable utility one, and across games utility is non-transferable as well.[8]

2.3. Set-valued and classical games

The values that an SVG takes are subsets of $\mathbb{R}^k_+$. A (classical) game is defined like an SVG on subsets of N, but opportunities are modeled with a single dimension, and thus each coalition $S \subseteq N$ corresponds to a single number reflecting its (maximum) payoff.

2.4. A multi-game based SVG

Consider a family of $k \in \mathbb{N}$ games $v_1, \dots, v_k$, and define a SVG $\mathbf{v}$ by

$$\mathbf{v}(S) := \left\{ (\alpha_1 v_1(S), \dots, \alpha_k v_k(S)) \in \mathbb{R}^k_+; \ \alpha_j \geq 0 \, \forall j, \ \sum_{j=1}^{k} \alpha_j \leq 1 \right\}. \tag{2}$$

This definition is interpreted as follows. Consider $v_j(S)$ to be the amount of project j, $j = 1, \dots, k$ coalition S can complete in one unit of time. When the coalition invests $\alpha_j(S)$ in project j, where $\sum_{j=1}^{k} \alpha_j(S) \leq 1$, a vector $(\alpha_1 v_1(S), \dots, \alpha_k v_k(S))$ can be produced by coalition S in one unit of time. An SVG of this kind is referred to as *multi-game based*.

The base games v_j $j = 1, \dots, k$ capture the stand-alone productivity (or, value) for each project. The SVG defined through these games, $\mathbf{v}$, reflects the natural time constraint and tradeoffs between investing in the different projects. Clearly, not every SVG is multi-game based, but the interpretation remains: $\mathbf{v}(S)$ captures the productivity of coalition S and reflects

[6] For any coalition $S \subseteq N$, the indicator $\mathbf{1}_S$ stands for the characteristic function of S, where $\mathbf{1}_S(i) = 1$ if $i \in S$ and 0 otherwise.

[7] $y \leq x$ means that $y_j \leq x_j$ for every $j = 1, \dots, k$.

[8] While not explicitly described as such, van den Nouweland et al. (1989) study a model of multiple non-transferable utility set-valued games in that spirit and characterize different notions of a core.

the time constraints faced by the coalition when contemplating how much time (or, more generally, resources) to invest in each project.

The following example illustrates the structure of a multi-game based SVG. It will play an important role in our subsequent analysis.

Example 1. Let $N=\{1,2,3\}, k=2$, and consider the following *SVG*: $\mathbf{v}(N)=\{(w_1,w_2);\ w_1+w_2\le 3\}, \mathbf{v}(1,2)=\{(w_1,w_2);\ 2w_1+w_2\le 3\}, \mathbf{v}(2,3)=\{(w_1,w_2);\ w_1+2w_2\le 3\}, \mathbf{v}(1,3)=\{(w_1,w_2);\ w_1+w_2\le 2\}$ and $\mathbf{v}(i)=\{(w_1,w_2);\ w_1+w_2\le 1\}, i=1,2,3$.

This *SVG* is multi-game based. Indeed, define two games v_1, v_2 as follows. $v_1(N)=3, v_1(1,2)=1.5, v_1(2,3)=3, v_1(1,3)=2$, and $v_1(i)=1$ for every player $i\in N$. Similarly, $v_2(N)=3, v_2(1,2)=3, v_2(2,3)=1.5, v_2(1,3)=2$, and $v_2(i)=1$ for every player $i\in N$. It is easy to verify that $\mathbf{v}$ is a multi-game based on v_1, v_2.

2.5. Additive SVGs

An SVG $\mathbf{v}$ is *additive* if for every two disjoint coalitions, $S, T\subseteq N$,

$$\mathbf{v}(S)+\mathbf{v}(T)=\mathbf{v}(S\cup T), \tag{3}$$

where on the left-hand side of the equality we mean the Minkowski sum. It is clear that $\mathbf{v}$ is additive if and only if $\mathbf{v}(S)=\sum_{i\in S}\mathbf{v}(i)$ for every $S\subseteq N$. In terms of productivity, each worker's productivity is independent of the group she is working with: her contribution to a group is always the set of vectors she can produce alone.

2.6. The induced game

The following definition will be helpful in analyzing the core of an SVG. Let $\lambda\neq 0$ be a non-negative vector in $\mathbb{R}^k$ (i.e., $\lambda\in\mathbb{R}^k_+$) and $\mathbf{v}$ be an SVG over N.

Definition 2. The λ*-induced-game of* $\mathbf{v}$ is the game v_λ defined by[9]

$$v_\lambda(S)=\max_{y\in\mathbf{v}(S)} y\cdot\lambda \tag{4}$$

for every $S\subseteq N$.

Remark 1. $\mathbf{v}(S)\subseteq\left\{y\in\mathbb{R}^k_+;\ y\cdot\lambda\le v_\lambda(S)\right\}$ for every $S\subseteq N$.

3. The core of an SVG

This section provides the definition of a central concept in the theories of cooperative games and capacities: the *core*.

Definition 3. The core of an *SVG* $\mathbf{v}$ is defined as

$$Core(\mathbf{v}):=\left\{\begin{aligned}&(x^i)_{i\in N};\ \text{(a) for every } i\in N, x^i\in\mathbb{R}^k_+;\\ &\qquad\text{(b)} \sum_{i\in N}x^i\in\mathbf{v}(N);\ \text{and}\\ &\qquad\text{(c)}\ \forall S\subseteq N,\ \text{if } y\in\mathbf{v}(S) \text{ and } y\ge\sum_{i\in S}x^i,\ \text{then } y=\sum_{i\in S}x^i\end{aligned}\right\}. \tag{5}$$

When $(x^i)_{i\in N}$ is in the core of $\mathbf{v}$, member i of N will get the share x^i, which is a vector in $\mathbb{R}^k_+$. That is, a core member is a "payoff" to each player for each of the k agendas. The total share of all the members of N is a feasible vector, i.e. it lies in $\mathbf{v}(N)$. Finally, it maintains stability in the sense that there is no coalition S that could find a better $y\in\mathbf{v}(S)$. That is, there is no $y\in\mathbf{v}(S)$ that dominates (Pareto) the total share of the S-members, $\sum_{i\in S}x^i$. Note that when $k=1$ the definition of the core of an SVG as in Eq. (5) coincides with the classical definition of the core (as in Eq. (1)).

Example 1 *(cont.).* Recall the SVG in Example 1. Consider $x^1=(0,1), x^2=(1,0)$, and $x^3=(0,1)$. The sum, $x^1+x^2+x^3=(1,2)$ is on the Pareto frontier of $\mathbf{v}(N)$. Furthermore, $x^1+x^2=(1,1)$, $x^1+x^3=(0,2)$, $x^2+x^3=(1,1)$, x^1, x^2, and x^3 are on the Pareto frontiers of $\mathbf{v}(1,2)$, $\mathbf{v}(1,3)$, $\mathbf{v}(2,3)$, $\mathbf{v}(1)$, $\mathbf{v}(2)$ and $\mathbf{v}(3)$, respectively. Thus, (x^1,x^2,x^3) is in the core of $\mathbf{v}$.

