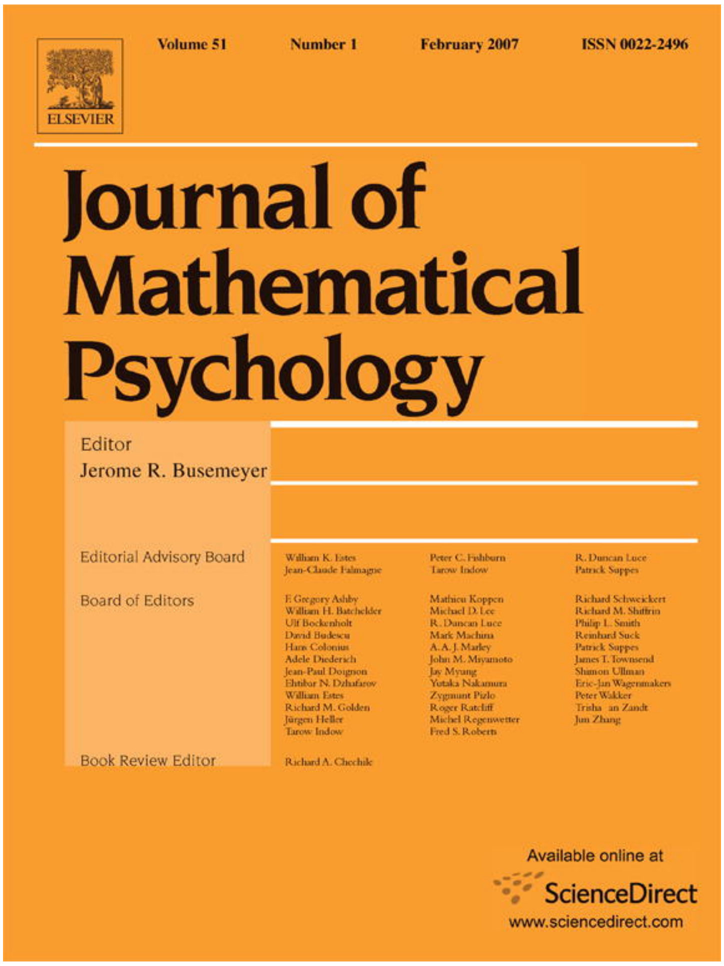
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Categorization generated by extended prototypes—An axiomatic approach

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Abstract

We suggest a model of categorization based on prototypes. A set of entities, identified with some finite dimensional Euclidian space, is partitioned into a finite number of categories. Such a categorization is said to be *generated by extended prototypes* if there is a set of distinguished entities, one for each category, such that the categorization is determined by proximity to these prototypical entities. The prototypes are called extended since they are described by points in a higher dimensional space than the entities. Sufficient conditions for a categorization to be generated by extended prototypes are provided. These conditions are also necessary if the prototypes are in general position.

Keywords: Categorization; Prototypes; Conceptual spaces; Prototype-oriented decision making

1. Introduction

Categorization is an act by which we sort things out (mentally). Some entities are members of a particular category, others are not. For instance, anchovy, bull shark and flying fish belong to the category "fish".

Categorization is one of the most fundamental tasks human beings engage in. Whenever we use the word "dog" to refer to two different animals, we actually perform an act of categorization. Categories are sets of entities to which we react in an identical or a similar way. For instance, we smile at all their members, or we call all their members by the same name. Classifying the world around us into categories is an efficient way to store and to access quickly a great deal of information using minimal resources. Indeed, if a creature belongs to the category "dog", we know that it probably has a tail, barks, and if you annoy it, may bite.

The classical perspective of categorization is that items are classified into their proper categories on the basis of features. Every category is characterized by a list of features. The entities belonging to a particular category are those, and only those, having all the appropriate features. For instance, mammals are creatures that (1) give birth, and (2) suckle their young. Therefore, a cat belongs to the category "mammal". On the other hand, sea perches of the Pacific coast give birth to live young, but do not suckle them and, thus, they are not considered mammals. A famous line that perfectly reflects this theory is:

> What kind of bird are you, if you cannot fly, said the little bird to the duck. What kind of bird are you, if you cannot swim, said the duck and dived (Sergei Prokofiev, Peter and the Wolf).

Wittgenstein (1953) disagreed with this perspective. He examined the category of "game" and claimed that there is nothing common to board games, card games, ball games and Olympic games.

The work of Rosch (see, Rosch & Lloyd, 1978) challenged the classical theory. She demonstrated that when people label an object, they rely less on abstract definitions than on a comparison with what they regard as the best representative of the category designated by that word. In a series of experiments, Rosch and her colleagues showed that people could not tell what features they rely

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on when performing the act of categorizing. Furthermore, people regarded members of the same category differently: some being considered more representative of a category than others. This means that the membership of an entity in a category is to a matter of degree. In particular, Rosch found that people categorize more typical members faster than less typical ones.

Rosch concluded that people do not perform a categorization on the basis of features. They categorize on the basis of proximity to a prototypical or ideal member of a category. An anchovy is closer to the fish prototype than a Pacific coast sea perch, but both are closer to it than they are to the mammal prototype. Therefore, people refer to them both as fish.

Following Rosch, a rich literature on prototypes has developed and many experiments to check the validity of the theory have been conducted. Tverski and Gati (1978) showed asymmetry in similarity ratings between prototypical and non-prototypical members of a category. Rips (1975) found that new information about prototypical members of a category is more likely to be generalized to the entire category than if the same information had been provided about non-prototypical members. Other examples can be found in Lakoff (1987).

Nevertheless, there is firm opposition to the prototypes theory. Osherson and Smith (1981), for instance, use fuzzy logic tools to claim that the prototypes theory can lead to ridiculous conclusions. Several alternative explanations to the experimental findings had been suggested. Armstrong et al. (1983) proposed a hybrid theory that combines the classical theory of categories with a theory of identification procedures which explains the prototypical effects.

Whether people categorize on the basis of proximity to prototypes or not is still subject to debate. The main task of this paper is to find conditions which characterize those categorizations that are generated by prototypes. Such conditions may be useful in experimental examinations of the validity of the prototype model.

To formulate the notion of categorization we use a geometric model of partitions of a finite dimensional Euclidian space. The set of *entities* to be categorized is identified with $\mathbb{R}^d$, where $d \in \mathbb{N}$ $(d \geqslant 2)$ is the number of *attributes* used in describing an entity. Every entity is defined by the *intensity* of each attribute, which is a real number. For instance, different people may be described by different intensity levels of the attributes "speaks Swahili", "likes red wine" and "dresses elegantly". A categorization (with respect to a given finite set of categories) is a partition of the set of entities ($\mathbb{R}^d$) into pairwise disjoint sets, one for each category.[1]

We study a special kind of such categorizations—those that are *generated by extended prototypes*. An extended prototype is a distinguished member of a category. The term "extended prototypes" is used to emphasize that these members are described by $d+1$ attributes, and not by d attributes like the rest of the entities. Formally, an extended prototype is a point in $\mathbb{R}^{d+1}$ (and not in $\mathbb{R}^d$). We say that a categorization is generated by extended prototypes if the closest extended prototype to a member of a certain category is the designated prototype of that category.[2] Thus, categorizations of this form are easy to describe: every entity is assigned to the category that corresponds to its closest extended prototype.

The representation of a categorization as a partition of a geometrical structure is thoroughly discussed in Gardenfors (2000). There, a categorization is an assignment of points in a "conceptual space"[3] to a set of categories. Gardenfors (2000, pp. 138–141) studies categorizations generated by extended prototypes (though he calls them "generalized Voronoi categorizations"[4]). The extra dimension of the prototypes might be used, for instance, to control the size of the various categories. This allows, in turn, to take into account the variability within categories: more diverse categories (such as "duck") will cover a larger area than less diverse ones (such as "ostrich").[5]

In order to find necessary and sufficient conditions for a categorization to be generated by extended prototypes, we consider the categorization of the same individual in various situations. If L is the (finite) set of all relevant categories then the primitive of our model is a collection of categorizations, one for each subset $A \subseteq L$. That is, the categorizer is asked to perform his categorization when only the categories in A are available, for every $A \subseteq L$. The resulting family of partitions is called *a categorization system*.

We introduce four axioms of categorization systems, namely, *Hierarchical Consistency*, *Convexity*, *Non-redundancy* and *Variety*. Our main result states that if a categorization system satisfies these four axioms then it is generated by extended prototypes. Moreover, every categorization system which is generated by extended prototypes satisfies *Hierarchical Consistency* and *Convexity*. The other two axioms are not necessary, but they are both satisfied when the extended prototypes are in general position. The exact definition of the term "general position" appears in Section 2.3, and we comment further on this issue in Section 3.4.

As shown in Section 6.2, the four axioms do not guarantee that a categorization system is generated by non-extended prototypes (these are prototypes described by only d attributes). Thus, an additional attribute is necessary. What characterizes categorization systems that are generated by non-extended prototypes is yet to be discovered.

[1]For technical reasons, we will assume particular restrictions on the structure of the partitions.

[2]Example 1 in Section 2.3 illustrates this definition.

[3]From a mathematical point of view, a "conceptual space" is a metric space.

[4]The geometrical object called Voronoi diagram is discussed in Section 5.3.

[5]See Section 3.1 for more on this issue.

The paper is organized as follows. Section 2 describes the model, the axioms and the main result. In Section 3 we elaborate on several aspects of the model and the result. Section 4 describes how the main result can be applied to experimental psychology and to the theory of decision making. Section 5 surveys related literature and Section 6 provides several concrete examples of categorization systems. We conclude in Section 7 where the proof of the main result is presented.

2. The model and the main result

2.1. Categorization system

Any entity is characterized by the intensity of d attributes. Thus, an entity is represented by a vector in $\mathbb{R}^d$. An *open partition* of $\mathbb{R}^d$ is a finite collection of non-empty, pairwise disjoint open sets, say, $A_1, A_2, \ldots, A_n$ such that[6] $\mathrm{cl}(\bigcup_{i=1}^n A_i) = \mathbb{R}^d$.

Consider a set of categories $L = \{1, 2, \ldots, \ell\}$. An agent is asked to classify the set of entities. When a subset $A \subseteq L$ is considered, the agent divides the set of entities among the categories of A. That is, for any $A \subseteq L$ (with $|A| \geqslant 2$), there is an open partition of $\mathbb{R}^d$, denoted by P_A, that consists of $|A|$ sets, one for each category in A. Such a system of partitions is called a *categorization system* and is a primitive of our model. Formally,

Definition 1. 1. A *categorization system* is a collection of open partitions $P_A = \{P_A(i)\}_{i \in A}$ of $\mathbb{R}^d$, one associated with each $A \subseteq L$ with $|A| \geqslant 2$.

2. When $x \in P_A(i)$ we say that x is *categorized as i when A is considered.*

2.2. Axioms

This subsection describes four axioms of categorization systems. The first two properties seem to be natural from a behavioral point of view. The other two are of a technical nature and their aim is to prevent degenerate cases.

