

Interest Rate Policies in Consumption-Savings Dynamic Models *

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ABSTRACT:

We examine the effect of interest rate policies on an agent's consumption and savings when her income is fluctuating. In each period a long-lived agent decides how much to save (i.e., invest in a risk-free bond) and how much to consume while her income is uncertain and depends on the state of the economy. The central bank, on the other hand, fixes a policy, called an *interest rate policy*, which determines the state-dependent interest rate. This interest rate is the rate of return on the risk-free bond which the agent takes into consideration when deciding on her savings. We compare interest rate policies and show that under concavity of the consumption function, a condition that ensures that the substitution effect dominates the income effect, lower interest rates encourage the agent's consumption across all states.

Keywords: Consumption; Savings; Interest rate policy; Income fluctuation problem; Comparative statics; Monetary policy; Dynamics.

JEL classification: C70; C78; D51; D58

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1 Introduction

The influence of aggregate demand on the economy is undisputed. Given the dependence of the consumption function on current interest rates and future interest rates, it is worth examining whether monetary policy affects aggregate consumption through different interest rate policies. While many empirical studies have tried to estimate the effect of a change in interest rates on savings and consumption,¹ we do not know of any analytical results in frameworks similar to ours that do not rely on closed form solutions² or numerical methods. Our goal in this paper is to analytically examine the effect of interest rate policies on savings and consumption decisions. Even in a two period model with no uncertainty, consumption may increase or decrease when interest rates increase. Both the income effect and the substitution effect influence the agent's consumption choice. The income effect increases consumption if the agent has positive savings, since her savings are worth more with higher interest rates. The substitution effect decreases the agent's consumption since higher interest rates raise the price of consumption. Thus, a necessary condition for an increase in consumption in response to an increase in interest rates is that the substitution effect dominates the income effect. We offer a condition on the agent's utility function that ensures such domination in Section 4.

In this paper we examine the effect of different interest rate policies on savings and consumption decisions in a standard consumption-savings model with income uncertainty, sometimes referred to as the "savings problem" (see Ljungqvist and Sargent (2004) and Chamberlain and Wilson (2000)) or the "income fluctuation problem" (see Schechtman and Escudero (1977)). The income fluctuation problem is fundamental in modern macroeconomics.³ In an income fluctuation problem, the agent receives a state-dependent income in each period. The states follow a stochastic process, so the agent's income follows a stochastic process as well. The agent solves an infinite horizon consumption-savings problem. That is, the agent decides how much to save and how much to consume in each period. In contrast to complete markets models (e.g., Arrow-Debru models) where the agent can insure herself against any realization, the income fluctuation problem is an incomplete markets model. The agent can transfer assets from one period to another only by investing in a risk-free bond and either is not allowed to borrow or has some borrowing limit. Usually the interest rate in the

¹For example Attanasio and Weber (1993), and Campbell and Mankiw (1989). Nabar (2011) finds that savings and interest rates in China were actually negatively correlated. Recently, Di Maggio et al. (2014) and references therein study the effect of interest rates on household consumption and savings decisions.

²For example, Weil (1993) finds a closed form solution to the consumption function showing that consumption increases with higher interest rates if the income effect dominates the substitution effect.

³The income fluctuation problem is used to study many macroeconomic phenomena. For example, the permanent income hypothesis (Bewley (1977)) wealth distribution (Benhabib et al. (2014) and Benhabib et al. (2011)), asset pricing (Huggett (1993)), fiscal policy (Heathcote (2005)) and many more.

income fluctuation problem is constant and is equal to or less than the discount factor (see Chamberlain and Wilson (2000)). Our model adds a central bank to the standard income fluctuation problem. In our model, a central bank decides on an interest rate policy. That is, the central bank chooses an interest rate for each state of the economy. Given an interest rate policy, the agent decides how much to consume and how much to save in each period. This is in contrast to general equilibrium models (for example Aiyagari (1994)) where the interest rate is determined in equilibrium and not by a central bank. The agent in our model has rational expectations: she knows the central bank's policy, the number of states, the value of her income in each state, and the distribution governing the states' dynamic.

This paper has two goals. First, to study the implications of a change in interest rate policy on savings and consumption. The consensus in the field of economics is that savings are positively correlated with the current interest rates. The economic logic is that the agent will be more willing to postpone consumption to the future if she is paid more to do so. In this paper we examine a more general question: how interest rate policies affect savings and consumption. An agent's savings are affected not only by the current interest rates, but also by the interest rates in different states of the economy. Thus, if a central bank were to announce that interest rates will be high in states with high incomes, it would influence the agent's optimal decision regarding how much to save in states with low incomes. Assume, for instance, an interest rate policy in which for every state (every possible income the agent can have) the interest rate is the same and is equal to zero. Denote this policy by ρ_1 . Assume that the agent decides to save $g(x, s_1, \rho_1)$ dollars when the state is s_1 and her wealth is x . It seems natural that under a different interest rate policy ρ_2 , that also assigns an interest rate of zero to state s_1 but in which other states have higher interest rates, the agent's optimal decision will change in state s_1 . That is, $g(x, s_1, \rho_1) \neq g(x, s_1, \rho_2)$. The question we address in this paper is what conditions on the agent's preferences are sufficient to conclude that $g(x, s, \rho_1) \leq g(x, s, \rho_2)$ for any state and wealth.

The second goal is to apply a comparative statics technique to dynamic economies. Comparative statics techniques help to determine whether an endogenous variable or a control variable is monotone with respect to a certain parameter. We focus in this paper on deriving comparative statics results that have a wide range of applications in a consumption-savings problem. The technique employed in this paper can also be implemented in other macroeconomic models, specifically when a question is being asked about comparative statics in dynamic programming models where the utility function is concave and the value function is differentiable. In Hopenhayn and Prescott (1992) (HP) a technique to apply comparative statics to dynamic economies is offered. HP's approach employs lattice programming techniques developed by Topkis (1978). Miao (2002) applies HP's technique and derives comparative statics results in a model similar to ours. We present HP's approach in detail in Section 7 and compare it to our approach. While HP's approach does not require concavity

or differentiability of the utility function, we show that if concavity and differentiability are assumed our approach is stronger than HP's approach in the following sense: if one can derive a comparative statics result with HP's approach, then one can derive this result with our approach, and the opposite is not true. Acemoglu and Jensen (2015) (AJ) offer a comparative statics approach that focuses on a change in aggregate variables. AJ's approach is suited to large dynamic economies where there is a shock to aggregate variables that affects the general equilibrium.

The rest of the paper is organized as follows. Section 2 presents the income fluctuation problem. We add a central bank to the income fluctuation problem and discuss some basic results in dynamic programming models that we use in the following sections. Section 3 discusses the agent's optimal consumption strategy and the differentiability of the value function. We present our main comparative statics theorem which shows how to determine whether the endogenous variables (consumption and savings) are monotone with respect to a change in some parameter. To be more precise, let \leq be a partial order and I a poset. For a level of cash-on-hand x , a state s and a parameter $e \in I$ denote the optimal savings policy by $g(x, s, e)$. We are interested in the following question: is it true that $e_2 \geq e_1$ implies $g(x, s, e_2) \geq g(x, s, e_1)$ for all (x, s) ? In section 3.3 we introduce a simple condition that ensures that the answer to the last question is positive.

Our main theorem deals with the effect of interest rate policies on consumption and savings decisions. In section 4 we prove that if the consumption policy function is concave in the agent's cash-on-hand and the substitution effect dominates the income effect then an interest rate policy with higher interest rates increases the agent's savings. Many other factors affect savings decisions. In section 5 we show that an increase in the agent's permanent income increases her consumption. In section 6.1 we show that a higher discount factor increases the agent's savings. In sections 6.2 and 6.3 we compare the influence on savings decisions of different transition matrices that govern the states' dynamic. In section 7 we will compare our comparative statics approach to the approach of Hopenhayn and Prescott (1992). In section 8 we provide some final remarks.