9 $y\cdot\lambda$ denotes the inner product of y and λ.

4. Multi-game based SVGs and logrolling

SVGs that are multi-game based were introduced in Section 2.4. For an SVG in this family it is natural to study the relation between the cores of the base games and the core of the SVG itself.

Bloch and de Clippel (2010) and Gayer and Persitz (2016) also study multi-agenda disputes, but, as explained in the Introduction, take a different approach. Both these papers are conceptually different from the current study: they consider the sum of the base games and study notions of the core in this set up. The core of the sum and the core of an SVG live in different spaces. Nevertheless, the relation between these approaches can be studied, as done below.

4.1. Base games, cores, and their sums

Let us formally introduce the ideas in Bloch and de Clippel (2010) and Gayer and Persitz (2016). Let $v_1, \ldots, v_k$ be a family of (base) games, and $v = v_1 + \cdots + v_k$ be the sum of these games. Bloch and de Clippel (2010) study the relation between the cores of the base games and the core of v. Gayer and Persitz (2016) then introduce the notion of a multi-core of $v_1, \ldots, v_k$. Conceptually, a multi-core element is a payoff to each individual for participating in the game v, where: 1) for each individual there may be a (different) subjective assessment of how these individual payoffs are generated from the payoffs for the base games; and 2) given these subjective assessments, no individual thinks that she can jointly deviate with a group of some other agents and increase their payoffs. That is, a payoff $(w^i)_{i\in N}$ is in the multi-core of v if for each agent $p \in N$ there is a feasible payoff vector $y(p) \in \mathbb{R}^{n\times k}$, where $y(p)^i_j$ is agent p's subjective assessment of the i's payoff in game j, such that:

- $\sum_{j=1}^{k} y(p)^i_j = w_i$ for every $p, i \in N$;
- $\sum_{i=1}^{N} y(p)^i_j = v_j(N)$ for every $p \in N$ and $j \in \{1, \ldots, k\}$;
- For every coalition $S \subseteq N$ and $p \in S$, $\sum_{i\in S} y(p)^i_j \geq v_j(S)$ for every $j \in \{1, \ldots, k\}$.

A result of interest is Theorem 3 in Gayer and Persitz (2016), stating that given any collection of base games, the sum of their cores is a subset of the multi-core, which is a subset of the core of the sum of the base games.

To begin understanding the relationship between all of these concepts and the core of an SVG, consider the following example which shows that it is possible that the cores of the v_j's, the sum of the cores, the core of the sum, and the multi-core of the v_j's, are all empty, while the core of the multi-game SVG based on $v_1, \ldots, v_k$ is not.

Indeed, consider again the SVG presented in Example 1. We saw in Section 3 that the core of the SVG is not empty. Now, both v_1 and v_2 are not balanced. For example,

$$\hat{v_1}(N) \geq v_1(2,3) + v_1(1) = 4 > 3 = v_1(1,2,3).$$

A similar calculation holds for the game v_2. Thus, the cores of v_1 and v_2 are empty, and consequently the sum of the cores is empty. In addition, the core of the sum $v_1 + v_2$ is also empty. Letting $v = v_1 + v_2$, we have that

$$\hat{v}(N) \geq v(2,3) + v(1) = 6.5 > 6 = v(1,2,3),$$

implying that v is not balanced, and thus has an empty core. By Theorem 3 in Gayer and Persitz (2016), the multi-core of v_1, v_2 is empty.

Now consider the agendas $v_1, \ldots, v_k$. We show that if the core of each v_j is not empty, then every "convex combination" of elements (across the k cores) is in the core of the multi-game SVG based on $v_1, \ldots, v_k$.

Proposition 1. *Let* $\mathbf{v}$ *be a multi-game* SVG *over* N *based on* $v_1, \ldots, v_k$*, and assume* x_j *is in the core of* v_j *for every* j*. Let* $z^i = (\alpha_j x^i_j)_{j=1}^k$ *(where* $\alpha_j \geq 0$ *for every* j *and* $\sum_{j=1}^k \alpha_j = 1$*). Then,* $(z^i)_{i\in N}$ *is in the core of* $\mathbf{v}$*.*

Proof. For each j, let x_j be a core member of v_j. For each player $i \in N$, let $z^i = (\alpha_j x^i_j)_{j=1}^k$. We claim that $(z^i)_{i\in N}$ satisfies the three conditions in Eq. (5). Indeed, (a) is clearly satisfied. As for (b), since x_j is in the core of v_j, we have $\sum_{i\in N} z^i = (\alpha_1 v_1(N), \ldots, \alpha_k v_k(N))$, and so

$$\sum_{i\in N} z^i \in \left\{ (\beta_1 v_1(N), \ldots, \beta_k v_k(N)) \in \mathbb{R}^k_+;\ \beta_j \geq 0\, \forall j,\ \sum_{j=1}^k \beta_j \leq 1 \right\} = \mathbf{v}(N).$$

Finally, to see that condition (c) holds, fix a coalition $S \subseteq N$ and consider a vector $(\beta_1 v_1(S), \ldots, \beta_k v_k(S)) \in \mathbf{v}(S)$ such that

$$(\beta_1 v_1(S), \ldots, \beta_k v_k(S)) \geq \sum_{i\in S} z^i = (\alpha_1 \sum_{i\in S} x^i_1, \ldots, \alpha_k \sum_{i\in S} x^i_k).$$

Since x_j is in the core of the respective v_j, we have $\sum_{i\in S} x^i_j \geq v_j(S)$. Therefore, $(\beta_1, \ldots, \beta_k) \geq (\alpha_1, \ldots, \alpha_k)$, which (since the sum of the β_j's does not exceed 1 while that of the α_j's equals 1), implies that the two vectors coincide. Thus, $\sum_{i\in N} z^i = (\beta_1 v_1(S), \ldots, \beta_k v_k(S))$, (c) is satisfied and the proof is complete. □

A similar result can be stated regarding the multi-core of the sum of the base games. The following proposition states that every subjective assessment of a member in the multi-core is also a member in the core of the SVG. The proof is identical to that of Proposition 1 and is therefore omitted.