Convexity: For every $A \subseteq L$ and for each $i \in A$, $P_A(i)$ is a convex set.

Convexity states that if two entities y and y' are categorized as i when A is considered, then so is any convex combination of y and y'.

Hierarchical Consistency: For every $A \subseteq L$ with $|A| \geqslant 3$, and for each $i \in A$, $P_A(i) = \bigcap_{B \subsetneq A, i \in B} P_B(i)$.

Hierarchical Consistency states that the entities categorized as i when A is considered are those entities categorized as i when B is considered for every $B \subsetneq A$. In Lemma 1 we prove that if a categorization system satisfies *Hierarchical Consistency* then all the partitions $\{P_A\}_{A \subseteq L}$ are uniquely determined by the partitions when only pairs of categories are considered.

[6] $\mathrm{cl}(B)$ denotes the closure of B.

Non-Redundancy: For every three categories $\{i, j, k\} \in L$, $P_{\{i,j\}}(i) \nsubseteq P_{\{i,k\}}(i)$.

Non-Redundancy states that there is always an entity categorized as i, when the two categories i, j are considered, but categorized as k when i, k are considered.

Variety: For every four distinct categories $\{i, j, k, m\} \in L$,

$$\mathrm{cl}(P_{\{i,j,k\}}(i)) \cap \mathrm{cl}(P_{\{i,j,k\}}(j)) \cap \mathrm{cl}(P_{\{i,j,k\}}(k)) \neq \mathrm{cl}(P_{\{m,j,k\}}(m)) \cap \mathrm{cl}(P_{\{m,j,k\}}(j)) \cap \mathrm{cl}(P_{\{m,j,k\}}(k)).$$

The set $\mathrm{cl}(P_{\{i,j,k\}}(i)) \cap \mathrm{cl}(P_{\{i,j,k\}}(j)) \cap \mathrm{cl}(P_{\{i,j,k\}}(k))$ consists of all entities that belong to (the closure of) the three categories $\{i, j, k\}$. *Variety* states that this set and the set of entities which are in the (closure of the) three categories $\{m, j, k\}$ are not the same.

2.3. Axiomatization

Definition 2. A categorization system $\{P_A\}_{A \subseteq L}$ is *generated by extended prototypes*, if there are ℓ points $x_1, x_2, \ldots, x_\ell$ in $\mathbb{R}^{d+1}$, such that for any $A \subseteq L$ and any $i \in A$, $P_A(i) = \{y \in \mathbb{R}^d : \mathbf{d}_i(y) < \mathbf{d}_j(y)$ for every $j \in A,\ j \neq i\}$, where $\mathbf{d}_i(y) = \|(y, 0) - x_i\|^2$ $(i = 1, \ldots, \ell)$ and $(y, 0)$ is the vector in $\mathbb{R}^{d+1}$ whose first d coordinates coincide with y and the last coincides with 0.

When a categorization system is generated by extended prototypes every category i has an extended prototype, $x_i \in \mathbb{R}^{d+1}$ (i.e., an entity characterized by $d+1$ attributes). When the subset $A \subseteq L$ of categories is considered, the open partition P_A of $\mathbb{R}^d$ is determined by the distance functions $\mathbf{d}_j, j \in A$. That is, an entity $y \in \mathbb{R}^d$ is in the set $P_A(i)$ if, and only if, the (Euclidean) distance $\mathbf{d}_i(y)$ between $(y, 0)$ and the extended prototype x_i is strictly smaller than the distance $\mathbf{d}_j(y)$ between $(y, 0)$ and the extended prototype x_j, for every $j \in A$ other than i. The following example illustrates Definition 2.

Example 1. Let $L = \{1, 2, 3\}$ be the set of categories, and assume that $d = 2$. Let the three extended prototypes be $x_1 = (0, 0, 0)$, $x_2 = (2, 0, 0)$ and $x_3 = (0, 2, 1)$. Thus, the intensity in the extra attribute of the first two prototypes is equal to zero, while it is equal to one for the third prototype. Consider the categorization system generated by these prototypes. When the set of categories considered is $A = \{1, 2\}$ the induced partition is $P_A(1) = \{(y_1, y_2) \in \mathbb{R}^2; y_1 < 1\}$ and $P_A(2) = \{(y_1, y_2) \in \mathbb{R}^2; y_1 > 1\}$. When $A = \{1, 3\}$ is considered we get[7] $P_A(1) = \{(y_1, y_2) \in \mathbb{R}^2; y_2 < 1.25\}$ and $P_A(3) = \{(y_1, y_2) \in \mathbb{R}^2; y_2 > 1.25\}$. Similarly, for $A = \{2, 3\}$ a simple computation gives $P_A(2) = \{(y_1, y_2) \in \mathbb{R}^2; y_2 < y_1 + 0.25\}$ and $P_A(3) = \{(y_1, y_2) \in \mathbb{R}^2; y_2 > y_1 + 0.25\}$. When all three categories are considered the partition is $P_L(1) = \{(y_1, y_2) \in \mathbb{R}^2; y_1 < 1, y_2 < 1.25\}$, $P_L(2) = \{(y_1, y_2) \in \mathbb{R}^2; y_1 > 1, y_2 < y_1 + 0.25\}$, and $P_L(3) = \{(y_1, y_2) \in \mathbb{R}^2; y_2 > 1.25, y_2 > y_1 + 0.25\}$.

[7] Indeed, assume that $y = (y_1, y_2) \in \mathbb{R}^2$ is closer to the extended prototype x_1 than to x_3. This means that $y_1^2 + y_2^2 + 0^2 < y_1^2 + (y_2 - 2)^2 + 1^2$, or equivalently $y_2 < 1.25$.

For the sake of exercise, assume that, instead of x_3, the extended prototype of the third category would have been $x'_3 = (0, 2, 0)$. In this case, the partition when all 3 categories are considered would become $P_L(1) = \{(y_1, y_2) \in \mathbb{R}^2; y_1 < 1, y_2 < 1\}$, $P_L(2) = \{(y_1, y_2) \in \mathbb{R}^2; y_1 > 1, y_2 < y_1\}$, and $P_L(3) = \{(y_1, y_2) \in \mathbb{R}^2; y_2 > 1, y_2 > y_1\}$. This illustrates the fact that as the intensity of the additional dimension of a prototype increases, the corresponding category contracts.

As we show in Section 6.2, the additional dimension is crucial for our main result. Its meaning is discussed in Section 3.1. Since we measure distance between extended prototypes (which are points in $\mathbb{R}^{d+1}$) and entities (which only have d coordinates), we need to embed $\mathbb{R}^d$ in $\mathbb{R}^{d+1}$. From a technical point of view, it is convenient to identify the set of entities $\mathbb{R}^d$ with the set of vectors in $\mathbb{R}^{d+1}$ whose last coordinate is 0. However, one can replace 0 with any other real number without affecting the results.

Before stating the main result we need one more definition.

Definition 3. Let $x_1, x_2, \ldots, x_\ell$ in $\mathbb{R}^{d+1}$ be a set of extended prototypes. Denote by x'_i $(i = 1, \ldots, \ell)$ the orthogonal projection of x_i on the set of entities.[8] We say that $x_1, x_2, \ldots, x_\ell$ are in *general position* if the following two conditions hold:

(i) x'_i, x'_j, x'_k are not collinear for every three extended prototypes x_i, x_j and x_k.
(ii) For every four extended prototypes x_i, x_j, x_k and x_m the dimension of the set of entities $\{y \in \mathbb{R}^d : \mathbf{d}_i(y) = \mathbf{d}_j(y) = \mathbf{d}_k(y) = \mathbf{d}_m(y)\}$ is at most $d - 3$.

Informally speaking, the extended prototypes are in general position if no "unlikely coincidences" happen in their configuration. In Section 3.4 we elaborate further on this issue, and explain the interrelations between the axioms of *Non-Redundancy* and *Variety* and sets of extended prototypes which are in general position.

We use the four axioms described in the previous subsection in order to axiomatize the categorization systems that are generated by extended prototypes. Our main result is:

Theorem 1.

1. *If* $\{P_A\}_{A \subseteq L}$ *is a categorization system generated by extended prototypes, then it satisfies* Convexity *and* Hierarchical Consistency. *If the extended prototypes are in general position then* Non-Redundancy *and* Variety *are also satisfied.*
2. *If a categorization system* $\{P_A\}_{A \subseteq L}$ *satisfies* Convexity, Hierarchical Consistency, Non-Redundancy *and* Variety, *then it is generated by extended prototypes.*

[8] x'_i is the vector which equal to x_i in its first d coordinates and its last coordinate is equal to 0.

3. Discussion

3.1. Extended and non-extended prototypes

Every entity in our model is described by the intensity of d attributes, while the prototype of each category has an additional attribute. We propose two possible cognitive interpretations for the extra dimension. The first is based on the fact that a categorizer may not have an entire description of the object he categorizes. This may happen, for instance, when the categorization is done in a matter of an instant and under risk. In such circumstances, the attributes available to the categorizer might provide merely a partial picture of an entity. A complete description of an entity requires more attributes, unobserved by the categorizer.

While obtaining just a partial description of the observed entity, the categorizer may have in his mind a more comprehensive description of prototypes. This description may include the intensity of more than the d attributes observed. According to this interpretation, the additional dimension is a completion of the information that the categorizer has on the prototypes, but not on the entities to be categorized.

Such an interpretation immediately raises another question. If the categorizer has a finer description of the prototypes than of the entities then why is only one additional attribute sufficient? It might be that the description of the prototypes contains many attributes which are unavailable for the entities. The answer to this question is that, from a mathematical point of view, *it does not matter whether the prototypes have one or many additional attributes.* That is, any categorization system which is generated by extended prototypes having $d + n$ attributes (for some $n \in \mathbb{N}$) is also generated by extended prototypes having $d + 1$ attributes. Thus, considering only one additional dimension does not restrict the generality of our result. The proof of this statement is simple and is, therefore, omitted.

The second interpretation is adopted from Gardenfors (2000, pp. 138–140). The idea is that the additional attribute of a prototype reflects the size of the category it represents. He writes:

> A drawback of the standard Voronoi tessellation is that it is only the prototype that determines the partitioning of the conceptual space. It is quite clear that for many natural categorizing systems, however, some concepts correspond to "larger" regions than others. For example, the concept "duck" covers a much larger variety of birds than "ostrich"... The question then arises whether there is some way of generalizing the Voronoi tessellation that can account for varying sizes of concepts in a categorization...

As a solution of this problem Gardenfors (2000) suggests to add to each prototype an extra dimension which corresponds to the size of the category it represents.

Different values attached to this dimension would induce different partitions. This extra dimension controls the size of the various categories. For instance, the "duck" category can be designed to be relatively large while that corresponding to "ostrich," relatively small.

By adding an extra dimension, prototypes become extended prototypes. Gardenfors (2000, p. 39) suggests that, when the prototypes are learned from a set of exemplars, the size of a category might be defined by a function of the standard deviation of this set. We comment further on this interpretation in Section 4.1.