2 The model: a dynamic consumption-savings problem

We consider a discrete time dynamic consumption-savings model. Let $S = \{s_1, \dots, s_n\}$ be a finite set of possible states of the economy. At any time $t = 1, 2, 3, \dots$ a state s_t is realized and the agent gets an income that depends on this state. Let $y : S \rightarrow \mathbb{R}_+$ be the labor income function: $y(s_i)$ indicates the agent's income in state s_i . For the sake of simplicity,

we assume that the states are ordered in accordance with the corresponding incomes:

$$(1) \quad y(s_1) < y(s_2) < \dots < y(s_n).$$

The evolution of the states follows a Markov chain with transition probabilities $P = (P_{ij})$. The probability of moving from state s_i to state s_j is denoted by P_{ij} .

The central bank determines the interest rate policy. That is, the central bank assigns an interest rate to each state of the economy.

Definition 1. (i) An interest rate policy is a function that assigns an interest rate to each state. That is, $\rho : S \rightarrow [1, K]$.

(ii) We say that an interest rate policy ρ is Keynesian if $\rho(s)$ is increasing in s . That is, $\rho(s_i) \geq \rho(s_j)$ whenever $i > j$.

Note that the interest rate policy takes values in the interval $[1, K]$, where 1 is the lowest real interest rate and K is the highest. We do not allow negative interest rates. A Keynesian interest rate policy is expansionary when the agent has a low income and is tight when the agent's income is high. Thus, a Keynesian interest rate policy encourages the agent to consume when her income is low and discourages consumption when her income is high.

We now consider the agent's consumption-savings decision problem. Suppose that the agent's initial savings is a_0 . In each period $t = 1, 2, \dots$ the agent receives an income⁴ $y(s(t))$ that depends on the state of the economy. The rate of return on her savings at time t , $\rho(s(t))$, is determined by the central bank's interest rate policy ρ as well as by the state of the economy $s(t)$.

Denote the agent's cash-on-hand at time $t = 1$ by $x(1) = \rho(s(1))a_0 + y(s(1))$. Suppose that at time t the agent's cash-on-hand is $x(t)$. Based on $s(t)$ and $x(t)$, the agent decides how much to consume at time t , which we denote by $c(t)$, and thereby how much to save for future consumption. Thus, her cash-on-hand at time $t + 1$ when $s(t + 1)$ is the realized state is

$$x(t + 1) = \rho(s(t + 1))(x(t) - c(t)) + y(s(t + 1)).$$

We assume for now that the agent cannot borrow and thus, $x(t) \geq c(t)$ in every period. We denote by $C(x) = [0, x]$ the interval from which the agent may choose her consumption level.

At each time t , the agent gets to know a pair consisting of the cash-on-hand and the realized state: $z(t) := (x(t), s(t))$, and when deciding on the consumption and savings levels at that time, the agent may condition her decision on the entire history $z^t := (z(1), \dots, z(t))$. For notational simplicity, denote $X = [0, \infty)$ and $Z = X \times S$.

⁴Note the distinction between $s(t)$ which is the realized state at time t , and s_i which is the i -th state as ordered in Eq. (1).

Definition 2. (i) A consumption strategy is a function π that assigns to every finite history $z^t = (z(1), \dots, z(t))$ in $Z^t := \underbrace{Z \times \dots \times Z}_{t \text{ times}}$ an action $\pi(z^t)$ in $C(x(k))$, where $z(t) = (x(t), s(t))$.

(ii) A stationary consumption strategy is a consumption strategy that depends only on the last pair of cash-on-hand and state: $\pi(z(1), \dots, z(t)) = \pi(z(t))$ for every $(z(1), \dots, z(t)) \in Z^t$.

Note that a consumption strategy induces a *savings strategy* that assigns to every finite history z^k a savings level $x(k) - \pi(z(1), \dots, z(k))$ in $C(x(k))$. Furthermore, a consumption strategy π induces a stochastic process over the extended set of states in Z , as follows. The initial state is determined by the agent's initial cash-on-hand $x(1)$ and the initial state $s(1)$. Let $z(1) = (x(1), s(1))$. Suppose that the history of cash-on-hand and state pairs up to time $t - 1$ is $(z(1), \dots, z(t - 1))$. Then, the probability of the next cash-on-hand and state pair being $(x(t), s(t))$ is given by,

$$\mathbb{P}(z(t)) = (\pi(z(1), \dots, z(t - 1)) \text{ and } s(t) = s_j | z(1), \dots, z(t - 1)) = P_{ij},$$

where $z(t - 1) = (x(t - 1), s_i)$. That is, the evolution of the cash-on-hand and state pairs is dictated both by the Markov chain that governs the states' evolution and by the consumption strategy.

The agent's utility from consumption in each period is given by a bounded utility function $u : [0, \infty) \rightarrow [0, M]$. We assume that u is strictly increasing, strictly concave, continuously differentiable and that $u'(0) = \infty$. The agent's utility derived from the sequence $c(1), c(2), \dots$ of consumption is the present discounted value $\sum_{t=1}^{\infty} \beta^{t-1} u(c(t))$, where $\beta \in (0, 1)$ is the agent's discount rate. When the agent follows a consumption strategy π , and the initial state is (x, s) her expected present discounted value is

$$V_{\pi}(x, s) = \mathbb{E}_{\pi} \left(\sum_{t=1}^{\infty} \beta^{t-1} u(\pi(z(1), \dots, z(t))) \right),$$

where $\mathbb{E}_{\pi}(\cdot)$ is the expectation operator with respect to the probability measure determined by the transition probabilities P and the consumption strategy π . Denote

$$V(x, s) = \sup_{\pi} V_{\pi}(x, s).$$

That is, $V(x, s)$ is the maximal expected utility that the agent can have when the initial cash-on-hand and state pair is (x, s) . We call $V : Z \rightarrow \mathbb{R}$ the *value function* and a strategy π attaining it *optimal*.

We denote by b the savings of the agent. For every $(x, s_i) \in Z$ and $b \in C(x)$ define the following function:

$$(2) \quad h(x, b, s_i, V) = u(x - b) + \beta \sum_{j=1}^n P_{ij} V(\rho(s_j) b + y(s_j), s_j).$$

When the initial cash-on-hand and state pair is (x, s_i) and the agent decides to consume $x - b$ and to save b , her present utility is $u(x - b)$. If in the next period the realized state is s_j then the next period's cash-on-hand is $\rho(s_j)b + y(s_j)$. If the agent employs the optimal savings strategy, her present discounted value utility is $\beta V(\rho(s_j)b + y(s_j), s_j)$. However, s_j is realized with probability P_{ij} and therefore when saving b , the expected present discounted value of all future utilities is $\beta \sum_{j=1}^n P_{ij} V(\rho(s_j)b + y(s_j), s_j)$, which is the second summand on the RHS of Eq. (2).

The agent's goal is to find the optimal possible division between savings b and consumption c , provided that the continuation value is given by V . That is, to find $\max_{b \in C(x)} h(x, b, s_i, V)$. Bellman's equality connects the functions V and h , as stated in the following well known proposition. For the proofs of parts (i), (ii) and (iii) see Blackwell (1965). For the proof of part (iv) see Stokey and Lucas (1989).

Proposition 1. (i) *The value function $V : Z \rightarrow \mathbb{R}$ satisfies the Bellman equation,*

$$(3) \quad V(x, s_i) = \max_{b \in C(x)} h(x, b, s_i, V).$$

(ii) *The Bellman equation has a unique solution. That is, if $\bar{V} : Z \rightarrow \mathbb{R}$ satisfies the Bellman equation, then $\bar{V}(z) = V(z)$ for every $z \in Z$.*

(iii) *There is a unique optimal stationary strategy.*

(iv) *If u is continuous, strictly concave and strictly increasing, then V is continuous, strictly concave and strictly increasing in x .*

The function $h(x, b, s_i, V)$ is the sum of two functions. One is strictly concave in b , $u(x - b)$, and the other is a summation of βV 's, all are, by Proposition 1, concave in their first argument. Thus, h itself is strictly concave in b .