Proposition 2. *Let* $\mathbf{v}$ *be a multi-game based* SVG *over* N *based on* $v_1, \ldots, v_k$. *Assume that* $(y(p))_{p\in N}$ *is a subjective justification vector for some member of the multi-core of* $(v_1, \ldots, v_k)$. *For every player* $p \in N$ *and game* j, *let* $x(p)_j = (y(p)^i_j)_{i\in N}$ *and* $z(p)^i = (\alpha_j x(p)^i_j)^k_{j=1}$, *where* $\alpha_j \geq 0$ *for every* j *and* $\sum_{j=1}^k \alpha_j = 1$. *Then,* $(z(p)^i)_{i\in N}$ *is in the core of* $\mathbf{v}$, *for every* $p \in N$.

The converse statements of Propositions 1 and 2 do not hold. We saw in Example 1 that the base games can an empty core, whereas the core of the multi-game based SVG is not empty. The following example shows that even if the base games have non-empty cores, they do not necessarily span the core of the multi-game based SVG.

Example 2. Consider the following variation of the game described in Example 1. Let $N = \{1, 2, 3\}$, and set $u_1(N) = 3, u_1(1, 2) = u_1(1, 3) = 1.5, u_1(2, 3) = 3$, and $u_1(i) = 0$ for all $i \in N$. Similarly, set $u_2(N) = 3, u_2(1, 3) = u_2(2, 3) = 1.5, u_2(1, 2) = 3$, and $u_2(i) = 0$ for all $i \in N$.

The core of u_1 consists only of the allocation $(0, 1.5, 1.5)$, and the core of u_2 consists only of the allocation $(1.5, 1.5, 0)$. It is readily verified that $x^1 = (0, 1), x^2 = (1, 0)$, and $x^3 = (0, 1)$ is in the core of the multi-game *SVG* $\mathbf{u}$ based on u_1 and u_2. However, (x^1, x^2, x^3) is not spanned in the sense of Proposition 1 by the cores of u_1 and u_2.

In addition, defining $u = u_1 + u_2$, we obtain that $u(N) = 6, u(1, 2) = u(2, 3) = 4.5, u(1, 3) = 3$, and $u(i) = 0$ for all $i \in N$. We see that the only core member of u is $(1.5, 3, 1.5)$, which is also the sum of the cores of u_1 and u_2. Again, by Theorem 3 in Gayer and Persitz (2016), we conclude that the multi-core consists only of $(1.5, 3, 1.5)$.

4.2. Logrolling

In the remainder of this section we show that for a class of multi-game based SVGs, the core of the SVG is not a convex span of the cores of the base games. This class consists of base games with a unique core element. An example for such games is the family of additive games: a game v is additive if $v(S) = \sum_{i\in S} v(i)$ for every coalition $S \subseteq N$. In this case the unique core element, x, of v is given by $x^i = v(i)$ for every player $i \in N$. Note also that a sum of additive games is additive, implying that the core of the sum is the sum of the cores of the base games. In particular, for such games the multi-core coincides with the sum of the cores, and consists of a single element.

Example 3. Consider two additive two-player base games, u, v, where $v(i) = u(i) = 1$ for every player i. The core element of each of these games assigns payoff 1 for each of the players.

From the propositions above, the core of the *SVG* based on these two games contains the convex hull of the cores of the base games. In particular, (x^1, x^2), where $x^1 = x^2 = (0.5, 0.5)$, is in the core of the *SVG*. However, notice that $(\hat{x}^1, \hat{x}^2)$, *where* $\hat{x}^1 = (0, 1), \hat{x}^2 = (1, 0)$ is also in the core of the *SVG*, and is not in the convex hull of the cores of the two base games.

The example shows that by reallocating payoffs between players the SVG provides a space for advantageous cooperation beyond what is given by the separate base games, or their sum. The next example shows that not every reallocation is possible if one wishes to maintain stability.

Example 4. Consider two additive base games, u and v, both with three players. Let $v(1) = u(3) = 1, v(2) = u(2) = 2$, and $v(3) = u(1) = 3$. Suppose that the players try to reallocate their payoffs as follows: player 3 gives player 1 one payoff unit in game u, and obtains one unit payoff in game v in return. Consider the $0.5 - 0.5$ mixture between the two games. The resulting payoffs are $x^1 = (0, 2)$, where the left coordinate corresponds to the payoff in agenda v, $x^2 = (1, 1)$, and $x^3 = (2, 0)$. We show that this allocation is not in the core of the *SVG* based on v and u.

The frontier of the set $\mathbf{v}(12)$ is given by the line $5x + 3y = 15$, while $x^1 + x^2 = (1, 3)$ is a strict interior point. Thus, the coalition consisting of the players 1 and 2 has an incentive to deviate and play on their own. It is possible, however, to find another payoff reallocation between players 1 and 3 that would give rise to a new core element which is not in the convex span of the cores of the base games. Indeed, $x^1 = (0.75, 1.25)$, $x^2 = (1, 1)$, and $x^3 = (1.25, 0.75)$ is such a reallocation.[10] No coalition has a profitable deviation. For instance, players 1 and 2 together get $x^1 + x^2 = (1.75, 2.25)$, which lies beyond the frontier of $\mathbf{v}(12)$.

[10] These payoffs correspond to reallocation of a quarter of a payoff unit between players 1 and 3 across the two base games (player 3 provides 0.25 in game u for player 1 in favor of 0.25 in game v), and considering the $0.5 - 0.5$ mixture between the two games.

The next theorem states that there exists a core member of an SVG, based on additive games (with strictly positive individual payoffs), that cannot be obtained by the convex span of the base games. The proof relies on a simple economic intuition behind the notion of comparative advantage adapted to the current setup (that is, when there are more than two agents). It shows that we can always reallocate some "value" between two of the players—the one with the lowest comparative advantage in one agenda and the one with the highest comparative advantage in the same agenda—and construct a new core member that is not in the span of the cores of the base games.

Theorem 1. *Let* $\mathbf{v}$ *be an* SVG *based on the additive games* $v_1, ..., v_k$*, not all are equivalent*[11] *and* $v_j(i) > 0$ *for every* $j = 1, ..., k$ *and* $i \in N$*. Then, there exists a core element of* $\mathbf{v}$*,* $x^1, ..., x^n$*, such that there are no numbers* $\alpha_j \geq 0\ \forall j, \sum_{j=1}^k \alpha_j = 1$*, such that* $x^i = (\alpha_1 v_1(i), \alpha_2 v_2(i), ..., \alpha_n v_n(i)), i \in N$*.*

Proof. We first assume that $k = 2$. Let $r = (v_1(N), -v_2(N)) \in \mathbb{R}^2$, and for $i \in N$ let $\beta_i = v_2(i) - \frac{v_2(N)}{v_1(N)} v_1(i)$ and $y^i = (\frac{1}{2}v_1(i), \frac{1}{2}v_2(i))$. For $S \subseteq N$, let $y(S) = \sum_{i \in S} y^i$. Note that $y(S)$ is on the Pareto efficient frontier of $\mathbf{v}(S)$ and is strictly positive in both coordinates. Thus, $y(S) + \gamma \sum_{i \in S} \beta_i r$ is also strictly positive in both coordinates for sufficiently small $\gamma > 0$.