An interesting question is what characterizes those categorization systems that are generated by (non-extended) prototypes. By non-extended prototype we mean an entity in $\mathbb{R}^d$ and not in $\mathbb{R}^{d+1}$. We say that a categorization system is generated by (non-extended) prototypes if it is generated by extended prototypes, and in addition, the extended prototypes can be chosen such that the last coordinate in all of them is 0. Obviously, this is a subset of the categorization systems that are generated by extended prototypes. We do not know what additional axioms are needed in order to get a representation by (non-extended) prototypes, and we leave this question open.

3.2. Measuring similarity

Even when there is a prototypical representative for each category, in order to determine the category attribution, one should specify the meaning of closeness. What does it mean that a bat is more similar to a cow than to a hoopoe? The metric by which closeness is measured determines the partition into different categories.

The psychology literature shows (see Gardenfors, 2000, pp. 24–26 for a short review) that the Euclidean metric, which is the one used in this paper, is most suitable when the various dimensions (attributes) are *integral*. Integral dimensions, as opposed to *separable* dimensions, are interdependent and have the property that we assign values to all of them simultaneously. For instance, it is commonly accepted (Gardenfors, 2000, pp. 9–12) that a human being's representation of color can be described by three dimensions: hue, chromaticness and brightness. When we see a certain color, we simultaneously assign values to all three dimensions. These three dimensions are, therefore, integral.

Upon fixing a metric by which the distance between entities is measured, it is natural to define similarity between entities by a monotonic decreasing function of the distance. That is, for every two entities $x, y \in \mathbb{R}^d$, the similarity between x and y is $s(x,y) = f(\|x-y\|)$, where $f : [0,\infty) \to \mathbb{R}$ is monotonically decreasing. Usually, f is taken to be some exponentially decaying function. It should be noted that the choice of f cannot affect the induced partition (as long as f is monotonically decreasing).

With the same set of prototypes, different metrics would induce different partitions. It would be interesting to find conditions that axiomatize categorization systems that are generated by prototypes (or extended prototypes), using other metrics than the Euclidean. In this regard, we note that the fact that categories are convex sets (Part 1 of Theorem 1) is quite unique to the Euclidian metric. Other metrics typically induce non-convex categories. For instance, the city-block[9] metric induces star-shaped[10] (but not necessarily convex) categories (see Boots et al., 1992, p. 187).

3.3. The hierarchical consistency axiom

We demonstrate the role of *Hierarchical Consistency* with the aid of a simple example. Suppose that an individual has in mind prototypes of Spanish, French and Italian. Thus, according to her categorization, a member of the "French" category would be closer to, or better match, the French prototype than the other two. If asked which of two categories an entity belongs to, say, Spanish or French, a member of the "French" category would be closer to the French prototype than to the Spanish one. However, a member of the French category, when only French and Spanish are considered, may be categorized as Italian, if only French and Italian are considered, or when all three are considered.

When the categorization is based on prototypes (or extended prototypes), those people who are categorized as French, rather than Spanish or Italian, are precisely those who were categorized as French when examined separately versus Spanish or Italian. More generally, this means that if a categorization system is generated by extended prototypes then it must satisfy *Hierarchical Consistency*. See also Section 4.1 for an explanation of how this axiom can be tested and Section 4.2 for its relation to the Independence of Irrelevant Alternatives axiom (IIA).

3.4. Non-Redundancy, Variety and degenerate prototypes

As shown in Section 6.3, *Convexity* and *Hierarchical Consistency* alone do not guarantee that a categorization system is generated by extended prototypes. The cause of this insufficiency rises from the degenerate structure that some categorization systems which satisfy *Convexity* and *Hierarchical Consistency* have. The purpose of the axioms of *Non-Redundancy* and *Variety* is to exclude such degenerate cases.

Even if a categorization system is generated by extended prototypes, it can still have anomalistic structure. This happens when the prototypes are not in *general position*. The precise meaning of the term general position is not fully standard and depends on the context in which it is used. The key property of general position configurations is

[9]According to the city-block metric, the distance between $x, y \in \mathbb{R}^d$ is $\sum_{i=1}^{d} |x_i - y_i|$. This metric is known also as ℓ_1.

[10]A set D in a linear space is star-shaped if there is a point $p \in D$ such that the line segment between p and any other point of D lies within D.

that they lie arbitrarily close to any given configuration. This property is satisfied by our definition of general position. To be precise, assume that a set of extended prototypes $\{x_1, \dots, x_\ell\} \subseteq \mathbb{R}^{d+1}$ is given. We claim that, for every $\varepsilon > 0$, there is a set of extended prototypes $\{y_1, \dots, y_\ell\} \subseteq \mathbb{R}^{d+1}$ such that $\|x_i - y_i\| < \varepsilon$ for $i = 1, 2, \dots, \ell$ and $\{y_1, \dots, y_\ell\}$ is in general position according to our definition.

To see that this is indeed so, one should look at the examples in Sections 6.4 and 6.5. There, the two typical cases where the prototypes are *not* in general position are shown. Since the dimension of the set of entities in these examples is 2 it is easy to see what can go wrong. In Section 6.4, the three prototypes are on the same line, while in Section 6.5 the 4 prototypes are on the same sphere with center at the point $(1, 1)$. It is clear in these cases that by arbitrary small perturbations of the given prototypes one can obtain prototypes in general position.

A formal proof of the above claim involves standard geometrical arguments. We choose not to go into more details. The reader is referred to Matoušek (2002, pp. 3–5), where the concept of general position is discussed in greater detail.

In summary, categorization systems which does not satisfy the axioms of *Non-Redundancy* and *Variety*, and sets of extended prototypes which are not in general position are the two sides of the same coin. When a categorization system is generated by extended prototypes, *Non-Redundancy* and *Variety* will be satisfied if and only if the generating prototypes are in general position. Since sets of prototypes which are not in general position are "rare", this means that "most" of the categorization systems which are generated by extended prototypes satisfy *Non-Redundancy* and *Variety*.

3.5. Context-dependent prototypes

There is evidence in the psychology literature (see e.g., Labov, 1973) that the process of categorization is context-dependent. That is, people categorize differently the same set of entities when the categorization is done in different circumstances. If the categorization is based on similarity to prototypical cases, then it might be that the cause for such a phenomenon is that *the prototypes are context dependent*.

Our model assumes that the prototype of a category is not influenced by the set of categories considered. That is, if x_i is the prototype of a category i when the set of categories considered is A $(i \in A)$, then it is also the prototype of i when the set of categories considered is B $(B \neq A, i \in B)$.

Consider the following example: A school has basketball and volleyball teams. A student is included in the basketball team if he is closer to some prototypical basketball player than to a prototype of a volleyball player, and vice versa. If the same school had to classify the students into basketball and soccer teams, the prototypical players would probably be different.

For instance, the speed of the prototypical basketball player in the first categorization (basketball vs. volleyball) would be higher than in the second (basketball vs. soccer), because speed is important in basketball and in soccer, and less so in volleyball. On the other hand, height would be a significant attribute of the prototypical basketball player in the second categorization, because height is an advantageous attribute in both basketball and in volleyball, and less so in soccer.

This phenomenon can result in a violation of the *Hierarchical Consistency* axiom. Indeed, it may happen that a student is assigned to the basketball team when only basketball and volleyball are considered, but when all three sports are considered, he is assigned to the volleyball team. Thus, if one tries to check the validity of the prototypes theory through our axioms then context-related effects must be removed.

3.6. Prototypical sets

Our main result axiomatizes categorization systems generated by extended prototypes. Every category is represented by one prototype. However, there may be cases where categories are defined by a closeness relation to one of a few typical representatives of a category. For instance, the category "French" may be defined by a proximity to either Charles de Gaulle, Brigitte Bardot or Gerard Depardieu. In such a case the category is generated by a set of prototypes rather than by one. Formally:

Definition 4. An open partition $P = (P(i))_{i \in L}$ is *generated by finite sets of prototypes*, if there are ℓ finite sets $B_1, B_2, \dots, B_\ell$ in $\mathbb{R}^d$, such that for any $i \in L$, $P(i) = \{y \in \mathbb{R}^d : f_i(y) < f_j(y)$ for every $j \in L, j \neq i\}$, where $f_i(y) = \min_{z \in B_i} \|y - z\|^2$ $(i = 1, \dots, \ell)$.

When the open partition is generated by finite sets of prototypes the resulting cells of the partition may not be convex sets. Thus, the convexity axiom may be violated if the categorizer has in mind several prototypes for each category. This might be the case if the prototypes are learned from a set of exemplars, as illustrated in the following example.

A board of managers examines a few candidates. It may have a few prototypical examples of what constitutes a good CEO and some others that are prototypical examples of a bad CEO. These prototypes may have been collected from the list of the firm's ex-CEOs, judged according to their performance. Candidates are sorted according to their resemblance to one of the prototypical individuals. It may happen that two individuals categorized as potentially good CEOs resemble different prototypes: one resembles a former CEO in her assertiveness and the other resembles another former CEO in his business creativity. A combination of these two may resemble a former bad CEO. In other words, the category of potentially good CEOs is not necessarily convex.

It is clear that an open partition that is generated by finite sets of prototypes consists of categories which are finite unions of convex sets. Further investigation of this subject is beyond the scope of this paper.

3.7. Other domains of entities

In this discussion we restricted our attention to the case where the set of entities is a finite dimensional Euclidian space. That is, an entity is defined by its intensity in any attribute, and the intensity is not bounded in both directions. However, there are cases where other domains of entities seem more appropriate.

One example for such a case is when the intensity of one or a few dimensions is bounded. It is reasonable, for instance, that the attribute that stands for the age of a person would be bounded. In other words, the age dimension should be described by a finite interval rather than by an infinite line. Another example is when a certain attribute is discrete. For instance, if a relevant dimension is the number of legs that each entity has, then the possible values are only integers.

Most of our results still hold if the domain of entities $\mathbb{R}^d$ is replaced by some other convex subset of a finite dimensional Euclidian space.[11] Specifically, the second part of Theorem 1 is not affected by such a change of domain. In addition, *Convexity* and *Hierarchical Consistency* will be satisfied whenever the categorization system is generated by extended prototypes. However, in some domains, the fact that the prototypes are in general position does not insure that *Non-Redundancy* and *Variety* will be satisfied.

3.8. Unassigned entities

If $\{P_A(i)\}_{i\in A}$ is an open partition of $\mathbb{R}^d$ then the union $\bigcup_{i\in A} P_A(i)$ does not cover the entire space. Indeed, the boundary of any one of these disjoint open sets is not contained in their union. Thus, there are entities which remain unassigned in the categorization process. This is also reflected in Definition 2, where an entity $y \in \mathbb{R}^d$ which is equidistant from two extended prototypes x_i and x_j is not included in $P_{\{i,j\}}(i)$ nor in $P_{\{i,j\}}(j)$. However, we believe that this issue is of minor significance because the set of unassigned entities is very "small". By this we mean that both, the Lebesgue measure of this set is 0, and that it is nowhere dense (its closure has an empty interior).