3 Preliminaries

3.1 Optimal savings and consumption stationary strategy

Let $z = (x, s) \in Z$. Define $g(x, s)$ to be the savings level of the next period that attains the maximum of $h(x, b, s, V)$. That is,

$$(4) \quad g(x, s) = \operatorname{argmax}_{b \in C(x)} h(x, b, s, V).$$

As a strictly concave function, h has a unique maximum, denoted as $g(x, s)$. The "argmax" function defined in Eq. (4) is called the *savings policy function*. The choice $g(x, s)$ is the best savings level in $C(x)$ that the agent has in a period where the state

is s and her cash-on-hand is x . Note that the savings policy function induces an optimal consumption policy function $\sigma(x, s)$ by the equation $\sigma(x, s) = x - g(x, s)$. $\sigma(x, s)$ is the best consumption level in $C(x)$ that the agent has in a period where the state is s and her cash-on-hand is x .

Let $B(Z)$ be the space of all bounded real valued functions defined on Z . Define the operator $T : B(Z) \rightarrow B(Z)$ by

$$TV(x, s_i) = \max_{b \in C(x)} h(x, b, s_i, V).$$

A standard argument⁵ shows that there is a unique function $V \in B(Z)$ such that $TV = V$. Furthermore, for every $f_1 \in B(Z)$ if we define $f_n = Tf_{n-1}$ for every $n = 2, \dots$ then $f_n \rightarrow V$. Assume that V is the fixed point of T . The following lemma helps to prove properties of the value function.

Lemma 1. *Let $\emptyset \neq D \subseteq B(Z)$. Suppose that D satisfies the following properties:*

(1) *If $f \in D$ then $Tf \in D$.*

(2) *D is closed. If $f_n \in D$ for every n and $f_n \rightarrow f$ then $f \in D$.*

Then $V \in D$.

Proof. Let $f \in D$ and define $f_n = T^n f$. By (1) we have $f_n \in D$, and by the Banach theorem $f_n \rightarrow V$. From (2) $V \in D$, which proves the lemma. \square

3.2 Differentiability of the value function

In order to characterize the optimal consumption policy, it is often important to establish the differentiability of the value function. We use the differentiability of the value function in order to prove comparative statics results. Fix the interest rate policy ρ and the labor income function y . The *Envelope Theorem* (see Benveniste and Scheinkman (1979)) implies that the value function is differentiable. For each state $s \in S$ define

$$V'(x, s) := \frac{\partial V(x, s)}{\partial x}.$$

Denote by u' the derivative of u . In addition, the Envelope Theorem states that for every $(x, s) \in Z$,

$$V'(x, s) \geq u'(\sigma(x, s)),$$

⁵Define the sup norm of a function $f \in B(Z)$ by $\|f\| = \sup\{|f(z)| : z \in Z\}$. This induces the metric $d(f, g) = \|f - g\|$. The space $(B(Z), d)$ is a complete metric space. It can be shown that T is a contraction mapping, i.e., $d(Tf, Tg) < \beta d(f, g)$ for $0 < \beta < 1$. Note that every contraction mapping is continuous, i.e., if $f_n \rightarrow f$ then $Tf_n \rightarrow Tf$. The Banach fixed point theorem states that if $T : B(Z) \rightarrow B(Z)$ is a contraction mapping then T has a unique fixed point.

with equality if $\sigma(x, s) > 0$. The last equation is called the *envelope condition*.

Berge's maximum theorem (see Aliprantis and Border (2006) theorem 17.31) implies that $\sigma(x, s)$ is continuous in x . Thus, V' is continuous as the composition of continuous functions. Let $h'(x, b, s_i, V)$ be the derivative of h with respect to b . If the savings at (x, s_i) is an interior point (i.e., $g(x, s_i) \in (0, x)$), then the savings policy function must satisfy the first order condition $h'(x, g(x, s_i), s_i, V) = 0$. That is,

$$-u'(x - g(x, s_i)) + \beta \sum_{j=1}^n P_{ij} \rho(s_j) V'(\rho(s_j) g(x, s_i) + y(s_j), s_j) = 0.$$

3.3 General comparative statics theorem

Let (I, \leq) be a poset: I is a set and \leq is a binary relation over I that is reflexive, antisymmetric and transitive.⁶ The set I will serve as a set of parameters that affect the consumption-savings problem faced by the agent. In the applications below the parameter e will play several roles, such as the interest rate policy ρ , the discount factor β , the labor income function y and the distribution governing the states dynamic P .

Throughout the discussion we assume that the parameter e does not change the interval from which the agent chooses her level of consumption, i.e., for all $e \in I$, $C(x) = C(x, e) = [0, x]$. We slightly abuse the notations and allow an additional argument in the functions defined above. For instance, the value function of the parameterized consumption-savings problem $V \in B(Z \times I)$ is denoted by

$$V(x, s, e) = \max_{b \in C(x)} h(x, b, s, e, V).$$

Likewise, the savings policy function is denoted by $g(x, s, e)$, the consumption policy function by $\sigma(x, s, e)$ and $h(x, b, s, e, V)$ is the h function associated with the consumption-savings problem with parameter e , as defined above in Eq. (2).

We are interested in the question whether for every $e_1, e_2 \in I$ such that e_2 is greater than e_1 (i.e., $e_2 \geq e_1$) the savings related to e_2 , $g(x, s, e_2)$ are greater than or equal to those related to e_1 (i.e., $g(x, s, e_1)$) for all $(x, s) \in Z$.

We introduce a condition that ensures that the answer to the last question is affirmative.

Definition 3. *Consider a parameterized consumption-savings problem.⁷ We say that h has the complementation-preserving (CP) property if for every differentiable function $f \in B(Z \times I)$, $f'(x, s, e)$ increasing in e implies that $h'(x, b, s, e, f(x, s, e))$ is increasing in e for all $(x, s) \in Z$, and $b \in C(x)$.*

⁶Formally, \leq is a partial order that for every $e, e', e'' \in I$ satisfies (i) $e \leq e$; (ii) $e \leq e'$ and $e' \leq e$ imply $e = e'$; and (iii) $e \leq e'$ and $e' \leq e''$ imply $e \leq e''$.

⁷We omit the reference to the poset (I, \leq) , the set of parameters.

The CP property means that if f has increasing differences⁸ in (x, e) , i.e., the cash-on-hand x and the parameter e are complementary; then h has increasing differences in (b, e) , i.e., the savings b and the parameter e are also complementary. The next theorem states that if h has the CP property, then the savings are increasing in the parameter e , i.e., $g(x, s, e)$ is increasing in e for all $(x, s) \in Z$. As we shall see in Sections 4-6 it is often easy to verify that h satisfies the CP property.

Theorem 1. *Assume that h has the CP property. Then, for every $(x, s) \in Z$, $g(x, s, e)$ is increasing and $\sigma(x, s, e)$ is decreasing in e .*

Theorem 1 states that comparative statics results in consumption-savings dynamic models can be obtained by checking a simple property. Given a poset I , when the fact that $V'(x, s, e)$ is increasing in e for all $(x, s) \in Z$ implies that $h'(x, b, s, e, V(x, s, e))$ is increasing in e for all $(x, s) \in Z$ and $b \in C(x)$, one can conclude that $g(x, s, e)$ is increasing in e for all $(x, s) \in Z$. Similarly, when the fact that $V'(x, s, e)$ is decreasing in e for all $(x, s) \in Z$ implies that $h'(x, b, s, e, V(x, s, e))$ is decreasing in e for all $(x, s) \in Z$ and $b \in C(x)$, one can conclude that $g(x, s, e)$ is decreasing in e for all $(x, s) \in Z$.

The proof uses the following lemma, whose proof, like all other proofs, except for that of Theorem 1, are deferred to the Appendix.