For $i \in N$ define $x^i = y^i + \gamma \beta_i r$, and for $S \subseteq N$, let $x(S) = \sum_{i \in S} x^i$. By assumption not all of $v_1, ..., v_k$, are equivalent. This implies that $\beta_i \neq 0$ and thus $x^i \neq y^i$ for at least one i. We show that $x^1, ..., x^n$ is a core element of $\mathbf{v}$.

Note that $\sum_{i \in N} \beta_i = v_2(N) - \frac{v_2(N)}{v_1(N)} v_1(N) = 0$. Thus, $x(N) = y(N)$ and therefore $x(N) \in \mathbf{v}(N)$. Now fix $S \subseteq N$. We show that $x(S)$ is not dominated by any point in $\mathbf{v}(S)$. To this end we show that

$$\frac{v_2(S)}{v_1(S)} \geq \frac{v_2(N)}{v_1(N)} \text{ if and only if } \sum_{i \in S} \beta_i \geq 0.$$

This equivalence means that the frontier of $\mathbf{v}(S)$ is steeper than that of $\mathbf{v}(N)$ if and only if $x(S) - y(S) = \sum_{i \in S} \gamma \beta_i r$. That is, $x(S) - y(S)$ is a vector r multiplied by a nonnegative coefficient $(\gamma(\sum_{i \in S} \beta_i) r)$.[12] Since $y(S)$ is on the frontier of $\mathbf{v}(S)$, this implies that $x(S)$ lies beyond this frontier.

Indeed,

$$\sum_{i \in S} \beta_i = \sum_{i \in S} v_2(i) - \frac{v_2(N)}{v_1(N)} v_1(i) = v_2(S) - \frac{v_2(N)}{v_1(N)} v_1(S) \geq 0 \text{ if and only if } \frac{v_2(S)}{v_1(S)} \geq \frac{v_2(N)}{v_1(N)},$$

as needed.

As for a general k, one can restrict attention to two non-equivalent games, say w.l.o.g. v_1 and v_2, and consider $y^i = (\frac{1}{2}v_1(i), \frac{1}{2}v_2(i), 0, ..., 0), i \in N$ as the original core member of $\mathbf{v}$. Then define $r = (v_1(N), -v_2(N), 0, ..., 0) \in \mathbb{R}^n$ to be the vector by which y^i is shifted. Now the proof proceeds along the same line as in the case of $k = 2$. $\square$

In the case where there are only two players, Ricardo defined the notion of comparative advantage. In our context there are many players. The advantage of player i is measured in comparison with the grand coalition. We say that player i has a comparative advantage in project 2 if

$$\frac{v_2(i)}{v_1(i)} \geq \frac{v_2(N)}{v_1(N)}.$$

The proof of Theorem 1 is based on the idea that when player i has a comparative advantage in project 2, he can give up some value related to project 2 and get in return some value related to project 1. The rate at which this logrolling/trade-off is carried out is determined by the grand coalition, and formally by the vector r. Player i would receive (in addition to y^i) the vector $\gamma \beta_i r$.

The trade-off of player i depends not only on the project at which player i has a comparative advantage, but also on two additional factors: the magnitude of the comparative advantage (i.e., $\frac{v_2(i)}{v_1(i)} - \frac{v_2(N)}{v_1(N)}$) and the "power" of player i (measured as $v_1(i)$). This is the reason why player i's trade-off across all projects, represented by $x^i - y^i$, is proportional to $v_1(i)\left(\frac{v_2(i)}{v_1(i)} - \frac{v_2(N)}{v_1(N)}\right) r$.

Remark 2. The vector $r = (r_1, r_2)$ was chosen above so that $\frac{|r_2|}{r_1} = \frac{|v_2(N)|}{v_1(N)}$. The same proof goes through if r is such that

$$\frac{|r_2|}{r_1} \leq \min\left\{\frac{v_2(S)}{v_1(S)};\ \frac{v_2(N)}{v_1(N)} < \frac{v_2(S)}{v_1(S)},\ S \neq \emptyset\right\},$$

[11] Two additive games v and u are equivalent if $\frac{v(i)}{u(i)}$ is constant across i.

[12] In other words, it is vector in the "south-western" orthant whose slope (like that of r) is smaller than that of the frontier of $\mathbf{v}(S)$.

and

$$\frac{|r_2|}{r_1} \geq \max\left\{\frac{v_2(S)}{v_1(S)};\ \frac{v_2(N)}{v_1(N)} > \frac{v_2(S)}{v_1(S)},\ S \neq \emptyset\right\}.$$

We therefore obtain that the set of possible logrolling opportunities has a full dimension, namely two.

5. Non-Bondareva-Shapley theorem and SVGs

5.1. The cores of $\mathbf{v}$ and v_λ

In this section we discuss the relation between the core of $\mathbf{v}$ and that of the induced game v_λ. This relation will be significant in terms of finding conditions for the non-emptiness of the core of $\mathbf{v}$. Prior to this we need to introduce the notion of the cover of an SVG.

Definition 4. Let $\mathbf{v}$ be an *SVG*. The *cover* of $\mathbf{v}$ at the coalition N is

$$\hat{\mathbf{v}}(N) = \left\{\sum_{j=1}^{\ell}\alpha_j y_j;\ \sum_{j=1}^{\ell}\alpha_j \mathbf{1}_{A_j} = \mathbf{1}_N,\ y_j \in \mathbf{v}(A_j),\ \alpha_j \geq 0,\ A_j \subseteq N,\ j = 1, ..., \ell\right\}. \tag{6}$$

Notice that $\hat{\mathbf{v}}(N)$ is a convex set.