4. Applications

4.1. Experimental psychology

The purpose of this subsection is to illustrate how Theorem 1 can be applied to check the validity of the prototypes theory in general, and of the geometric model described here specifically. We emphasize that there is already a rich literature describing experimental study of issues related to prototypes-based categorization (Rosch & Lloyd, 1978 is one of the earliest). Moreover, Gardenfors and Holmqvist (1994, see also Gardenfors, 2000, pp. 142–150) conducted experiments in order to compare various geometric models of categorization. One of these models is the same as the one discussed in this paper. However, we claim that Theorem 1 may put such experiments in a new light.

A fundamental problem facing a researcher who wants to compare different theories of categorization is that people cannot tell what make them categorize one way or another. Thus, conclusions must be drawn based on the observable data alone, which is typically, just the categorization itself. In particular, one cannot expect that individuals who participate in an experiment would be able to tell whether or not their categorization was based on proximity to a set of prototypes.[12]

The approach that guided the experiments performed by Gardenfors and Holmqvist (1994) was based on the *assumption* that individuals use the set of exemplars of a category in a special way to create the prototype of that category. Specifically, they assumed that the prototype of a category is the average of the exemplars of that category. Based on this assumption, they check the predictions of the extended prototypes model[13] in a task of categorizing shell shapes.

But what if prototypes are generated in another way, and not specifically by averaging exemplars? For instance, it might be that exemplars recently encountered have a stronger effect than those encountered previously. Further, it might be that the standard deviation of the set of exemplars is not the appropriate assignment for the extra attribute of the prototype. Thus, it may be that the predictions of the model are inaccurate, not because the extended prototypes model fails, but rather because the parameters have been wrongly selected.

The axioms presented in Section 2.2 do not refer to any specific set of prototypes. Therefore, Theorem 1 can be used in order to determine whether the model of extended prototypes is accurate with *some* set of prototypes. The experiment should be designed in a way that allows us to check whether the axioms are satisfied. Obviously, the main interest is in the *Convexity* and *Hierarchical Consistency* axioms.

Checking whether *Hierarchical Consistency* is satisfied seems to be a rather simple matter. Consider, for instance, the experiment of categorizing shell shapes, as in Gardenfors and Holmqvist (1994). After the learning stage of the various categories (say, categories A–C) is over, individuals

[11] Notice that this excludes the possibility of discrete attributes.

[12] Though, individuals may be able to tell which members of a certain category are more prototypical than others (see Rosch & Lloyd, 1978).

[13] The intensity of the extra attribute was taken to be the standard deviation of the exemplars of the category (see also Section 3.1).

should be asked to categorize shells when only pairs of categories are allowed. That is, the shapes tested should first be partitioned into the categories A and B, among A and C, and among B and C (the same set of shells should be used in each partition). Then, the same set of shells should be categorized when all three categories are permitted. *Hierarchical Consistency* is satisfied if, and only if, the shells which were categorized as A (B, C) in the final stage are exactly those that were categorized as A (B, C) in the two relevant categorizations of the first stage.

Checking *Convexity*, on the other hand, is trickier. Checking whether the categorization of a subject satisfies this axiom requires knowledge of the representation of every instance as a point in the entities space. However, there are domains in which the representation of entities is known.[14] In such domains *Convexity* can be examined by asking subjects to categorize an instance located between 2 other instances that were identically categorized.

4.2. Decision theory

Although our main interest is in the cognitive process of categorization, Theorem 1 can also be applied to the theory of decision making. Let D be a decision problem characterized by ℓ attributes. For instance, an individual needs to make a decision whether to take an umbrella or not. The relevant attributes in this case may be the outside temperature, the color of the sky, the time he plans to stay outside and so on. The decision maker may have two prototypical decision problems in mind, one in which he should take an umbrella and another in which he should not. The decision is taken according to which of the two prototypes is closer to the current situation. If this is the case we say that the decision process is *prototype-oriented*.

Categorization of decision problems may be done according to the actions taken. That is, all problems that share the same best response are lumped together. The decision process should not be consciously prototype-oriented, but it might seem that way. In fact, this is a case where the categorization can be clearly seen by an outside observer. Our axiomatization, therefore, can be used to determine whether the categorization is based on proximity to prototypical decision problems.

In the context of decision making, the *Hierarchical Consistency* axiom is closely related to the IIA axiom. If $A = \{a_1, \ldots, a_\ell\}$ is the set of all actions available to the decision maker (categories) then (almost) any decision problem (entity) induces a choice function over A. Restricting attention to the choice function induced by a given entity, one can see that *Hierarchical Consistency* implies that this choice function satisfy IIA. The converse is also true. That is, if the choice function over actions induced by every decision problem satisfies IIA then the categorization of decision problems according to their best response satisfies *Hierarchical Consistency*.

[14] Colors and sounds are two examples.

We would also like to relate our model to the theory of decision making with uncertainty. Suppose that a decision problem is defined by a distribution over the state space, Ω. As before, A stands for the (finite) action set. Suppose that the utility function u specifies the utility the decision maker derives from taking the action a when the distribution over states is P. That is, $u : \Delta(\Omega) \times A \to \mathbb{R}$, where $\Delta(\Omega)$ is the set of distributions over Ω. In other words, $u(P, a)$ is the utility derived from taking the action a when a state is drawn from Ω according to the distribution P.

In this setup, decision problems are merely distributions over the state space, Ω. Thus, the (convex) set of entities is $\Delta(\Omega)$. Consider a categorization of distributions in $\Delta(\Omega)$ according to their best response. P is classified to the category corresponding to a if a is the best response to P (i.e., $u(P, a) > u(P, b)$ for any $b \in A \backslash \{a\}$). If for any $P, P' \in \Omega$ and $\alpha \in (0, 1)$, u satisfies

$$u(P, a) \geqslant u(P, b) \quad \text{and} \quad u(P', a) \geqslant u(P', b) \quad \text{imply} \quad u(\alpha P + (1-\alpha)P', a) \geqslant u(\alpha P + (1-\alpha)P', b),$$

then this categorization satisfies *Convexity*. In particular, if u is defined according to the expected utility, then the best-response categorization satisfies *Convexity*.

As a last remark, we note that the entire discussion in this subsection is restricted to decision problems with finitely many available actions. This is because our model assumes that the set of categories in a categorization system is finite. A generalization of our results to infinite action sets is left for future research.

5. Related literature

5.1. Economics

An extensive amount of effort has been invested in finding alternatives to the classical expected utility theory. This is due to cumulative evidence that the predictions of this theory are not always consistent with the actual decisions being made by individuals. Some alternative theories are strongly inspired by psychological concepts, and categorization has an important role in many of them.

Fryer and Jackson (2003) use a model of categorization to explain discrimination against minorities. They claim that discrimination against minorities in employment is a result of a cognitive process whereby majority groups are better sorted on the basis of qualifications than minority groups.

Mullainathan (2002) suggests an alternative to Bayesian updating of probabilities based on the idea of coarse categories. He claims that people tend to consider similar (but not equal) cases as the same (i.e., belonging to the same category). Based on category estimates people obtain biased probabilities. Furthermore, upon observing new data, beliefs about the actual state of the world are not updated in a continuous manner, as in the Bayesian case.

Gilboa and Schmeidler (2001) developed a case-based decision theory based on past experience. The action chosen when a new decision problem is encountered is the one that performed best in past problems, which are weighted according to their similarity to the decision problem under consideration.

In view of the case-based decision theory, prototype-oriented decision making, as described in Section 4.2, may be considered as follows: All past experience is encapsulated in some prototypical cases. That is, the decision maker replaces the set of past decision problems by a relatively small set of prototypical decision problems. When a new decision problem is encountered, the action taken is the best response to the closest prototypical case (and not to the decision problem actually encountered).

5.2. Scoring rules and linear representation of orders

It turns out that our results are also related to some papers on scoring rules by Smith (1973), Young (1975) and Myerson (1995), and to the literature dealing with linear representation of orders (see, Gilboa & Schmeidler, 2001 and Ashkenazi & Lehrer, 2002).[15] While the model, interpretation of the results and the proofs in this paper significantly differ from those mentioned, there are some surprising similarities.

A society needs to choose from a number of alternatives. Each individual in the society submits a ballot which reflects his opinion on the matter. A *voting rule* is a correspondence between the set of possible collections of ballots and the set of alternatives, which assigns the set of winning alternatives to every list of ballots submitted.

A voting rule is induced by a *scoring rule* if for every alternative i and for every possible ballot b there exists a real number $x(i,b)$, such that the winning alternatives are exactly those that maximize $\sum_b x(i,b)y(b)$, where $y(b)$ is the number of individuals who submitted the ballot b. Defining y (x_i) to be the vector with coordinates $y(b)$ ($x(i,b)$), the above sum is the scalar product $\langle x_i, y\rangle$. Smith (1973), Young (1975) and Myerson (1995) provide various axiomatizations that characterize voting rules that are induced by scoring rules.

Gilboa and Schmeidler (2001) and Ashkenazi and Lehrer (2002) deal with a model where each point y in a subset of an Euclidean space induces a total order over a set of alternatives, L. They axiomatically derive the existence of a linear representation. That is, vectors x_i, $i \in L$, that induce the order in the sense that an alternative i is preferred over alternative j, with respect to the order induced by y, if and only if $\langle x_i, y\rangle > \langle x_j, y\rangle$.

A categorization system clearly induces a binary relation $\succ_y$ over categories for (almost) every entity y: $i \succ_y j$ if, when $\{i,j\}$ is considered, y is categorized as i. It turns out that when a categorization system satisfies *Hierarchical Consistency* this binary relation is a total order (see also Section 4.2).

Furthermore, when the categorization is generated by extended prototypes this total order has a linear representation. Indeed, let $\{x_i\}_{i\in L} \subset \mathbb{R}^{d+1}$ be the set of prototypes. An entity $y \in \mathbb{R}^d$ is categorized as i if $\|(y,0) - x_i\|^2 < \|(y,0) - x_j\|^2$ for every $j \in L\backslash\{i\}$. Denote by $\hat{x}_i$ the vector in $\mathbb{R}^{d+1}$ which coincides with x_i on the first d coordinates and whose last coordinate equals $\frac{-\|x_i\|^2}{2}$. Note that $\|(y,0) - x_i\|^2 < \|(y,0) - x_j\|^2$ if and only if $\langle (y,1), \hat{x}_i\rangle > \langle (y,1), \hat{x}_j\rangle$. Therefore, the order (over categories) induced by y is the one induced by $(y,1)$ and the linear representation of $\{\hat{x}_i\}_{i\in L}$.

The differences between our axiomatization and those mentioned above are due to the differences in the models and the motivations. The primitive in Smith (1973) and in Gilboa and Schmeidler (2001) is a total order over the set of alternatives (categories) induced by any profile (entity). Young (1975) and Myerson (1995) assume that for every profile there is a subset of winning alternatives. The approach in our paper resembles the later since the model only indicates the category that an entity belongs to.