Lemma 2. *Let $z : \mathbb{R} \rightarrow \mathbb{R}$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ be strictly concave, continuously differentiable functions. Let $\phi > 0$. Denote $x_f = \operatorname{argmax}_{x \in [0, \phi]} f(x)$ and $x_z = \operatorname{argmax}_{x \in [0, \phi]} z(x)$.*

(i) Assume $x_z \in (0, \phi)$. Then $f'(x_z) \geq z'(x_z)$ if and only if $x_f \geq x_z$.

(ii) If for all $x \in [0, \phi]$ we have $f'(x) \geq z'(x)$, then $x_f \geq x_z$. Furthermore, if $x_z \in (0, \phi)$ and $f'(x) > z'(x)$ for all $x \in [0, \phi]$, then $x_f > x_z$.⁹

Proof of Theorem 1. Assume that $f(x, s, e) \in B(Z \times I)$ is concave in the first argument and that the derivative of f is increasing in e (i.e., $e_2 \geq e_1$ implies $f'(x, s, e_2) \geq f'(x, s, e_1)$) for all $(x, s) \in Z$. The constant function $f \equiv 0$, for instance, satisfies these conditions.

Let $e_2 \geq e_1$, $(x, s) \in Z$ and $b \in C(x)$. A standard argument shows that Tf is strictly concave and bounded. The envelope theorem implies that Tf is differentiable. We now show that $(Tf)'$ is increasing in e . Define $g_f(x, s, e) = \operatorname{argmax}_{b \in C(x)} h(x, b, s, e, f)$. Since h has the CP property we have

$$h'(x, b, s, e_2, f(x, s, e_2)) \geq h'(x, b, s, e_1, f(x, s, e_1)).$$

⁸Let I be a poset and let $O \subseteq X \times I$. Let f be a real valued function defined on O . f is said to have increasing differences in (x, e) if for all $e_2 \geq e_1$, $f(x, e_2) - f(x, e_1)$ is increasing in x . Thus, if $\partial f(x, e)/\partial x$ is increasing in e then f has increasing differences in (x, e) .

⁹The condition $f'(x) \geq z'(x)$ is equivalent to the single crossing property in Milgrom and Shannon (1994): for all $x_2 \geq x_1$, $z(x_2) - z(x_1) \geq 0$ implies $f(x_2) - f(x_1) \geq 0$.

Lemma 2(ii) implies that $g_f(x, s, e_2) \geq g_f(x, s, e_1)$. Since for all $e \in I$ we have $C(x, e) = C(x) = [0, x]$, $g_f(x, s, e_2) \geq g_f(x, s, e_1)$ implies that $\sigma_f(x, s, e_1) = x - g_f(x, s, e_1) \geq x - g_f(x, s, e_2) = \sigma_f(x, s, e_2)$. The assumption $u'(0) = \infty$ implies that $\sigma_f(x, s, e_2) > 0$. From the envelope condition and the concavity of u we obtain

$$(5) \quad (Tf)'(x, s, e_2) = u'(\sigma_f(x, s, e_2)) \geq u'(\sigma_f(x, s, e_1)) = (Tf)'(x, s, e_1).$$

That is, $(Tf)'$ is increasing in e .

Define the sequence $f_n = T^n f$, $n = 1, 2, \dots$. Then f_n is strictly concave and f'_n is increasing in e for each n . From the Banach Theorem¹⁰ f_n converges uniformly to V . The envelope theorem implies that V is differentiable. Since $(Tf_n)' = u'(\sigma_{f_n})$ for each n and¹¹ $\sigma_{f_n} \rightarrow \sigma$ we have $(Tf_n)' \rightarrow V'$. Thus, $V'(x, s, e)$ is increasing in e . From the same argument as in Eq. (5), $\sigma(x, s, e)$ is decreasing in e and $g(x, s, e)$ is increasing in e for all $(x, s) \in Z$. \square

4 Comparing interest rate policies

Milgrom and Shannon (1994) develop a technique to derive comparative statics results without using the implicit function theorem or any differentiability assumptions. This technique is called monotone comparative statics.¹² In Topkis (1978), the properties of super-modular functions are studied and conditions for determining whether the argmax correspondence is monotone with respect to a parameter are established. In Hopenhayn and Prescott (1992) this technique is applied also to dynamic economies. For a survey of the use of super-modularity to derive comparative statics results in economics, see Amir (2005). These papers employ lattice programming in order to derive comparative statics results. We, in contrast, use the concavity and differentiability of the objective function h and obtain comparative statics results in dynamic economies in a simple manner. We elaborate on this subject in Section 7. In this section we present the comparative statics result regarding the effect of interest rate policies on consumption and savings. Let ρ_1, ρ_2 be two interest rate policies. We order the set of interest rate policies with the usual product order, i.e., we write $\rho_2 \geq \rho_1$ if $\rho_2(s) \geq \rho_1(s)$ for each state $s \in S$. Let $\sigma(x, s, \rho)$ be the optimal consumption policy function when the interest rate policy is ρ .

The following definition relates to a comment we made in the introduction regarding the fact that the substitution effect must dominate the income effect in order for an increase in interest rates to result in an increase in consumption.

¹⁰See Theorem 3.48 in Aliprantis and Border (2006).

¹¹See Theorem 3.8 in Stokey and Lucas (1989).

¹²See also Athey (2002), Quah (2007) and Quah and Strulovici (2009).

Definition 4. Let $W = (0, w]$. We say that the utility function exhibits substitution effect domination (SED) over W , if $cu'(c)$ is increasing on W .¹³

The SED is strongly related to the notion of relative risk aversion (RRA).¹⁴ Indeed, when u is twice continuously differentiable, u exhibiting SED over W is equivalent to the RRA being less than one on W . The main theorem, Theorem 2, states that if the utility function exhibits SED over W and the consumption policy function is concave in the cash-on-hand, then $\rho_2 \geq \rho_1$ implies $\sigma(x, s, \rho_1) \geq \sigma(x, s, \rho_2)$ for all $(x, s) \in W \times S$.

In order to provide a motivation for the condition that $cu'(c)$ is increasing, we show that this condition is equivalent to an increasing differences condition on the utility function that implies that wealth and interest rates are complementary.

Suppose that an agent lives for one period and has an indirect utility from an amount of wealth $a \in W$ and an interest rate $r \in [1, K]$. This agent who lives for one period consumes all her wealth, her consumption is given by ra and her indirect utility function is given by $u(ra)$. When the interest rate rises it is not clear whether the marginal contribution of wealth to the agent's indirect utility is positive. Let $a_2 \geq a_1, r_2 \geq r_1$. With a higher interest rate, the marginal contribution is higher in terms of consumption, i.e., $r_2a_2 - r_2a_1 \geq r_1a_2 - r_1a_1$. The question is what the marginal utility would then be.

Since the marginal utility from consumption is strictly decreasing, the difference in the LHS of the last inequality could, in principle, be smaller than the difference in the RHS in terms of utility: $u(r_2a_2) - u(r_2a_1) \leq u(r_1a_2) - u(r_1a_1)$. The condition that u exhibit SED over W implies that the marginal contribution of wealth to the utility is increasing with the interest rate. In other words, $u(r_2a_2) - u(r_2a_1) \geq u(r_1a_2) - u(r_1a_1)$, which is the property known as *increasing differences in (r, a)* .

To see the equivalence between increasing differences in (r, a) and the SED property, fix $a_1, a_2 \in W$ such that $a_2 \geq a_1$. Define the marginal contribution of wealth to the utility to be $m(r) := u(ra_2) - u(ra_1)$. Note that $m(r)$ is increasing in r if and only if $a_2u'(ra_2) \geq a_1u'(ra_1)$, which in turn is true if and only if $c_2u'(c_2) \geq c_1u'(c_1)$. Thus, the function $u(ra)$ has increasing differences in (r, a) on $[1, K] \times W$ if and only if u exhibits SED over W .

As noted in the introduction, even in a two period consumption-savings model, it could be the case that savings decrease when the interest rate increases. This happens when the income effect dominates the substitution effect. The following example shows, however, that when u exhibits SED over W , the substitution effect dominates the income effect.