Proposition 3. *Fix $\lambda \in \mathbb{R}^k_+$, $\lambda \neq 0$. The core of v_λ is not empty if and only if there exists $x \in \mathbf{v}(N)$ such that*

$$x \in \operatorname{argmax}\left\{y \cdot \lambda;\ y \in \hat{\mathbf{v}}(N)\right\}. \tag{7}$$

Proof. Assume first that there is $x \in \mathbf{v}(N)$ such that $x \in \operatorname{argmax}_{y \in \hat{\mathbf{v}}(N)} y \cdot \lambda$. The sets $\hat{\mathbf{v}}(N)$ and $C := \{y;\ y \geq x\}$ (the latter being the positive orthant relative to x) are convex. Due to Eq. (7), $\lambda \in \mathbb{R}^k_+$ is a separating vector:

$$y \cdot \lambda \leq z \cdot \lambda \text{ for every } y \in \hat{\mathbf{v}}(N) \text{ and } z \in C. \tag{8}$$

We show that the core of v_λ is non-empty. Suppose this is not true. Then, by the Bondareva-Shapley Theorem, $\hat{v_\lambda}(N) > v_\lambda(N)$. This implies that there is a decomposition of $\mathbf{1}_N$, $\mathbf{1}_N = \sum_{j=1}^{\ell}\alpha_j \mathbf{1}_{A_j}$, with $\alpha_j \geq 0$, $j = 1, ..., \ell$, such that $\sum_{j=1}^{\ell}\alpha_j v_\lambda(A_j) > v_\lambda(N)$. Thus, for every $j = 1, ..., \ell$ there is $x_j \in \mathbf{v}(A_j)$ such that

$$\sum_{j=1}^{\ell}\alpha_j x_j \cdot \lambda > \max_{w \in \mathbf{v}(N)} w \cdot \lambda \geq x \cdot \lambda. \tag{9}$$

Since $\sum_{j=1}^{\ell}\alpha_j x_j \in \hat{v}(N)$, we reached a contradiction with Eq. (8), so the core of v_λ is not empty.

Now assume that no $x \in \mathbf{v}(N)$ satisfies Eq. (7). Since $\mathbf{v}(N) \subseteq \hat{\mathbf{v}}(N)$, it implies that $\max_{y \in \hat{v}(N)} y \cdot \lambda > \max_{y \in \mathbf{v}(N)} y \cdot \lambda$. Therefore, $\hat{v_\lambda}(N) > v_\lambda(N)$, which by the Bondareva-Shapley Theorem implies that the core of v_λ is empty. $\square$

The following proposition will be useful below. This result appears also in Fernandez et al. (2004). The proof is provided for completeness.

Proposition 4. *Suppose that there exists $\lambda \in \mathbb{R}^k_+$ such that the core of the induced game v_λ is not empty. Then,* $\mathrm{Core}(\mathbf{v}) \neq \emptyset$.

Proof. Let $(w_i)_{i\in N}$ be in the core of v_λ, in particular, $\sum_{i\in N} w_i = v_\lambda(N)$. Take $x \in \mathbf{v}(N)$ such that $v_\lambda(N) = x \cdot \lambda$. Let, $w := v_\lambda(N)$ and $x^i := \frac{w_i}{w}x$ (recall, $x \in \mathbb{R}^k_+$, $\frac{w_i}{w} \in \mathbb{R}$ and $\sum_i \frac{w_i}{w} = 1$). Then, $x^i \in \mathbb{R}^k_+$, $\sum_{i\in N} x^i = \sum_{i\in N}\frac{w_i}{w}x = x \in \mathbf{v}(N)$ and for every $S \subseteq N$, $\sum_{i\in S} w_i \geq v_\lambda(S)$, and so

$$(\sum_{i\in S} x^i) \cdot \lambda = (\sum_{i\in S}\frac{w_i}{w}x) \cdot \lambda = (\sum_{i\in S}\frac{w_i}{w})v_\lambda(N) = \sum_{i\in S} w_i \geq v_\lambda(S) = \max_{y\in\mathbf{v}(S)} y \cdot \lambda.$$

This implies that if $y \in \mathbf{v}(S)$ and $y \geq \sum_{i\in S} x^i$, then $y = \sum_{i\in S} x^i$. $\square$

5.2. A sufficient condition for the non-emptiness of the core

First, a useful notion.

Definition 5. Let $D \subseteq \mathbb{R}^k$ be compact. We say that $x \in D$ is *D-efficient* if $y \in D$ and $y \geq x$ imply $y = x$.

Theorem 2. *Let* $\mathbf{v}$ *be an* SVG. *If there exists* $x \in \mathbf{v}(N)$ *which is* $\hat{\mathbf{v}}(N)$*-efficient, then the core of* $\mathbf{v}$ *is not empty.*

Before presenting the proof, note that the condition in the theorem is a generalization of the classical notion of balancedness of a game: a game v is said to be *balanced* if $\hat{v}(N) = v(N)$.

Proof. Let $x \in \mathbf{v}(N)$ be $\hat{\mathbf{v}}(N)$-efficient. Define, $C := \{z; z \geq x\}$. The sets $\hat{\mathbf{v}}(N)$ and C are convex. Since x is $\hat{\mathbf{v}}(N)$-efficient, the intersection $\hat{\mathbf{v}}(N) \cap C$ contains only x. The set C is of full dimension and therefore there is a non-zero vector $\lambda \in \mathbb{R}^k_+$ separating $\hat{\mathbf{v}}(N)$ and C. That is, $y \cdot \lambda \leq z \cdot \lambda$ for every $y \in \hat{\mathbf{v}}(N)$ and $z \in C$. We have that x satisfies Eq. (7) with respect to λ. Due to Proposition 3, the core of v_λ is not empty, which in turn implies, by Proposition 4, that the core of $\mathbf{v}$ is not empty. $\square$

The condition of Theorem 2 has the same flavor as that of the Bondareva-Shapley Theorem. Indeed, when $k = 1$, this condition is equivalent to that of the Bondareva-Shapley Theorem and is therefore not only sufficient, but also necessary. The following example shows that the core of $\mathbf{v}$ can be non-empty, while the core of v_λ is empty for every $\lambda \in \mathbb{R}^k_+$. This implies that the condition of Theorem 2 is not necessary. Our example requires only two agendas.

5.3. The sufficient condition is not necessary

In Section 3 we saw that the core of the SVG in Example 1 is not empty. On the other hand, we show now that for every $\lambda \in \mathbb{R}^k_+$, the core of v_λ is empty. To this end, it is sufficient, by Proposition 3, to show that the efficient frontier of $\mathbf{v}(N)$ is strictly dominated by a point in $\hat{\mathbf{v}}(N)$. That is, for every $x \in \mathbf{v}(N)$ there is $y \in \hat{\mathbf{v}}(N)$ such that $y_i > x_i, i = 1, 2$. Since the efficient frontier of $\mathbf{v}(N)$ is a straight line, since $\hat{\mathbf{v}}(N)$ is convex, it is sufficient to show that this statement is true for the two $\mathbf{v}(N)$-efficient points that lie on the axes. The $\mathbf{v}(N)$-efficient point on the vertical axes is $(0, 3)$. Consider the decomposition $\mathbf{1}_N = \frac{1}{2}(\mathbf{1}_{\{1,2\}} + \mathbf{1}_{\{2,3\}} + \mathbf{1}_{\{1,3\}})$ and the most upper points in $\mathbf{v}(1, 2), \mathbf{v}(2, 3), \mathbf{v}(1, 3)$ that lie on the vertical axis. These points are, respectively, $(0, 3)$, $(0, 1.5)$ and $(0, 2)$. Thus, the point $\frac{1}{2}((0, 3) + (0, 1.5) + (0, 2)) = (0, 3.25) \in \hat{\mathbf{v}}(N)$. This point lies strictly above $(0, 3)$. A similar argument applies to the horizontal axis. We therefore conclude that the two $\mathbf{v}(N)$-efficient points that lie on the vertical axis are strictly dominated by points in $\hat{\mathbf{v}}(N)$ and the argument is complete.