The axiomatizations of Gilboa and Schmeidler (2001) and Ashkenazi and Lehrer (2002) resort to the anti-symmetric orders over alternatives induced by entities. In contrast, our axiomatization, specifically *Non-Redundancy* and *Variety*, makes use of the equivalence relations between categories which can be expressed in terms of the primitives of the model.

5.3. Voronoi diagrams

Partitions of the space that are generated by a set of center points are well known as Voronoi diagrams or Dirichlet tessellations. This concept is fundamental to computational geometry (see Preparata & Shamos, 1985) and has applications in almost every field of science. To appreciate the variety of topics where this idea is used, see Boots et al. (1992).

In its simplest form, a Voronoi diagram is a partition of some Euclidean space into a finite number of sets. Given the set of center points (generators), every point in the space is assigned to its closest generator (in terms of the Euclidean distance). However, there are many different generalizations of this concept. One such generalization considers generators that are not necessarily points. These generators may be subsets of the space such as lines, arcs and circles. Other examples use metrics other than the Euclidean metric.

In this paper we use a generalization of the Voronoi diagram known as the power diagram (Aurenhammer, 1987) or sectional Dirichlet tessellation (Ash & Bolker, 1986). Every generator x_i has a "weight" w_i (this is the intensity of the prototype x_i along the additional dimension), and the distance between some point y and x_i is $\mathbf{d}_i(y) = \|y - x_i\|^2 + w_i^2$. This distance has a strong connection to Laguerre geometry and therefore the resulting

[15]We thank Itzhak Gilboa for pointing out the relation between scoring rules and categorizations generated by prototypes.

diagram is also referred to as a Laguerre diagram (Imai et al., 1985).

A relatively small amount of the literature is dedicated to the problem of characterizing and recognizing Voronoi diagrams. Ash and Bolker (1985) give a geometrical characterization of those partitions of the plane which are Voronoi diagrams. For algorithms that check whether or not a given partition is a Voronoi diagram the reader is referred to Evans and Jones (1987) and to Heath and Kasif (1993).

6. Examples

6.1. Categorization which is not generated by extended prototypes

Let the set of entities be $\mathbb{R}^2$ and consider a set of three categories $L = \{i, j, k\}$. Assume that the partitions, when only pairs of categories are considered, are as follows (we write only one of the cells in each partition; the other one is the interior of the complementary set):

$$P_{\{i,j\}}(i) = \{(y_1, y_2) \in \mathbb{R}^2; y_1 < 1\},$$
$$P_{\{i,k\}}(i) = \{(y_1, y_2) \in \mathbb{R}^2; y_2 < 1\},$$
$$P_{\{j,k\}}(j) = \{(y_1, y_2) \in \mathbb{R}^2; y_2 < 2 - y_1\}.$$

By part (1) of Theorem 1, if a categorization system is generated by prototypes then it must satisfy *Hierarchical Consistency*. Thus, when all three categories are considered, the entities categorized as, say i, are those that are categorized as i when both $\{i,j\}$ and $\{i,k\}$ are considered. It follows that when L is considered:

$$P_L(i) = \{(y_1, y_2) \in \mathbb{R}^2; y_1 < 1, y_2 < 1\},$$
$$P_L(j) = \{(y_1, y_2) \in \mathbb{R}^2; y_1 > 1, y_2 < 2 - y_1\},$$
$$P_L(k) = \{(y_1, y_2) \in \mathbb{R}^2; y_2 > 1, y_2 > 2 - y_1\}.$$

However, the resulting P_L is not an open partition of $\mathbb{R}^2$, since the closure of the union of these three sets is not $\mathbb{R}^2$ (see Fig. 1).

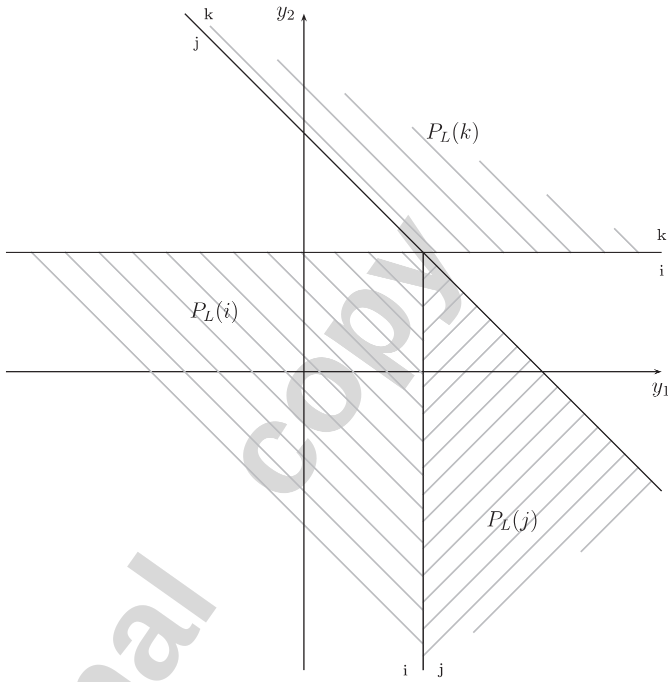

Fig. 1. Categorization which is not generated by extended prototypes.

6.2. The additional attribute

Consider the following partitions when pairs of categories are considered ($L = \{i, j, k, m\}$ and the set of entities is $\mathbb{R}^2$):

$$P_{\{i,j\}}(i) = \{(y_1, y_2) \in \mathbb{R}^2; 2y_2 > 4 - y_1\},$$
$$P_{\{i,k\}}(i) = \{(y_1, y_2) \in \mathbb{R}^2; y_2 > 1\},$$
$$P_{\{i,m\}}(i) = \{(y_1, y_2) \in \mathbb{R}^2; 2y_2 > y_1 - 1\},$$
$$P_{\{j,k\}}(j) = \{(y_1, y_2) \in \mathbb{R}^2; 2y_2 > y_1\},$$
$$P_{\{j,m\}}(j) = \{(y_1, y_2) \in \mathbb{R}^2; 2y_1 < 5\},$$
$$P_{\{k,m\}}(k) = \{(y_1, y_2) \in \mathbb{R}^2; 2y_2 < 5 - y_1\}.$$

When more than two categories are considered the partitions are determined by *Hierarchical Consistency*. One may verify that for any subset A of L the resulting P_A is indeed an open partition of $\mathbb{R}^2$. Moreover, this categorization system satisfies the four axioms of Section 2.2. Therefore, by the second part of Theorem 1, this categorization system is generated by extended prototypes.

However, this categorization system is not generated by non-extended prototypes. That is, if we restrict ourselves to the original two attributes without allowing for an additional attribute, this categorization system is not generated by prototypes. The reason is that the sum of the angles $\alpha + \beta$ (see Fig. 2) is less than π. It is shown in Ash and Bolker (1985, Corollary 10) that this cannot be the case when the partition is a Voronoi diagram. The reader is referred to Ash and Bolker (1985) for a proof of this point, as well as to a thorough discussion of the geometry of Voronoi diagrams in the plane.

6.3. Convexity and Hierarchical Consistency are insufficient

Consider a categorization system where ($L = \{i, j, k\}$ and the set of entities is $\mathbb{R}^2$):

$$P_{\{i,j\}}(i) = \{(y_1, y_2) \in \mathbb{R}^2; y_1 < 1\},$$
$$P_{\{i,k\}}(i) = \{(y_1, y_2) \in \mathbb{R}^2; y_1 < 1\},$$
$$P_{\{j,k\}}(j) = \{(y_1, y_2) \in \mathbb{R}^2; y_2 > y_1\}.$$

Assuming that the categorization system satisfies *Hierarchical Consistency*, the partition when the three categories are considered should be

$$P_L(i) = \{(y_1, y_2) \in \mathbb{R}^2; y_1 < 1\},$$
$$P_L(j) = \{(y_1, y_2) \in \mathbb{R}^2; y_1 > 1, y_2 > y_1\},$$
$$P_L(k) = \{(y_1, y_2) \in \mathbb{R}^2; y_1 > 1, y_2 < y_1\}.$$

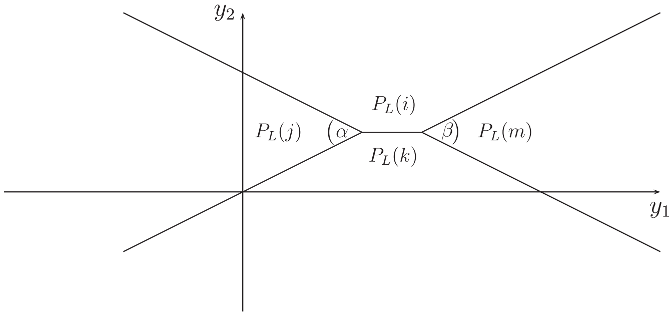

Fig. 2. The additional attribute is necessary.

It is clear that this categorization system satisfies *Convexity* and *Hierarchical Consistency*. However, *Non-Redundancy* is not satisfied since $P_{\{i,j\}}(i) \subseteq P_{\{i,k\}}(i)$.

This categorization system is not generated by extended prototypes. Suppose, on the contrary, that $x_i = (x_i^1, x_i^2, x_i^3)$, $x_j = (x_j^1, x_j^2, x_j^3)$ and $x_k = (x_k^1, x_k^2, x_k^3)$ are the extended prototypes of the categories i, j and k, respectively. Then, the lines $(x_i^1, x_i^2) - (x_j^1, x_j^2)$ and $(x_i^1, x_i^2) - (x_k^1, x_k^2)$ are perpendicular to the line $y_1 = 1$ (which is the border between $P_L(i)$ and $P_L(j)$ and the border between $P_L(i)$ and $P_L(k)$). However, the line $(x_j^1, x_j^2) - (x_k^1, x_k^2)$ is perpendicular to the line $y_1 = y_2$, which contradicts the previous conditions (see Fig. 3).

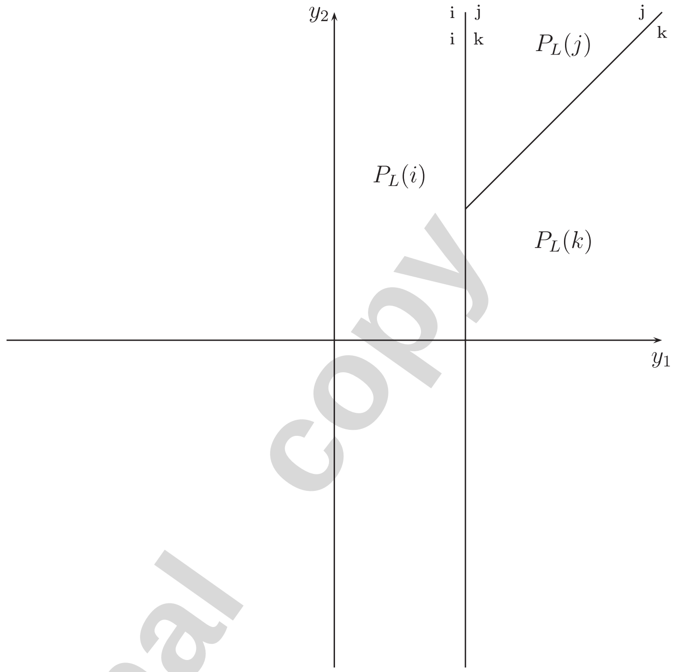

Fig. 3. Convexity and hierarchical consistency are insufficient.