¹³We restrict the condition $cu'(c)$ to be increasing on $W = (0, w]$ because of technical reasons. $w > 0$ is a real number that can be as large as we want. The reason for this restriction is that we assume that u is bounded in order to use the contraction mapping theorem. However, for every bounded utility function it cannot be the case that $cu'(c)$ is increasing on the domain $[0, \infty)$.

¹⁴The measure of RRA is defined as $R(c) = -\frac{cu''(c)}{u'(c)}$.

Example 1. Consider an agent who lives for two periods. Let $\phi > 0$. The agent has $x \in (0, \phi)$ dollars at the start of the first period and she receives an income of $y(s_j)$ with probability θ_j in the second period. The agent decides how much to consume in each period. If the agent consumes $0 \leq c \leq x$ in the first period and her income in the second period is $y(s_j)$ then the agent's consumption in the second period is $r(x - c) + y(s_j)$, where $r \in [1, K]$ is the interest rate on the agent's savings $x - c$. The agent chooses $0 \leq c \leq x$ to maximize her expected utility

$$U(x, c, r) := u(c) + \sum_{j=1}^n \theta_j u(r(x - c) + y(s_j)).$$

When the interest rate increases the income and substitution effects have their impact on the agent's consumption choice. The substitution effect dominates the income effect if the agent's consumption $\sigma(x, r)$ is decreasing in r . We show that if u exhibits SED over $W := [0, K\phi + y(s_n)]$, then $\sigma(x, r)$ is decreasing in r for all $x \in (0, \phi)$. Note that u exhibiting SED over W together with the fact that u' is decreasing imply that for all $x \in (0, \phi)$ and each $s \in S$ the function $ru'(rx + y(s))$ is increasing in r on $[1, K]$. Let $r_2 \geq r_1$, we have

$$\begin{aligned} U'(x, c, r_2) &= u'(c) - \sum_{j=1}^n \theta_j r_2 u'(r_2(x - c) + y(s_j)) \\ &\leq u'(c) - \sum_{j=1}^n \theta_j r_1 u'(r_1(x - c) + y(s_j)) = U'(x, c, r_1). \end{aligned}$$

Lemma 2 implies that $\sigma(x, r_1) \geq \sigma(x, r_2)$ for all $x \in (0, \phi)$. Therefore, if the utility function exhibits SED then the substitution effect dominates the income effect.

Given an interest rate policy ρ we say that the consumption function is concave if for each state $s \in S$, $\sigma(x, s, \rho)$ is concave in x on $(0, \infty)$. Zeldes (1989) and Deaton (1991) have noted in numerical studies that when the agent's labor income function is uncertain, the consumption function is concave in the cash-on-hand. Carroll and Kimball (1996) prove concavity of the consumption function analytically for the key class of hyperbolic absolute risk aversion (HARA)¹⁵ utility functions. Jensen (2015) offers a simpler proof of Carroll and Kimball's concavity result and generalizes the result to a framework similar to ours, with borrowing constraints and a Markov earnings process. Concavity of the consumption function is also consistent with empirical evidence. Mian et al. (2013) provide detailed empirical evidence that the consumption function is concave in the cash-on-hand and show empirically that the average marginal propensity to consume (MPC) decreases with the cash-on-hand. They show that the average MPC for households living in ZIP codes with

¹⁵Recall that a utility function is in the class of HARA utility functions if its absolute risk aversion $A(c)$ is hyperbolic. That is, $A(c) := -\frac{u''(c)}{u'(c)} = \frac{1}{ac+b}$ for $c > \frac{-b}{a}$.

an average annual income of less than 35,000 dollars is three times as large as the MPC for households living in ZIP codes with an average annual income of more than 200,000 dollars. Jappelli and Pistaferri (2014) also find that households with low cash-on-hand exhibit a higher MPC than households with high cash-on-hand.

Theorem 2. *Let $\phi > 0$. Assume that u exhibits SED on $W := (0, K\phi + y(s_n)]$ and that the consumption policy function is concave in x . Then $\rho_2 \geq \rho_1$ implies that $\sigma(x, s, \rho_1) \geq \sigma(x, s, \rho_2)$ for all $x \in (0, \phi]$ and $s \in S$.*

Let $\alpha \geq 0$. A utility function is in the CRRA class if $u(c) = c^{1-\alpha}/(1-\alpha)$ for $\alpha \neq 1$ and $u(c) = \ln(c)$ if $\alpha = 1$. It is known that the substitution effect dominates the income effect if the RRA is not greater than one,¹⁶ i.e., $\alpha \leq 1$. If $u(c)$ is in the CRRA class then the consumption policy function is concave in the cash-on-hand.¹⁷ From the above, we can conclude the next corollary.

Corollary 1. *Let $\alpha \in (0, 1]$. Assume that on W , $u(c) = c^{1-\alpha}/(1-\alpha)$ when $\alpha \neq 1$ and $u(c) = \ln(c)$ when $\alpha = 1$. Then, $\rho_2 \geq \rho_1$ implies $\sigma(x, s, \rho_1) \geq \sigma(x, s, \rho_2)$ for all $(x, s) \in W \times S$.*

The major implications of Theorem 2 relate to Keynesian policies, i.e., interest rate policies that assign higher interest rates to states with higher incomes and lower ones to states with lower incomes. These policies are Keynesian in the sense that when the economy is in a high-income state, the interest rate is high while in a low-income state the interest rate is low. Keynesian policies can be induced by a Taylor rule or by other monetary policy rules. Theorem 2 states that if the central bank changes its policy and assigns higher interest rates to high-income states, the agent's savings will increase also in low-income states. More precisely, consider two different Keynesian interest rate policies ρ_1, ρ_2 such that $\rho_1(s_i) = \rho_2(s_i)$ for $i \leq k$ and $\rho_1(s_i) \leq \rho_2(s_i)$ for $i > k$ for some $1 < k < n$. Both policies are identical except that in the second policy ρ_2 the interest rates are higher in the high-income states. Theorem 2 implies that the agent's savings are higher under the interest rate policy ρ_2 than under the interest rate policy ρ_1 in all states, including those states in which both policies agree.

From a monetary policy perspective consider a central bank that chooses an interest rate policy in order to increase consumption to fight a recession. The central bank considers two interest rate policies ρ_1, ρ_2 with a zero interest rate for the low-income states of the economy $\rho_1(s_i) = \rho_2(s_i) = 1$ for $i \leq k$. In effect, the central bank considers the interest rates of the high-income states of the economy. Will an interest rate policy that assigns lower interest rates to high-income states of the economy be more effective (in the sense that

¹⁶Thus, u exhibits SED if $\alpha \leq 1$.

¹⁷Recall that if a utility function is in the CRRA class it also belongs to the class of HARA utility functions.

it will increase consumption in the low-income states of the economy) than an interest rate policy that assigns higher interest rates to high-income states of the economy? Theorem 2 provides conditions under which the answer is affirmative. Diamond and Rajan (2012) study, in a different model from ours, the effects of policy interventions that alter interest rates on banks' optimal decisions. They find that the central bank's willingness to intervene when liquidity needs are high by pushing down interest rates encourages banks to make commitments that increase the need for intervention.

We end this section with the case in which the agent has negative cash-on-hand. Assume that $x_0 < 0$ is some borrowing limit. For a discussion on borrowing limits see Aiyagari (1994). Suppose that the agent with a cash-on-hand of x chooses her savings level from the set $[x_0, x]$. In particular she may choose to save a negative amount. It is clear that when the agent has a negative cash-on-hand, an increase in interest rates has a negative income effect. In this case the income effect and the substitution effect decrease consumption when the interest rate is higher, making the next proposition intuitive.