To summarize, this example shows that the core of $\mathbf{v}$ is not empty despite the fact that the core of any v_λ is empty and the fact that any $\mathbf{v}(N)$-efficient point is strictly dominated by a point in $\hat{\mathbf{v}}(N)$.

6. Generalized Bondareva-Shapley theorem

In this section we provide a necessary and sufficient condition for the non-emptiness of the core of an SVG in the case where the set $\mathbf{v}(N)$ is convex and, moreover, the set of points in $\mathbb{R}^k_+$ that dominate $\mathbf{v}(T)$ is convex for every $T \subsetneq N$.

6.1. The one-dimensional case

We start out with the one-dimensional case which is closely connected with classical cooperative games and the Bondareva-Shapley characterization of core non-emptiness. Section 6.3 below, and Theorem 3 therein, generalizes these results to multi-dimensions and SVGs.

Let $x_1, ..., x_l \in \mathbb{R}^n$ and $A_1, ..., A_l \subseteq \mathbb{R}$ be non-empty, convex and closed. Define X to be the $l \times n$ dimensional matrix whose i-th row is x_i. Let $A = A_1 \times \cdots \times A_l$ and $B = \{Xc;\ c \in \mathbb{R}^n\}$. Both A and B are closed and convex.

The following proposition is a (one-dimensional) consequence of Theorem 3. Due to its importance in the study of the core of classical cooperative games (see next subsection), we state it separately.

Proposition 5. *There exists* $c \in \mathbb{R}^n$ *such that* $x_i \cdot c \in A_i$ *for every* $i = 1, ..., l$ *if and only if for every vector* $s = (s_1, ..., s_l) \in \mathbb{R}^l$,

$$\sum_i s_i x_i = 0 \text{ implies } 0 \in \sum_i s_i A_i. \tag{10}$$

Note that $\sum_i s_i x_i = 0$ (the left-hand side of Eq. (10)) is equivalent to $sX = 0$. Furthermore, $0 \in \sum_i s_i A_i$ (the right-hand side of Eq. (10)) is equivalent to the condition that there exists $a \in A$ such that $s \cdot a = 0$. In other words, the condition in Eq. (10) can be restated as

$$sX = 0 \text{ implies that there exists } a \in A \text{ such that } s \cdot a = 0. \tag{11}$$

This condition is the one-dimensional version of the condition in Eq. (13) that appears in Theorem 3 below. In other words, Proposition 5 is a special case of Theorem 3.

6.2. A particular case: an implication to the core of a game

The problem related to the non-emptiness of the core of a game v with n players is defined by the following parameters. Consider the vectors $x_S = \mathbf{1}_S, S \subseteq N$, all in $\mathbb{R}^n$, and the sets $A_S = [v(S), \infty)$ for $S \subsetneq N$ and $A_N = \{v(N)\}$, all contained in $\mathbb{R}$. The existence of $c \in \mathbb{R}^n$ such that $x_S \cdot c \in A_S$ for every $S \subseteq N$ is equivalent to the existence of $c \in \mathbb{R}^n$ such that

$$\mathbf{1}_S \cdot c \geq v(S) \text{ for every } S \subsetneq N, \text{ and } \mathbf{1}_N \cdot c = v(N). \tag{12}$$

Such a vector c is a core member of v. Proposition 5 therefore states that the core of v is non-empty if and only if condition (10) is satisfied.

To see that this condition is equivalent to the Bondareva-Shapley condition, we show first that the Bondareva-Shapley condition implies that Eq. (10) is satisfied for every $s = (s_T)_{T \subseteq N}$. Consider the equation $sX = 0$. If there are two proper coalitions $R, T \subsetneq N$ such that $s_R s_T < 0$, then $s_R A_R + s_T A_T = \mathbb{R}$. This implies, in particular, that $0 \in \sum_T s_T A_T$. In other words, when the product $s_R s_T$ is negative, the condition of the proposition is fulfilled automatically.

We can therefore assume that $s_T, T \subsetneq N$, all non-negative or non-positive. Furthermore, we may assume that $s_T \geq 0$ for every proper coalition T (otherwise we would consider $-s_T, T \subsetneq N$). Then $s_N \leq 0$. Therefore, $sX = 0$ implies $\sum_{T \neq N} s_T x_T = |s_N| \mathbf{1}_N$. If $s_N \neq 0$, then we are dealing with a *balanced collection*, as referred to in the theory of cooperative games, with $s_T/|s_N|$ being the balancing coefficients. The Bondareva-Shapely condition dictates that a balanced collection must satisfy $\sum_{T \neq N} (s_T/|s_N|) v(T) \leq v(N)$. The latter inequality is equivalent to $0 \in \sum_{T \neq N} s_T A_T + s_N A_N$, which is what is stated in Eq. (11). If however, $s_N = 0$, then all $s_T = 0$, $T \subseteq N$. Thus, $0 \in \sum_{T \subseteq N} s_T A_T$ (which is the left-hand side of Eq. (10) in this particular case), as required.

Now suppose that (10) holds, and let us show that the Bondareva-Shapely condition is satisfied. Consider balanced coefficients: $s_T \geq 0, T \subsetneq N$, such that $\sum_{T \subsetneq N} s_T x_T = x_N$. We have to show that $\sum_{T \subsetneq N} s_T v(T) \leq v(N)$. Set $s_N = -1$ in order to get $\sum_{T \subsetneq N} s_T x_T + s_N x_N = 0$. By Eq. (10), $0 \in \sum_{T \subsetneq N} s_T A_T - A_N$. It means that there is a vector $(a_T)_{T \subsetneq N}$, where $a_T \geq v(T)$ for every $T \subsetneq N$, such that $\sum_{T \subsetneq N} s_T a_T = v(N)$. This implies that $\sum_{T \subsetneq N} s_T v(T) \leq v(N)$, as required

6.3. The multi-dimensional case

Let $x_1, \ldots, x_l \in \mathbb{R}^n$ and let $A_1, \ldots, A_l \subseteq \mathbb{R}^k$ be non-empty, convex and closed. As above, X denotes the $l \times n$ dimensional matrix whose i-th row is x_i. Let A be the set of all $l \times k$ matrices whose i-th row is a vector in A_i.