6.4. Non-Redundancy is not necessary

Let $L = \{i, j, k\}$ and $d = 2$. Consider the categorization system generated by the extended prototypes $x_i = (0, 0, 0)$, $x_j = (2, 0, 0)$ and $x_k = (4, 0, 0)$. We claim that, although it is generated by extended prototypes, this categorization system does not satisfy *Non-Redundancy*. Indeed,

$$P_{\{i,j\}}(i) = \{(y_1, y_2) \in \mathbb{R}^2; y_1 < 1\},$$
$$P_{\{i,k\}}(i) = \{(y_1, y_2) \in \mathbb{R}^2; y_1 < 2\}.$$

So $P_{\{i,j\}}(i) \subseteq P_{\{i,k\}}(i)$. Notice that the reason for the failure of *Non-Redundancy* is that the three points x_i, x_j and x_k are on the same line, which means that they are not in general position.

6.5. Variety is not necessary

Let $L = \{i, j, k, m\}$ and $d = 2$. Consider the categorization system generated by the extended prototypes $x_i = (0, 0, 0)$, $x_j = (2, 0, 0)$, $x_k = (0, 2, 0)$ and $x_m = (2, 2, 0)$. We will show that this categorization system does not satisfy *Variety*.

When the triplet $A = \{i, j, k\}$ is considered the induced partition is

$$P_A(i) = \{(y_1, y_2) \in \mathbb{R}^2; y_1 < 1, y_2 < 1\},$$
$$P_A(j) = \{(y_1, y_2) \in \mathbb{R}^2; y_1 > 1, y_2 < y_1\},$$
$$P_A(k) = \{(y_1, y_2) \in \mathbb{R}^2; y_2 > 1, y_2 > y_1\}.$$

When $B = \{j, k, m\}$ is considered the induced partition is

$$P_B(j) = \{(y_1, y_2) \in \mathbb{R}^2; y_2 < 1, y_2 < y_1\},$$
$$P_B(k) = \{(y_1, y_2) \in \mathbb{R}^2; y_1 < 1, y_2 > y_1\},$$
$$P_B(m) = \{(y_1, y_2) \in \mathbb{R}^2; y_1 > 1, y_2 > 1\}.$$

It is straightforward to check that $\mathrm{cl}(P_A(i)) \cap \mathrm{cl}(P_A(j)) \cap \mathrm{cl}(P_A(k)) = (1, 1)$ and $\mathrm{cl}(P_B(m)) \cap \mathrm{cl}(P_B(j)) \cap \mathrm{cl}(P_B(k)) = (1, 1)$. Thus, *Variety* is not a necessary condition for a categorization system to be generated by extended prototypes. Like in the previous example, the prototypes are not in general position since the point $(1, 1)$ is in equal distance (of $\sqrt{2}$) from all of them.

7. Proof of Theorem 1

7.1. Preliminary lemmas

Lemma 1. *Let $\{P_A\}_{A \subseteq L}$ be a categorization system. Then $\{P_A\}_{A \subseteq L}$ satisfies Hierarchical Consistency if and only if for every $A \subseteq L(|A| \geqslant 2)$ and for each $i \in A, P_A(i) = \bigcap_{j \in A \setminus \{i\}} P_{\{i,j\}}(i)$.*

Proof. Assume that for every $A \subseteq L(|A| \geqslant 2)$ and for each $i \in A, P_A(i) = \bigcap_{j \in A \setminus \{i\}} P_{\{i,j\}}(i)$. We show that the categorization system satisfies *Hierarchical Consistency*. Let $A \subseteq L$ be such that $|A| = 3$. Then for each $i \in A$,

$$P_A(i) = \bigcap_{j \in A \setminus \{i\}} P_{\{i,j\}}(i) = \bigcap_{B; i \in B \subsetneq A} P_B(i).$$

The proof proceeds by induction. Assume that A consists of more than three elements, and that the assertion holds for any $B \subsetneq A$. Then, for any $i \in A$,

$$\bigcap_{B; i\in B \subsetneq A} P_B(i) = \bigcap_{B; i \in B \subsetneq A} \bigcap_{j \in B\setminus\{i\}} P_{\{i,j\}}(i) = \bigcap_{j \in A\setminus\{i\}} P_{\{i,j\}}(i) = P_A(i),$$

as desired.

As for the inverse direction, assume that $\{P_A\}_{A\subseteq L}$ satisfies *Hierarchical Consistency*. We show that $P_A(i) = \bigcap_{j\in A\setminus\{i\}} P_{\{i,j\}}(i)$ for any $A \subseteq L (|A| \geqslant 2)$ and $i \in A$. The proof is by induction on $|A|$. The statement is obviously true for $|A| = 2$. Assume that it is true for all $A \subseteq L$ with $|A| \leqslant k$, and let $\hat{A}$ be a subset of L containing $k+1$ elements. Fix $i \in \hat{A}$. Then

$$\bigcap_{j\in \hat{A}\setminus\{i\}} P_{\{i,j\}}(i) = \bigcap_{B; i\in B\subsetneq \hat{A}} \bigcap_{j \in B\setminus\{i\}} P_{\{i,j\}}(i).$$

By the induction hypothesis, the right-hand side is equal to $\bigcap_{B; i \in B \subsetneq \hat{A}} P_B(i)$. Therefore, $\bigcap_{j\in\hat{A}\setminus\{i\}} P_{\{i,j\}}(i) = \bigcap_{B;i\in B\subsetneq\hat{A}} P_B(i)$, as required. □

Lemma 1 asserts that *Hierarchical Consistency* is equivalent to the following seemingly weaker condition: for every $A \subseteq L$ and for each $i \in A$, $P_A(i)$ is the set of entities categorized as i when all the pairs i,j ($j \in A\setminus\{i\}$) are considered.

A *hyperplane* H in $\mathbb{R}^d$ is a set of the type $\{y \in \mathbb{R}^d; \langle v, y\rangle = c\}$, where v is a non-zero vector in $\mathbb{R}^d$, $\langle v, y\rangle$ is the inner product of v and y, and c is a constant. We denote $H^+ = \{y \in \mathbb{R}^d; \langle v, y\rangle > c\}$ and $H^- = \{y \in \mathbb{R}^d; \langle v, y\rangle < c\}$. We say that a hyperplane $H_{i,j}$ *separates* $P_{\{i,j\}}(i)$ *from* $P_{\{i,j\}}(j)$ if $P_{\{i,j\}}(i) = H^-_{i,j}$ and $P_{\{i,j\}}(j) = H^+_{i,j}$.

Lemma 2. *Let $\{P_A\}_{A\subseteq L}$ be a categorization system which satisfies Convexity and let i,j be two categories in L. Then, there exists a hyperplane $H_{i,j}$ that separates $P_{\{i,j\}}(i)$ from $P_{\{i,j\}}(j)$.*

Proof. Follows from the fact that $P_{\{i,j\}}(i)$ and $P_{\{i,j\}}(j)$are open, non-empty disjoint convex sets such that the closure of their union is the entire $\mathbb{R}^d$. □

The hyperplane $H_{i,j}$ consists of all those entities that are in the boundary of both $P_{\{i,j\}}(i)$ and $P_{\{i,j\}}(j)$. These are the entities that are categorized neither as i nor as j, when the pair of categories i,j is considered.

We say that two hyperplanes H_1 and H_2 are parallel if there are two constants c_1 and c_2 and one vector v such that $H_i = \{y \in \mathbb{R}^d; \langle v, y\rangle = c_i\}, i = 1, 2$.

In case H_1 and H_2 are not parallel, there are two independent vectors v_1 and v_2 such that $H_i = \{y \in \mathbb{R}^d; \langle v_i, y\rangle = c_i\}, i = 1, 2$. Thus, $H_1 \cap H_2 = \{y \in \mathbb{R}^d; \langle v_i, y\rangle = c_i, i = 1, 2\}$. This means that when H_1 and H_2 are not parallel, $H_1 \cap H_2$ is an affine subspace of dimension $d-2$.

Lemma 3. Non-Redundancy *implies that for any three distinct categories i,j,k the hyperplanes $H_{i,j}$ and $H_{i,k}$ are not parallel.*

Proof. If $H_{i,j}$ and $H_{i,k}$ are parallel for some three categories i,j,k then there is a vector v and constants c_1, c_2 such that $H_{i,j} = \{y \in \mathbb{R}^d; \langle v, y\rangle = c_1\}$ and $H_{i,k} = \{y \in \mathbb{R}^d; \langle v, y\rangle = c_2\}$. Assume without loss of generality that $c_1 \geqslant c_2$. By Lemma 2, $P_{\{i,j\}}(i) = H^+_{i,j} = \{y \in \mathbb{R}^d; \langle v, y\rangle > c_1\} \subseteq \{y \in \mathbb{R}^d; \langle v, y\rangle > c_2\} = H^+_{i,k} = P_{\{i,k\}}(i)$ and this contradicts *Non-Redundancy*. □

Lemma 4. Non-Redundancy *and* Hierarchical Consistency *imply that $\emptyset \neq P_{\{j,k\}}(k) \cap P_{\{i,k\}}(i) \subseteq P_{\{i,j\}}(i)$ for any three distinct categories i,j,k.*

Proof. We start by showing that $P_{\{j,k\}}(k) \cap P_{\{i,k\}}(i) \neq \emptyset$. Indeed, if $P_{\{j,k\}}(k) \cap P_{\{i,k\}}(i) = \emptyset$, then $P_{\{j,k\}}(k) \subseteq P_{\{i,k\}}(k)$ which contradicts *Non-Redundancy*.

Next, if $P_{\{j,k\}}(k) \cap P_{\{i,k\}}(i) \nsubseteq P_{\{i,j\}}(i)$, then $B = P_{\{j,k\}}(k) \cap P_{\{i,k\}}(i) \cap P_{\{i,j\}}(j) \neq \emptyset$. B is an open set.

By *Hierarchical Consistency*, $P_{\{i,j,k\}}(i) = P_{\{i,k\}}(i) \cap P_{\{i,j\}}(i) \subseteq P_{\{i,j\}}(i)$and $B \subseteq P_{\{i,j\}}(j)$. Thus, $B \cap P_{\{i,j,k\}}(i) = \emptyset$. For similar reasons, $B \cap P_{\{i,j,k\}}(j) = B \cap P_{\{i,j,k\}}(k) = \emptyset$. Therefore, $B \cap [P_{\{i,j,k\}}(j) \cup P_{\{i,j,k\}}(j) \cup P_{\{i,j,k\}}(j)] = \emptyset$. However, since $P_{\{i,j,k\}}(i) \cup P_{ijk}(j) \cup P_{ijk}(k)$ is the union of an open partition, the intersection of this union with any open set is not empty. This is a contradiction and the lemma is proven. □

The next lemma expresses the main geometric property of categorizations which satisfy *Hierarchical Consistency*. This property is later used to prove Theorem 1.