Proposition 2. *Let x be such that $x_0 \leq x \leq 0$ (i.e., x is non-positive). Then, $\rho_2 \geq \rho_1$ implies that $\sigma(x, s, \rho_1) \geq \sigma(x, s, \rho_2)$ for all $s \in S$.*

5 Comparing labor income functions

Let y_1, y_2 be two labor income functions. We order labor income functions in a natural way: $y_2 \geq y_1$ if $y_2(s) \geq y_1(s)$ for each state $s \in S$. In this section we examine the impact on consumption of increasing the underlying labor income function. We first show that for two labor income functions y_1, y_2 , if $y_2 \geq y_1$ then the agent consumes more under y_2 than under y_1 in each state of the economy and at every cash-on-hand level. That is, $\sigma(x, s, y_2) \geq \sigma(x, s, y_1)$ for every $(x, s) \in Z$. This result is related to the permanent income hypothesis (Friedman (1957)) which claims that an agent's consumption is determined by the agent's expected future income. We say that $P > 0$ if the probability to move from any state s_i to any other one, s_j , is positive (i.e., $P_{ij} > 0$ for every ij). If $y_2 \geq y_1$, $y_1 \neq y_2$ and $P > 0$, then the agent's present discounted value of expected future income under the labor income function y_2 is *strictly* higher than under the labor income function y_1 . Thus, the permanent income hypothesis would predict that the agent's consumption is higher in each state and at every cash-on-hand level. We show that it is indeed the case. This result is consistent with empirical evidence, as noted in Jappelli and Pistaferri (2010). We formulate these statements in the following theorem.

Theorem 3. *Fix the interest rate policy ρ . Let $y_1 \neq y_2$ be two labor income functions such that $y_2 \geq y_1$. Then,*

(i) $\sigma(x, s, y_2) \geq \sigma(x, s, y_1)$ for all $(x, s) \in Z$.

(ii) If $P > 0$ then inequality (i) is strict for $(x, s) \in Z$ that satisfies $\sigma(x, s, y_1) \in (0, x)$.

Consider two agents that are identical except for their income function: they have the same amount of cash-on-hand and the same utility function, but one has an income function y_1 and the other y_2 . Assume that $y_2 \geq y_1$. We measure the utility inequality among the two agents by considering the relative values of their expected present discounted utility. Define, $r(x, s) := \frac{V(x, s, y_1)}{V(x, s, y_2)}$. The smaller $r(x, s)$ is, the larger the relative utility inequality. The next proposition shows that the utility inequality is getting smaller as the level of cash-on-hand is getting higher.

Proposition 3. *Let $y_2 \geq y_1$. Then, the expected utility inequality $r(x, s)$ is decreasing in x for each state $s \in S$.*

6 Other applications

In this section we apply Theorem 1 in order to derive additional comparative statics results.

6.1 Comparison between different discount factors

Miao (2002) proves that higher discounting leads to higher savings. Miao's proof relies on lattice programming arguments. We present here a simpler proof. We parameterize the consumption-savings problem with a parameter β , that stands for the discount factor. Let $I = (0, 1)$ be the set of possible discount factors, endowed with natural order: $\beta_2 \geq \beta_1$ if β_1 is lower than β_2 .

Theorem 4. *Let $\beta_2 \geq \beta_1$ be two discount factors, then $g(x, s, \beta_2) \geq g(x, s, \beta_1)$ for all $(x, s) \in Z$. Furthermore, if $g(x, s, \beta_1) \in (0, x)$ and $\beta_2 > \beta_1$, then $g(x, s, \beta_2) > g(x, s, \beta_1)$.*

6.2 Comparison between different transition probabilities

Throughout this section we assume that the interest rate policy is constant across states, i.e., $\rho(s) = r$ for all states $s \in S$. Let I be the set of all $n \times n$ transition matrices. We endow this set with a partial order \geq defined as follows. For $P, Q \in I$, we say that $P \geq Q$ if for every i the distribution $(P_{ij})_j$ stochastically dominates $(Q_{ij})_j$ in the sense that for every increasing and bounded function $f : S \rightarrow \mathbb{R}$,

$$\sum_{j=1}^n P_{ij} f(s_j) \geq \sum_{j=1}^n Q_{ij} f(s_j).$$

We show that the consumption policy function is increasing in the state of the economy when the probabilities are inertial. We use this result to show that for two transition matrices P, Q if P stochastically dominates Q , then the agent consumes more under the transition matrix P than under the transition matrix Q for every cash-on-hand level and each state of the economy. This result can be interpreted in two ways. First, the transition matrix can be thought of as the agent's subjective beliefs about her future income. The fact that $P \geq Q$ means the agent with beliefs P is more optimistic about her future income than the one with beliefs Q . As a result, the former consumes more. The second interpretation is that the economy might undergo a technological improvement and thereby move from a transition matrix Q to the "better" transition matrix P .

We say that V' is decreasing in s for all $x \in X$ if $V'(x, s_i) \leq V'(x, s_k)$ for all $i > k$ and $x \in X$. As $V(x, s)$ is the agent's maximal expected utility when the state is s and the cash-on-hand is x , $V'(x, s)$ decreasing in s means that the marginal contribution of cash-on-hand to the maximal expected utility decreases with the state of the economy. Thus, if $V'(x, s)$ is decreasing in s then the marginal contribution of cash-on-hand is higher when the agent has a lower income. This property seems intuitive if moving to higher income states is more likely from a high income state than from a low income state. Lemma 3 shows that this is indeed the case.

Definition 5. *The transition probabilities are inertial if for every ℓ , the sum $\sum_{j=\ell}^n P_{ij}$ is nondecreasing in i .*

When the transition matrix is inertial, moving to higher-income states is more likely from a high income state than from a low income state. Equivalently, if $i > k$, then the distribution $(P_{ij})_j$ stochastically dominates $(P_{kj})_j$ (i.e., for every increasing and bounded function $f : S \rightarrow \mathbb{R}$, $\sum_{j=1}^n P_{ij} f(s_j) \geq \sum_{j=1}^n P_{kj} f(s_j)$).

Lemma 3. *Assume that the transition probabilities are inertial and $\rho(s) = r$ for all states $s \in S$. Then, $V'(x, s)$ is decreasing in s for all $x \in X$ and the consumption policy $\sigma(x, s)$ is increasing in s for each $x \in X$.*

Theorem 5. *Assume that the transition probabilities P are inertial and $\rho(s) = r$ for all states $s \in S$. If $P \geq Q$, then $\sigma(x, s, P) \geq \sigma(x, s, Q)$ for all $(x, s) \in Z$.*

6.3 Prudence and precautionary savings

Precautionary savings are the savings that the agent sets aside in response to uncertainty regarding future income. An agent is prudent if the marginal utility u' is convex, i.e., if $u''' \geq 0$.¹⁸ The condition $u''' \geq 0$ implies that the absolute risk aversion $-\frac{u''(c)}{u'(c)}$ is decreasing.

¹⁸See Kimball (1990) for a measure of the agent's prudence.

In this section we show that prudence implies precautionary savings when the earnings process is i.i.d.

Schechtman (1976) shows that if the earnings process y is i.i.d. with some distribution θ , i.e., $P_{ij} = \theta_j$ for all i , and u' is convex, then the agent saves more when he has the deterministic income that coincides with the average income $\mathbb{E}_\theta(y)$, than when he has the uncertain income y .

Definition 6. (*Rothschild and Stiglitz (1970)*). We say that a distribution θ is riskier than θ' if θ' second order stochastically dominates θ . That is, for all concave, increasing and bounded functions f we have

$$\sum_{j=1}^n \theta'_j f(x_j) \geq \sum_{j=1}^n \theta_j f(x_j).$$

The intuition behind this definition is that θ is riskier than θ' if every risk averter (i.e., an agent whose utility function is concave, increasing and bounded) would prefer θ' to θ . Miller (1976) and Huggett (2004) show that for each level of cash-on-hand the agent's savings increase if θ is riskier than θ' . We provide here a simple proof that relies on Theorem 1. Let I be the set of all distributions on S . We endow I with a partial order. For two distributions $\theta, \theta' \in S$ we write $\theta \geq \theta'$ if θ is riskier than θ' .