Definition 6. We say that $x_1, \ldots, x_l; A_1, \ldots, A_l$ are *balanced* if for every $k \times l$-dimensional matrix S,

$$SX = \mathbf{0} \text{ implies that there exists } a \in A \text{ such that } Sa = \mathbf{0}. \tag{13}$$

The term 'balanced' is inspired by the similarity between Eqs. (11) and (13) and, in view of this, the following is a generalized Bondareva-Shapley theorem.

Theorem 3. *Assume that each A_i is a union of a bounded, convex, closed set and a polyhedron.*[13] *There exists an $n \times k$ dimensional matrix C such that*

$$x_i C \in A_i \text{ for every } i = 1, \ldots, l$$

if and only if $x_1, \ldots, x_l; A_1, \ldots, A_l$ are balanced.

Proof. The 'only if' direction is simple and is therefore omitted. For the 'if' direction let $B = \{XC;\ C \text{ is an } n \times k \text{ matrix}\}$. Note that both sets A and B are closed and convex.

In case $A \cap B \neq \emptyset$, the proof is complete. Suppose now that A and B are disjoint.

Lemma 1. *There exists a separating $k \times l$-dimensional matrix $S \neq 0$, such that*[14]

$$\mathrm{tr}(Sb) > \mathrm{tr}(Sa) \text{ for every } b \in B \text{ and } a \in A. \tag{14}$$

---

[13] A polyhedron is a set defined by a finite number of weak linear inequalities. Such a set might be unbounded.

[14] tr is the trace operator: $\mathrm{tr}(E)$ is the sum of the diagonal entries in the matrix E.

The proof is given in the Appendix.

Lemma 1 is actually a separation theorem. The matrix S serves as a separator between the sets A and B. Two points need to be clarified. First, the trace operator yields an inner product. Consider the matrix S. One can think of it as a vector of dimension $k \cdot l$: the top row occupies the first block of l coordinates, the second row takes the second block of l coordinates, and so on, until the last, the k-th, row which sits on the last block of l coordinates. Similarly, fix a matrix $b \in B$. It has the same dimension and can be also thought of as a $k \cdot l$-dimensional vector: the left column occupies the first block of l coordinates, the second column takes the second block of l coordinates, etc. $\mathrm{tr}(Sb)$ is the inner product of these two vectors.

The second point is the strict inequality in Eq. (14). In order to guarantee a strict separation (i.e., with a strict inequality) it is not sufficient to show that A and B are disjoint: we show that the distance between these sets is positive. This is done in the proof of Lemma 1 in the Appendix.

Lemma 1 states that there exists a matrix S such that $\mathrm{tr}(SXC) > \mathrm{tr}(Sa)$ for every $n \times k$-dimensional matrix C and $a \in A$. We claim that $SX = \mathbf{0}$ ($\mathbf{0}$ being the all-0 matrix). Otherwise, one can find a matrix C such that $\mathrm{tr}(SXC) < \mathrm{tr}(Sa)$ for some $a \in A$. Indeed, fix one $a \in A$. If $SX \neq 0$, then there is an entry, say (i, j), of the matrix SX which is not 0, namely $(SX)_{ij} \neq 0$. Now consider the matrix C whose (j, i) entry is equal to $\frac{-2|\mathrm{tr}(Sa)|}{(SX)_{ij}}$, and all other entries are 0. Then $\mathrm{tr}(SXC) = -2|\mathrm{tr}(Sa)|$, which is strictly smaller than $\mathrm{tr}(Sa)$.

As $SX = \mathbf{0}$, we obtain that for every $n \times k$ dimensional matrix C, $SXC = \mathbf{0}C = \mathbf{0}$ implying that

$$\mathrm{tr}(SXC) = 0. \tag{15}$$

On the other hand, the balancedness condition (i.e., Eq. (13)) guarantees that since $SX = \mathbf{0}$, there is $a \in A$ such that $Sa = \mathbf{0}$, and therefore

$$\mathrm{tr}(Sa) = 0. \tag{16}$$

Eqs. (15) and (16) violate Eq. (14). We conclude that $A \cap B \neq \emptyset$, which completes the proof. □

7. Non-emptiness of the core in the convex case

In this section we discuss the non-emptiness of the core of an SVG under some convexity conditions, and its characterization by the generalized Bondareva-Shapley condition presented in Theorem 3.

7.1. The problem behind core non-emptiness

In order to explain the difficulty behind the non-emptiness of the core, define (compare with Eq. (5))

$$C(\mathbf{v}, T) = \Big\{ (x^i)_{i \in N};\ \text{(a) for every } i \in N, x^i \in \mathbb{R}^k_+; \quad \text{(b)} \sum_{i \in N} x^i \in \mathbf{v}(N);\ \text{and} \quad \text{(c) if } y \in \mathbf{v}(T) \text{ and } y \geq \sum_{i \in T} x^i, \text{ then } y = \sum_{i \in S} x^i. \Big\} \tag{17}$$

Note that

$$\mathrm{Core}(\mathbf{v}) = \bigcap_{T \subseteq N} C(\mathbf{v}, T). \tag{18}$$

Thus, in order for the core of $\mathbf{v}$ to be non-void, the intersection of all $C(\mathbf{v}, T)$, $T \subseteq N$, must be non-empty. Condition (c) in Eq. (17) can be also expressed as

$$\min_{y \in \mathbf{v}(T)} \max_{j=1,\dots,k} \Big(\sum_{i \in T} x^i_j - y_j \Big) \geq 0,$$

which is equivalent to[15]

$$\forall S \subseteq N,\ \sum_{i \in T} x^i \in \mathrm{cl}(\mathbb{R}^k_+ \setminus \mathbf{v}(\mathrm{T})).$$

[15] cl() is the closure operator.

While the first two conditions in Eq. (17) are convex (when $\mathbf{v}(N)$ is convex), typically $\mathbb{R}^k_+ \setminus \mathbf{v}(T)$ is not convex, hence the last condition is not. Condition (c) makes it clear why the set defined in Eq. (17) is not convex. It therefore seems that finding a necessary and sufficient condition for the non-emptiness of the intersection of sets that are not convex (see Eq. (18)), goes beyond a straightforward duality.

In what follows we address this issue and provide a characterization of when the core of a SVG is not empty in cases where $\mathbf{v}(N)$ is convex, and $\mathbb{R}^k_+ \setminus \mathbf{v}(T), T \subsetneq N$ are convex, or a finite union of convex sets. The question remains open in more general cases.

7.2. Characterization of the core of a SVG: the convex case

Theorem 3 implies the following.