Lemma 5. *Let $\{P_A\}_{A\subseteq L}$ be a categorization system which satisfies* Convexity, Hierarchical Consistency *and* Non-Redundancy. *For any three distinct categories i,j and k*

$$H_{i,j} \cap H_{i,k} = H_{i,j} \cap H_{j,k} = H_{j,k} \cap H_{i,k}.$$

Proof. It is sufficient to prove that $H_{i,j} \cap H_{i,k} \subseteq H_{i,j} \cap H_{j,k}$. Obviously, $H_{i,j} \cap H_{i,k} \subseteq H_{i,j}$. Thus, it remains to show that $H_{i,j} \cap H_{i,k} \subseteq H_{j,k}$.

Denote $S = P_{\{i,k\}}(i) \cap P_{\{i,j\}}(j)$, $T = P_{\{i,j\}}(i) \cap P_{\{i,k\}}(k)$. By Lemma 4, S and T are both non-empty sets. Moreover, $S \subseteq P_{\{j,k\}}(j)$ and $T \subseteq P_{\{j,k\}}(k)$. It follows that $H_{j,k}$ separates S from T.

Next, notice that S and T are the non-empty intersection of two convex sets. It follows that (see Rockafellar, 1970, Theorem 6.5, p. 47) $\text{cl}(S) = \text{cl}(P_{\{i,k\}}(i)) \cap \text{cl}(P_{\{i,j\}}(j))$ and $\text{cl}(T) = \text{cl}(P_{\{i,j\}}(i)) \cap \text{cl}(P_{\{i,k\}}(k))$. Thus, $\text{cl}(S) \cap \text{cl}(T) = \text{cl}(P_{\{i,j\}}(i)) \cap \text{cl}(P_{\{i,j\}}(j)) \cap \text{cl}(P_{\{i,k\}}(i)) \cap \text{cl}(P_{\{i,k\}}(k)) = H_{i,j} \cap H_{i,k}$.

Lemma 3 implies that $H_{i,j} \cap H_{i,k}$ is an affine space of dimension $d-2$. Since, $H_{j,k}$ is a hyperplane, it follows that $H_{j,k}$ must contain $H_{i,j} \cap H_{i,k}$ which proves the lemma. □

Notation 1. *The hyperplane $H_{i,j}$ is defined by the vector $s_{i,j}$ and the constant $c_{i,j}$. That is, $H_{i,j} = \{y \in \mathbb{R}^d; \langle s_{i,j}, y\rangle = c_{i,j}\}$. Without loss of generality we may assume that $P_{\{i,j\}}(j) \subseteq H^+_{i,j}$. We denote $s_{i,j} = -s_{j,i}$ and $c_{i,j} = -c_{j,i}$.*

Corollary 1. *Convexity, Hierarchical Consistency and Non-Redundancy imply that $s_{i,j}, s_{i,k}$ and $s_{k,j}$ are linearly dependent for every three distinct categories i,j,k.*

Proof. Let $D_{i,j,k} = H_{i,j} \cap H_{i,k} \cap H_{j,k}$. By Lemma 5, $D_{i,j,k} = H_{i,j} \cap H_{i,k}$. By Lemma 3, $H_{i,j} \cap H_{i,k}$ is an affine subspace of dimension $d-2$. Thus, $s_{i,j}, s_{i,k}$ and $s_{k,j}$ are linearly dependent. □

Corollary 2. *Convexity, Hierarchical Consistency and Non-Redundancy imply that for any three distinct categories* i,j,k *there exists* $y \in \mathbb{R}^d$ *such that* $\langle y, s_{i,j}\rangle > c_{i,j}$ *and* $\langle y, s_{j,k}\rangle > c_{j,k}$. *Moreover, for every such* y, $\langle y, s_{i,k}\rangle > c_{i,k}$.

Proof. By Lemma 4, the set $P_{\{i,j\}}(j) \cap P_{\{j,k\}}(k)$ is not empty, so there exists some $y \in \mathbb{R}^d$ which satisfies the above inequalities. Every such y is by Lemma 4 in $P_{\{i,k\}}(k)$, so $\langle y, s_{i,k}\rangle > c_{i,k}$. □

Notation 2. *Let* t *and* s *be two vectors in* $\mathbb{R}^d$. *Denote the ray that starts at* t *and continues in the direction of* s *by* $R(t,s)$. *Formally,*

$$R(t,s) = \{t + as; a > 0\}.$$

Lemma 6. *Let* i,j *and* k *be three distinct categories and* t_i *and* t_j *be two points in* $\mathbb{R}^d$, *such that* $t_j - t_i = \gamma s_{ij}$, *where* $\gamma > 0$. *Then, assuming Convexity, Hierarchical Consistency and Non-Redundancy, the rays* $R(t_i, s_{ik})$ *and* $R(t_j, s_{jk})$ *intersect.*

Proof. By Corollary 1, $s_{i,j}, s_{i,k}$ and $s_{k,j}$ are linearly dependent. Lemma 3 implies that no two are linearly dependent. Thus, there are two non-zero constants α and β such that $s_{i,k} = \alpha s_{i,j} + \beta s_{j,k}$. Recall that $D_{i,j,k} = H_{i,j} \cap H_{i,k} \cap H_{j,k}$ and let $z \in D_{i,j,k}$. Then, $c_{i,k} = \langle z, s_{i,k}\rangle = \langle z, \alpha s_{i,j} + \beta s_{j,k}\rangle = \alpha c_{i,j} + \beta c_{j,k}$.

We prove that both α and β are positive. We prove first that it cannot be the case that both are negative. If, on the other hand, both are negative, then consider $y \in \mathbb{R}^d$ such that $\langle y, s_{i,j}\rangle > c_{i,j}$ and $\langle y, s_{j,k}\rangle > c_{j,k}$ (such y exists by Corollary 2).Then, $\langle y, s_{i,k}\rangle = \langle y, \alpha s_{i,j} + \beta s_{j,k}\rangle = \alpha\langle y, s_{i,j}\rangle + \beta\langle y, s_{j,k}\rangle < \alpha c_{i,j} + \beta c_{j,k} = c_{i,k}$. However, by Corollary 2, $\langle y, s_{i,k}\rangle > c_{i,k}$, which is a contradiction. This proves that both α and β cannot be negative.

It remains to show that it cannot be the case that either α or β is negative. Assume, on the contrary, that α is negative and β is positive. Consider $y \in \mathbb{R}^d$ such that $\langle y, s_{j,k}\rangle > c_{j,k}$ and $\langle y, s_{k,i}\rangle > c_{k,i}$ (again, such y exists by Corollary 2). Then, $\langle y, -\alpha s_{i,j}\rangle = \langle y, \beta s_{j,k} - s_{i,k}\rangle = \langle y, \beta s_{j,k} + s_{k,i}\rangle > \beta c_{j,k} + c_{k,i} = -\alpha c_{i,j}$. Thus, $\langle y, s_{i,j}\rangle > c_{i,j}$. However, by Corollary 2,$\langle y, s_{i,j}\rangle < c_{i,j}$, which is a contradiction. Similarly, it is impossible that α is positive and β is negative. We conclude that $s_{i,k} = \alpha s_{i,j} + \beta s_{j,k}$, where both α and β are positive. Thus, $s_{i,k} = \frac{\alpha}{\gamma}(t_j - t_i) + \beta s_{j,k}$ and therefore, $t_i + \frac{\gamma}{\alpha} s_{i,k} = t_j + \frac{\gamma\beta}{\alpha} s_{j,k}$. Since the left side is in $R(t_i, s_{ik})$ and the right side is in $R(t_j, s_{jk})$, the rays $R(t_i, s_{ik})$ and $R(t_j, s_{jk})$ intersect at this point, and the proof is complete. □

7.2. *Proof of part (1) of Theorem 1*

We prove first that if $\{P_A\}_{A\subseteq L}$ is generated by extended prototypes then it satisfies *Convexity* and *Hierarchical Consistency*. Let $\{x_1, \ldots, x_\ell\} \subseteq \mathbb{R}^{d+1}$ be the set of extended prototypes. First, by the definition of a categorization system generated by extended prototypes, we have for every $A \subseteq L$ and for each $i \in A$

$$P_A(i) = \bigcap_{j\in A,\ j\neq i} \{y \in \mathbb{R}^d : \mathbf{d}_i(y) < \mathbf{d}_j(y)\} = \bigcap_{j\in A,\ j\neq i} P_{\{i,j\}}(i).$$

Thus, by Lemma 1 the categorization system satisfies *Hierarchical Consistency*.

Second, let i,j be two different categories in L. The set of entities which are equidistant from these two prototypes is the set $H_{i,j} = \{y \in \mathbb{R}^d; \mathbf{d}_i(y) = \mathbf{d}_j(y)\}$. An elementary calculation shows that this set can be rewritten as $\{y \in \mathbb{R}^d; \langle y, x'_j - x'_i\rangle = \frac{1}{2}(w_j^2 - w_i^2 + \|x'_j\|^2 - \|x'_i\|^2)\}$, where $x_i = (x'_i, w_i), i = 1, \ldots, \ell$. Therefore, $H_{i,j}$ is the hyperplane perpendicular to $x'_j - x'_i$ which passes through the point $\frac{\|x'_j\|^2 - \|x'_i\|^2 + w_j^2 - w_i^2}{2\|x'_j - x'_i\|^2}(x'_j - x'_i)$. It follows that $P_{\{i,j\}}(i)$,which is the set of entities closer to x_i than to x_j, is an open half space. In particular, it is convex. $P_A(i)$ is the intersection of the sets $P_{\{i,j\}}(i)$ ($j \in A\backslash\{i\}$) and therefore is a convex set, which proves *Convexity*.

We now show that if the extended prototypes are in general position then *Non-Redundancy* and *Variety* are also satisfied. Indeed, if *Non-Redundancy* is violated then there are 3 categories i,j,k such that $P_{\{i,j\}}(i) \subseteq P_{\{i,k\}}(i)$. This means that the intersection of the sets $\{y \in \mathbb{R}^d; \mathbf{d}_i(y) < \mathbf{d}_j(y)\}$ and $\{y \in \mathbb{R}^d; \mathbf{d}_i(y) > \mathbf{d}_k(y)\}$ is empty. As shown in the previous paragraph each of these sets is an open half space. Two half spaces can be disjoint only if the two vectors defining them are dependent. It follows that $x'_k - x'_i = \alpha(x'_j - x'_i)$ for some constant $\alpha \in \mathbb{R}$. This means that x'_i, x'_j and x'_k are on the same line, a contradiction to the assumption of general position.