Theorem 6. Assume that $P_{ij} = \theta_j$ for all $s_i \in S$ and that the interest rate is r . In this case the savings policy function does not depend on the state s . Let θ and θ' be two distributions. Assume that $\theta \geq \theta'$ and that u' is convex. Then, for all $x \in X$, $g(x, \theta) \geq g(x, \theta')$.

Theorem 6 states that when earnings shocks are i.i.d. and the agent is prudent (i.e., the marginal utility from consumption is convex), the agent's savings increase when the earnings risk increases (in the sense of second order stochastic dominance). Thus, prudence generates precautionary savings when the earnings shocks are i.i.d.¹⁹ We note that it remains an open question whether this result can be extended to a case where the agent's earnings follow a Markov process. In other words, it is not known to us whether convexity of the marginal utility is sufficient to generate precautionary savings.

7 The approach of Hopenhayn and Prescott

The approach of Hopenhayn and Prescott (henceforth, HP) to deriving comparative statics results uses lattice programming techniques. Consider the parameterized consumption-savings problem introduced in Section 3.3. In our framework, Theorem 6.2 in Topkis (1978) implies that if the objective function h has increasing differences in (b, e) for all $(x, s) \in Z$

¹⁹This statement is true also when earnings shocks are not necessarily identical but independent over time (see Miller (1976)).

then $g(x, s, e)$ is increasing in e for all $(x, s) \in Z$. HP identify the conditions that imply that the value function has increasing differences in (b, e) and use Topkis's theorem to derive comparative statics results. HP's approach is summarized by the following Proposition. We phrase Proposition 2 of Hopenhayn and Prescott (1992) differently in order to fit it to our framework, and then compare it to our approach.²⁰

Proposition 4. *If V having increasing differences in (x, e) implies that h has increasing differences in (x, b, e) for each $s \in S$ then $g(x, s, e)$ is increasing in e for all $(x, s) \in Z$.*

The idea of the proof goes as follows. If h has increasing differences in (x, b, e) , then Lemma 1 in Hopenhayn and Prescott (1992) shows that TV has increasing differences in (x, e) . Since the set of functions that has increasing differences is closed (see Topkis (2011)), Lemma 1 implies that V has increasing differences in (x, e) . Thus, h has increasing differences in (x, b, e) which from Theorem 6.2 in Topkis (1978) implies that $g(x, s, e)$ is increasing in e for all $(x, s) \in Z$. This approach is more general than ours, as it does not require concavity of the utility function or differentiability. To apply Topkis's theorem one only needs h to have increasing differences in (b, e) . However, HP require that h have increasing differences in (x, b, e) . This is required in order to prove that TV has increasing differences. Our approach uses concavity and the envelope condition in order to show that TV preserves increasing differences.

If one assumes differentiability and concavity, our approach is stronger than that of HP in the following sense. HP's technique relies on the fact that whenever V has increasing differences in (x, e) , the function h has increasing differences in (x, b, e) . Our approach, by contrast, requires only that whenever V has increasing differences in (x, e) , the function h has increasing differences in (b, e) . This means that h has the CP property, which enables us to use Theorem 1. Thus, whenever HP's technique can be used, ours can be used as well.

In some applications, however, h does not have increasing differences in (x, b, e) , which does not allow the use of the lattice programming approach. In Section 5, for instance, we show that for two labor income functions y_1, y_2 if $y_2(s) \geq y_1(s)$ for each state $s \in S$, then for every $(x, s_i) \in Z$ the agent's consumption is higher under y_2 than under y_1 . We prove this statement by showing that whenever V has increasing differences in $(-x, y)$, the function h has increasing differences in $(-b, y)$. However, h does not necessarily have increasing differences in $(-x, -b, y)$. It is therefore impossible to implement HP's approach in this case.

²⁰The condition in Proposition 2 that the correspondence C is ascending is omitted here because it is always satisfied in our framework.

8 Final comments

8.1 Summary

In this paper we find conditions that guarantee that consumption increases as a result of lowering interest rates. We introduce a method to derive general comparative statics results in dynamic economies and apply it to derive a few other comparative statics results related to income functions, discount factors and states-dynamics.

8.2 Stationary vs. non-stationary interest rate policies

In our framework the interest rate policy is stationary: it is a function that depends only on the state of the economy and it does not depend on time (in particular, it does not depend on the history of states.) We leave it to future research to find out the effect of non-stationary interest rate policies on consumption and savings decisions.

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9 Appendix

Lemma 2. *Let $z : \mathbb{R} \rightarrow \mathbb{R}$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ be strictly concave, continuously differentiable functions. Let $\phi > 0$. Denote $x_f = \operatorname{argmax}_{x \in [0, \phi]} f(x)$ and $x_z = \operatorname{argmax}_{x \in [0, \phi]} z(x)$.*

(i) *Assume $x_z \in (0, \phi)$. Then $f'(x_z) \geq z'(x_z)$ if and only if $x_f \geq x_z$.*

(ii) *If for all $x \in [0, \phi]$ we have $f'(x) \geq z'(x)$ then $x_f \geq x_z$. Furthermore, if $x_z \in (0, \phi)$ and $f'(x) > z'(x)$ for all $x \in [0, \phi]$ then $x_f > x_z$.*

Proof. Recall that if f is continuously differentiable and concave then for all x_1, x_2 we have $f(x_1) \leq f(x_2) + f'(x_2)(x_1 - x_2)$. Furthermore, if $f : \mathbb{R} \rightarrow \mathbb{R}$ is strictly concave and continuous on a compact subset $[0, \phi] \subseteq \mathbb{R}$ then f has a unique maximizer on $[0, \phi]$.

(i) First assume that $x_z \in (0, \phi)$ which implies from the optimality of x_z that $z'(x_z) = 0$. Since f is concave we have

$$f(x_f) \leq f(x_z) + f'(x_z)(x_f - x_z).$$

From the optimality of x_f we have $f(x_f) > f(x_z)$, thus $f'(x_z)(x_f - x_z) \geq 0$ which implies that $f'(x_z) \geq 0 = z'(x_z)$ if and only if $x_f \geq x_z$. Furthermore, if $f'(x_z) > z'(x_z) = 0$ then $x_z \neq x_f$. For part (ii) consider two cases. The first is where $x_z = 0$. In this case $x_f \geq 0 = x_z$. The second case is $x_z = \phi$ which implies $z'(x) > 0$ for all $x \in (0, \phi)$. Thus, $f'(x) > 0$ for all $x \in (0, \phi)$, implying that $x_f = \phi$. \square

Theorem 2. *Let $\phi > 0$. Assume that u exhibits SED on $W := (0, K\phi + y(s_n)]$ and that the consumption policy function is concave in x . Then $\rho_2 \geq \rho_1$ implies that $\sigma(x, s, \rho_1) \geq \sigma(x, s, \rho_2)$ for all $x \in (0, \phi]$ and $s \in S$.*

We first need the following two Lemmas.