Theorem 4. *Let $\mathbf{v}$ be an SVG defined on the set N. Define $x_T = \mathbf{1}_T$ for every $T \subseteq N$. Also, for every $T \subsetneq N$, define $A_T = \mathrm{cl}(\mathbb{R}^k_+ \setminus \mathbf{v}(T))$ and $A_N = \mathbf{v}(N)$.*[16] *Suppose that A_T is convex for every $T \subseteq N$. Then, the core of $\mathbf{v}$ is not empty if and only if $(x_T, A_T)_{T \subseteq N}$ is balanced.*

Proof. For every $T \subseteq N$ the set $\mathbf{v}(T)$ is bounded. Thus, there is a constant $m > 0$ such that set $M := \{x \in \mathbb{R}^k_+; x \le (m, \dots, m)\}$ contains $\mathbf{v}(T)$ for every $T \subseteq N$: $\mathbf{v}(T) \subseteq M$. Denote $D^1_T := \mathrm{cl}(M \cap A_T)$ and $D^2_T := \mathrm{cl}(\mathbb{R}^k_+ \setminus D^1_T)$. By assumption, for every $T \subsetneq N$, A_T is convex, hence D^1_T is compact and convex. Finally, D^2_T is a polyhedron and $A_T = D^1_T \cup D^2_T$. This feature of A_T is required in order to employ Theorem 3.

By Theorem 3, there exists an $n \times k$ dimensional matrix C such that

$$x_T C \in A_T \text{ for every } T \subseteq N \tag{19}$$

if and if $(x_T, A_T)_{T \subseteq N}$ is balanced. To complete the proof note that C is a core member if and only if it satisfies condition Eq. (19). $\square$

In the present context, the equation $SX = 0$ is interpreted as follows. As we have seen above, in the one-dimensional case $sX = 0$ means that s is a vector of balanced coefficients. Here $SX = 0$ is interpreted as a collection of balanced coefficients: every row of the matrix S is a vector of balanced coefficients.

Remark 3. In the case where for every $T \subsetneq N$ the set of the points in $\mathbb{R}^k_+$ that dominate $\mathbf{v}(T)$ is a union of finitely many convex sets, one can list a finite number of problems, namely, $A^j_T \subseteq \mathbb{R}^k$, $T \subseteq N$, $j = 1, \dots, L$, such that the core of $\mathbf{v}$ is not empty if and only if there exists at least one j and a matrix C such that $x_T C \in A^j_T$ for every $T \subsetneq N$ and $x_T C \in \mathbf{v}(N)$. An example of such a case is where $\mathbf{v}(N)$ is convex and each $\mathbf{v}(T), T \subsetneq N$ is a polytope i.e., bounded polyhedron.

We conclude that Theorem 4 provides a characterization of the non-emptiness of the core also in cases where thew sets $\mathbb{R}^k_+ \setminus \mathbf{v}(T)$, $T \subsetneq N$ and $\mathbf{v}(N)$ are finite unions of convex sets.

Appendix A

For any two convex sets E, F in a Euclidean space, define

$$d(E, F) := \inf\{\|e - f\|;\ e \in E,\ f \in F\}.$$

Proof of Lemma 1. A and B are disjoint. Thus, in order to prove the lemma[17] it is sufficient to show that $d(A, B) > 0$ and then use a separation theorem (see for instance, Rockafellar, 1970) in order to find a separating matrix S between A and B. If, on the contrary $d(A, B) = 0$, then for every $i = 1, \dots, l$,

$$d(A_i, B_i) = 0, \tag{20}$$

where $B_i = \{x_i C;\ C \text{ is an } n \times k \text{ matrix}\}$. We show that Eq. (20) implies $A_i \cap B_i \neq \emptyset$. This would imply that A and B are not disjoint, which is a contradiction.

B_i is a subspace while (by assumption) A_i is a union of a convex and compact set, say D_i, and a polyhedron, say P_i. If $d(D_i, B_i) = 0$, then due to the compactness of D_i, $D_i \cap B_i \neq \emptyset$ and the proof is done. It must be then that $d(D_i, B_i) > 0$, while $d(P_i, B_i) = 0$.

By the Minkowski-Weyl Theorem (see Jünger et al., 2010), P_i, as a polyhedron, is the Minkowski sum of a convex hull of finitely many points,

[16] Note that here l is equal to $2^n - 1$, the number of coalitions in the game.

[17] Recall, we need to show a strict inequality in Eq. (14).

$$C = \mathrm{conv}\{c_1, ..., c_s\} = \left\{ \sum_{j=1}^{s} \alpha_j c_j;\ \alpha_j \geq 0,\ \sum_{j=1}^{s} \alpha_j = 1 \right\},$$

and a finitely generated cone,

$$V = \mathrm{cone}\{v_1, ..., v_t\} = \left\{ \sum_{j=1}^{t} \beta_j v_j;\ \beta_j \geq 0 \right\}.$$

In other words, $P_i = \{c + v;\ c \in C, v \in V\}$. Since $d(P_i, B_i) = d(C + V, B_i) = 0$, we obtain $d(C, B_i - V) = 0$. However, C is convex and compact. Furthermore, $B_i - V$ is convex and, as a finitely generated cone, closed. Hence, $C \cap (B_i - V) \neq \emptyset$. This, in turn, implies that $P_i \cap B_i = (C + V) \cap B_i \neq \emptyset$, which completes the proof that Eq. (20) implies $A_i \cap B_i \neq \emptyset$, as desired. $\square$

References

Assa, H., Elliston, S., Lehrer, E., 2016. Joint games and compatibility. Econ. Theory 61, 91–113.
Bloch, F., de Clippel, G., 2010. Cores of combined games. J. Econ. Theory 145, 2424–2434.
Bondareva, Olga N., 1963. Some applications of linear programming methods to the theory of cooperative games. Probl. Kibern. 10, 119–139 (in Russian).
Fernandez, F.R., Hinojosa, M.A., Puerto, J., 2004. Set-valued TU-games. Eur. J. Oper. Res. 159, 181–195.
Gayer, G., Persitz, D., 2016. Negotiation across multiple issues. Theor. Econ. 11, 937–969.
Jünger, M., Liebling, T.M., Naddef, D., 2010. 50 Years of Integer Programming 1958-2008: From the Early Years to the State-of-the-Art. Springer, Heidelberg.
Lehrer, E., Teper, R., 2018. Set-valued capacities: multi-agenda decision making. Econ. Theory, 1–16.
Lind, M., 1996. Cooperative Game Theory and Multiple Criteria Decision Making. Ph.D Thesis, working paper No. 6. Aarhus University.
Predtetchinski, A., Herings, P.J.J., 2004. A necessary and sufficient condition for non-emptiness of the core of a non-transferable utility game. J. Econ. Theory 116, 84–92.
Rockafellar, R.T., 1970. Convex Analysis. Princeton Math. Series, vol. 28. Princeton Univ. Press.
Shapley, L.S., 1967. On balanced sets and cores. Nav. Res. Logist. Q. 14, 453–460.
van den Nouweland, A., Aarts, H., Borm, P., 1989. Multi-commodity games. Methods Oper. Res. 63, 329–338.