To see that *Variety* is satisfied, fix some four categories i,j,k,m. Since the categorization system is generated by extended prototypes, and by what we already proved, *Hierarchical Consistency*, *Convexity* and *Non-Redundancy* are satisfied. Thus, by the auxiliary lemmata (which does not use *Variety*), each of the sets $D_{i,j,k} = H_{i,j} \cap H_{i,k} \cap H_{j,k}$ and $D_{m,j,k} = H_{m,j} \cap H_{m,k} \cap H_{j,k}$ are of dimension $d-2$. If *Variety* is not satisfied then $D_{i,j,k} = D_{m,j,k}$ so the intersection $D_{i,j,k} \cap D_{m,j,k}$ is also of dimension $d-2$. However, this last intersection is exactly the set of entities $\{y \in \mathbb{R}^d : \mathbf{d}_i(y) = \mathbf{d}_j(y) = \mathbf{d}_k(y) = \mathbf{d}_m(y)\}$. This violates the assumption of general position.

7.3. *Proof of part (2) of Theorem 1*

The proof of this part is divided into two propositions.

Proposition 1. *If a categorization system* $\{P_A\}_{A\subseteq L}$ *satisfies Convexity, Hierarchical Consistency, Non-Redundancy and Variety, then there are* ℓ *points* $x'_1, \ldots, x'_\ell \in \mathbb{R}^d$ *such that*:

1. $x'_i - x'_j$ *is perpendicular to the hyperplane* $H_{i,j}$, *for every* $i,j \in L$, *and*

2. $\langle x'_j - x'_i, s_{ij}\rangle \geqslant 0$ *for every* $i,j \in L$. *That is, the direction from* x'_i *to* x'_j *is the same as the direction from* $P_{\{i,j\}}(i)$ *to* $P_{\{i,j\}}(j)$ (*we call such points "well oriented*").

Proof. The proof is constructive. We select ℓ points $x'_1, \dots, x'_\ell$, sequentially. We show that after $x'_1, \dots, x'_{k-1}$ have been selected, it is possible to find x'_k such that $x'_k - x'_j$ is perpendicular to $H_{j,k}$ for every $j = 1, \dots, k-1$.

Let x'_1 be an arbitrary point in $\mathbb{R}^d$. Define $x'_2 = x'_1 + s_{12}$. Since s_{12} is perpendicular to $H_{1,2}$, so is $x'_2 - x'_1$. Lemma 6 ensures that the rays $R(x'_1, s_{13})$ and $R(x'_2, s_{23})$ intersect. The third point, x'_3, is placed at the intersection of these rays.

Now comes the key argument of the proof. Similar to x'_3, we place x'_4 at the intersection point of the rays $R(x'_2, s_{24})$ and $R(x'_3, s_{34})$, whose existence is guaranteed by Lemma 6. In particular, $x'_4 - x'_2$ is perpendicular to $H_{2,4}$ and therefore to$D_{1,2,4}$ and $x'_4 - x'_3$ is perpendicular to $H_{3,4}$ and therefore to $D_{1,3,4}$. We show now that $x'_1 - x'_4$ is perpendicular to $H_{1,4}$, and moreover, that x'_1 and x'_4 are well oriented.

The hyperplane $H_{1,4}$ contains both $D_{1,2,4}$ and $D_{1,3,4}$. *Variety* implies that $D_{1,2,4}$ and $D_{1,3,4}$ are not equal. Furthermore, the dimensions of $D_{1,2,4}$ and $D_{1,3,4}$ are $d-2$. Therefore, in order to show that $x'_1 - x'_4$ is perpendicular to $H_{1,4}$ it is sufficient to show that $x'_1 - x'_4$ is perpendicular to any vector of the type $y - y'$, where $y, y' \in D_{1,2,4} \cup D_{1,3,4}$. Let $y, y' \in D_{1,2,4}$. By construction, since $y, y' \in H_{1,2}$, $\langle x'_1 - x'_2, y - y'\rangle = 0$. Similarly, $\langle x'_2 - x'_4, y - y'\rangle = 0$. Summing up these equations, we obtain that $\langle x'_1 - x'_4, y - y'\rangle = 0$. For similar reasons, if $y, y' \in D_{1,3,4}$, then $\langle x'_1 - x'_4, y - y'\rangle = 0$.

It remains to show that if $y \in D_{1,2,4}$ and $y' \in D_{1,3,4}$, $\langle x'_1 - x'_4, y - y'\rangle = 0$. Let $z \in D_{1,2,3}$ and $w \in D_{2,3,4}$. Since both z and y are in $H_{1,2}$,

$$\langle z - y, x'_1 - x'_2\rangle = 0. \tag{1}$$

Similarly,

$$\langle z - y', x'_1 - x'_3\rangle = 0, \tag{2}$$

$$\langle w - y, x'_2 - x'_4\rangle = 0 \tag{3}$$

and

$$\langle w - y', x'_3 - x'_4\rangle = 0. \tag{4}$$

By summing up Eqs. (1) and (3) and subtracting Eqs. (2) and (4) we obtain,

$$\langle y - y', x'_1 - x'_4\rangle = \langle w - z, x'_2 - x'_3\rangle. \tag{5}$$

Since w and z are in $H_{2,3}$, $w - z$ is perpendicular to $x'_2 - x'_3$. Thus, (5) implies that $x'_1 - x'_4$ is perpendicular to $H_{1,4}$.

It remains to show that every pair of $x'_1, \dots, x'_4$ is well oriented. By construction, every pair of x'_1, x'_2, x'_3 is well oriented. Furthermore, by the choice of x'_4, both $\langle x'_2 - x'_4, s_{24}\rangle$ and $\langle x'_3 - x'_4, s_{34}\rangle$ are positive. We therefore only need to show that x'_1 and x'_4 are well oriented.

Consider the triplet x'_1, x'_2, x'_4. By construction, x'_4 is on the line perpendicular to $H_{2,4}$ which passes through x'_2. By what we showed earlier, x'_4 is also on the line perpendicular to $H_{1,4}$ which passes through x'_1.It means that x'_4 is the intersection point of these two lines. However, by Lemma 6, the rays $R(x'_1, s_{14})$ and $R(x'_2, s_{24})$ intersect, so x'_4 is the intersection point of these rays. It follows that x'_1 and x'_4 are well oriented.

The procedure is then continued. After $x'_1, \dots, x'_{k-1}$ have been fixed, we place x'_k at the intersection of the rays $R(x'_{k-2}, s_{k-2,k})$ and $R(x'_{k-1}, s_{k-1,k})$. For every $j < k-2$, we use the same argument as before (this time with the categories $j, k-2, k-1, k$), to show that $x'_k - x'_j$ is perpendicular to $H_{j,k}$ and that x'_j, x'_k are well oriented. □

Proposition 2. *Let* $\{x'_1, \dots, x'_\ell\} \subseteq \mathbb{R}^d$ *be points that satisfy Proposition* 1. *There are numbers* $w_1, \dots, w_l$, *such that* $P_{\{i,j\}}(i) = \{y \in \mathbb{R}^d; \mathbf{d}_i(y) < \mathbf{d}_j(y)\}$ *for every* $i,j \in L$, *where* $\mathbf{d}_i(y) = \|(y,0) - (x'_i, w_i)\|^2$.

Proof. As mentioned before, for extended prototypes $x_i = (x'_i, w_i)$ and $x_j = (x'_j, w_j)$ the set of points in $\mathbb{R}^d$ which are equidistant from x_i and x_j, is the hyperplane

$$T_{i,j} = \{y \in \mathbb{R}^d; \mathbf{d}_i(y) = \mathbf{d}_j(y)\}$$
$$= \{y \in \mathbb{R}^d; \langle y, x'_j - x'_i\rangle = \tfrac{1}{2}(w_j^2 - w_i^2 + \|x'_j\|^2 - \|x'_i\|^2)\}.$$

$T_{i,j}$ is an hyperplane perpendicular to $x'_j - x'_i$. When w_j grows to infinity, $T_{i,j}$ moves in one direction, while, when w_i grows to infinity, $T_{i,j}$ moves in the other direction. It follows that every hyperplane perpendicular to $x'_j - x'_i$ can be written with appropriate w_i and w_j.

We sequentially choose the numbers $w_1, \dots, w_\ell$ and show that they satisfy the proposition. Let $x'_1, x'_2, \dots, x'_\ell$ be the prototypes found in Proposition 1. Choose w_1 and w_2 such that $T_{1,2} = H_{1,2}$ (recall that $H_{1,2}$ is the hyperplane separating $P_{\{1,2\}}(1)$ and $P_{\{1,2\}}(2)$). Since x'_1, x'_2 are well oriented, the set of points closer to x'_1 is exactly $P_{\{1,2\}}(1)$ and the set of points closer to x'_2 is $P_{\{1,2\}}(2)$.

Next, choose w_3 such that $T_{2,3} = H_{2,3}$. We need to show that w_3 is consistent with w_1. That is, $T_{1,3} = H_{1,3}$. $T_{1,3}$ is perpendicular to $x'_1 - x'_3$ by definition, while $H_{1,3}$ has the same property by Proposition 1. Moreover, for every $y \in T_{1,2} \cap T_{2,3}$ we have $\mathbf{d}_1(y) = \mathbf{d}_2(y)$ and $\mathbf{d}_2(y) = \mathbf{d}_3(y)$. Thus, $\mathbf{d}_1(y) = \mathbf{d}_3(y)$. It means that $T_{1,3}$ is the unique hyperplane perpendicular to $x'_1 - x'_3$ which contains $T_{1,2} \cap T_{2,3} = H_{1,2} \cap H_{2,3}$. Lemma 5 states that $H_{1,3}$ also contains $H_{1,2} \cap H_{2,3}$. Therefore, $T_{1,3}$ coincides with $H_{1,3}$.

Assume that $w_1, \dots, w_k$ have already been found. We choose w_{k+1} such that $T_{k,k+1} = H_{k,k+1}$. For every $j = 1, 2, \dots, k-1$ we have $T_{j,k} = H_{j,k}$ and $T_{k,k+1} = H_{k,k+1}$, so similarly to the argument in the case of the first three categories, $T_{j,k+1} = H_{j,k+1}$. □

Conclusion of the proof: We have shown that if a categorization system satisfies *Convexity*, *Hierarchical Consistency*, *Non-Redundancy* and *Variety* then it is possible to find points $x_1 = (x'_1, w_1), \dots, x_\ell = (x'_\ell, w_\ell)$ in $\mathbb{R}^{d+1}$ such that for every two categories $i,j \in L$, $P_{\{i,j\}}(i) = \{y \in \mathbb{R}^d; \|(y,0) - x_i\|^2 < \|(y,0) - x_j\|^2\}$. By Lemma 1, for any $A \subseteq L$ and any $i \in A$,

$$P_A(i) = \bigcap_{j \in A \setminus \{i\}} P_{\{i,j\}}(i) = \bigcap_{j \in A \setminus \{i\}} \{y \in \mathbb{R}^d; \|(y,0) - x_i\|^2 < \|(y,0) - x_j\|^2\} = \{y \in \mathbb{R}^d; \|(y,0) - x_i\|^2 < \|(y,0) - x_j\|^2 \text{ for every } j \in A, j \neq i\}.$$

Thus, the categorization system is generated by extended prototypes.

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