Lemma 4. *Let $f : [0, \infty) \rightarrow [0, \infty)$ be a concave function that satisfies $f(0) = 0$. Then, the function $\frac{k}{f(k)}$ is increasing on $(0, \infty)$.*

Proof. Since f is concave then the function $\frac{f(k)-f(k_1)}{k-k_1}$ is decreasing in k for a fixed k_1 . For $k_1 = 0$ we obtain that $\frac{f(k)-f(0)}{k-0} = \frac{f(k)}{k}$ is decreasing in k , implying that $\frac{k}{f(k)}$ is increasing in k . \square

Lemma 5. *Fix the interest rate policy ρ and the labor income function y . Let $\phi > 0$. Assume that u exhibits SED on $W := (0, K\phi + y(s_n)]$ and that the consumption policy is concave. Then, for every $s \in S$, $a_1 \in (0, \phi]$, $a_2 \in [0, y(s_n)]$, the function $kV'(a_1k + a_2, s)$ is increasing in k on $[1, K]$.*

Proof. First we show that for every state $s \in S$, $xV'(x, s)$ is increasing in x on W . Let $s \in S$, $a_1 \in (0, \phi]$ and $a_2 \in [0, y(s_n)]$. The envelope condition implies

$$xV'(x, s) = xu'(\sigma(x, s)) = \frac{x}{\sigma(x, s)}\sigma(x, s)u'(\sigma(x, s)).$$

Let $\alpha(x) := \frac{x}{\sigma(x, s)}$ and $\delta(x) := \sigma(x, s)u'(\sigma(x, s))$. Since $\sigma(x, s)$ is concave in x and $\sigma(0, s) = 0$, Lemma 4 implies that $\alpha(x)$ is increasing in x . Since $\sigma(x, s)$ is strictly increasing²¹ in x , and $cu'(c)$ is increasing on W , $\delta(x)$ is also increasing in x , over the range W . Thus, $xV'(x, s)$ is increasing in x on W as the product of two positive increasing functions. Thus, $(a_1k + a_2)V'(a_1k + a_2, s) = a_1kV'(a_1k + a_2, s) + a_2V'(a_1k + a_2, s)$ is increasing in k on $[1, K]$. Since V is concave in the first argument, $a_2V'(a_1k + a_2, s)$ is decreasing in k . This implies that $kV'(a_1k + a_2)$ is increasing in k on $[1, K]$, which proves the lemma. \square

Proof of Theorem 2. Let I be the set of interest rate policies. We show that h has the CP property. Let $\rho_2 \geq \rho_1$ and assume that $V'(x, s, \rho_2) \geq V'(x, s, \rho_1)$ for all $(x, s) \in W \times S$. Fix $x \in (0, \phi]$, $s \in S$ and $b \in [0, x]$. From Lemma 5,

$$\rho_2(s)V'(\rho_2(s)b + y(s), s, \rho_2) \geq \rho_1(s)V'(\rho_1(s)b + y(s), s, \rho_2).$$

This inequality and $V'(x, s_i, \rho_2) \geq V'(x, s_i, \rho_1)$ for all $(x, s_i) \in W \times S$ imply that

$$\begin{aligned} \beta \sum_{j=1}^n P_{ij}\rho_2(s_j)V'(\rho_2(s_j)b + y(s_j), s_j, \rho_2) &\geq \beta \sum_{j=1}^n P_{ij}\rho_1(s_j)V'(\rho_1(s_j)b + y(s_j), s_j, \rho_2) \\ &\geq \beta \sum_{j=1}^n P_{ij}\rho_1(s_j)V'(\rho_1(s_j)b + y(s_j), s_j, \rho_1). \end{aligned}$$

Adding $-u'(x - b)$ to both sides of the last inequality yields

$$h'(x, b, s_i, \rho_2, V(x, s, \rho_2)) \geq h'(x, b, s_i, \rho_1, V(x, s, \rho_1)).$$

We conclude that h has the CP property. Theorem 1 proves the theorem. \square

Proposition 2. *Let x be such that $x_0 \leq x \leq 0$. Then, $\rho_2 \geq \rho_1$ implies that $\sigma(x, s, \rho_1) \geq \sigma(x, s, \rho_2)$ for all $s \in S$.*

²¹Recall that $V'(x, s) = u'(\sigma(x, s))$ is strictly decreasing in x for each $s \in S$.

Proof. Let $x \in [x_0, 0]$. Since V' is decreasing and $b \leq x \leq 0$ we have

$$\rho_2(s) V'(\rho_2(s)b + y(s), s, \rho_2) \geq \rho_1(s) V'(\rho_1(s)b + y(s), s, \rho_2)$$

for each state $s \in S$. We can continue as in the proof of Theorem 2 and prove the proposition. \square

Theorem 3 *Fix the interest rate policy ρ . Let $y_1 \neq y_2$ be two labor income functions such that $y_2 \geq y_1$. Then,*

(i) $\sigma(x, s, y_2) \geq \sigma(x, s, y_1)$ for all $(x, s) \in Z$.

(ii) If $P > 0$ (i.e., $P_{ij} > 0$ for every i, j) then inequality (i) is strict for $(x, s) \in Z$ that satisfies $\sigma(x, s, y_1) \in (0, x)$.

Proof. Let $y_2 \geq y_1$. Assume $V'(x, s_i, y_2) \leq V'(x, s_i, y_1)$ for all $(x, s_i) \in Z$. Let $(x, s_i) \in Z$ and $b \in C(x)$. For each state $s \in S$ we have the following inequality

$$\rho(s) V'(\rho(s)b + y_1(s), s, y_1) \geq \rho(s) V'(\rho(s)b + y_1(s), s, y_2).$$

Thus, we have

$$\begin{aligned} \sum_{j=1}^n P_{ij} \rho(s_j) V'(\rho(s_j)b + y_1(s_j), s_j, y_1) &\geq \sum_{j=1}^n P_{ij} \rho(s_j) V'(\rho(s_j)b + y_1(s_j), s_j, y_2) \\ &\geq \sum_{j=1}^n P_{ij} \rho(s_j) V'(\rho(s_j)b + y_2(s_j), s_j, y_2), \end{aligned}$$

where the second inequality follows from the strict concavity of V and the fact that $y_2 \geq y_1$. Multiplying by β and adding $-u'(x - b)$ to each side of the last inequality yields

$$(6) \quad h'(x, b, s_i, y_1, V(x, s, y_1)) \geq h'(x, b, s_i, y_2, V(x, s, y_2)).$$

This proves that h has the CP property. Theorem 1 implies that $g(x, s, y_1) \geq g(x, s, y_2)$ and $\sigma(x, s, y_2) \geq \sigma(x, s, y_1)$ for all $(x, s) \in Z$. Note that inequality (6) is strict if $y_2 \neq y_1$, $P > 0$. Now apply Lemma 2 to prove that $\sigma(x, s, y_2) > \sigma(x, s, y_1)$ for all (x, s) such that $\sigma(x, s, y_1) \in (0, x)$. \square

Proposition 3. *Let $y_2 \geq y_1$. Then the expected utility inequality $r(x, s)$ is decreasing in x for each state $s \in S$.*

Proof. First we show that $r(x, s) \leq 1$ for all $(x, s) \in Z$. The proof is by induction. Let $U_1 = U_2 = 0$. Define $U_i^t(x, s) = \max_{b \in C(x)} h(x, b, s, y_i, U_i^{t-1})$ for all $(x, s) \in Z$, $i = 1, 2$ and $t = 1, 2, \dots$. Then for every t , U_i^t is strictly increasing, continuous, strictly concave and

bounded. Assume that for some $t \geq 1$ we have $U_2^t \geq U_1^t$. Let $(x, s_i) \in Z$ and $b \in C(x)$. We have

$$\begin{aligned} U_2^{t+1}(x, s_i) &\geq u(x-b) + \beta \sum_{j=1}^n P_{ij} U_2^t(\rho(s_j)b + y_2(s_j), s_j) \\ &\geq u(x-b) + \beta \sum_{j=1}^n P_{ij} U_1^t(\rho(s_j)b + y_1(s_j), s_j). \end{aligned}$$

The first inequality follows from the definition of U_2^{t+1} . The second inequality follows from the induction hypothesis and the fact that U_1^t is increasing in the first argument. Taking the maximum in the RHS of the last inequality yields $U_2^{t+1} \geq U_1^{t+1}$. We conclude that for every $t \geq 1$ we have $U_2^t \geq U_1^t$.

The Banach Theorem and the continuity of V imply that $V(x, s, y_2) \geq V(x, s, y_1)$. Thus, $r(x, s) \leq 1$. To prove that $r(x, s)$ is increasing in x , let $s \in S$ and note that

$$\frac{\partial r(x, s)}{\partial x} = \frac{V'(x, s, y_1)V(x, s, y_2) - V(x, s, y_1)V'(x, s, y_2)}{[V(x, s, y_2)]^2} \geq 0,$$

where the inequality follows from $V(x, s, y_2) \geq V(x, s, y_1) \geq 0$ and from Theorem 3, which implies that $V'(x, s, y_1) \geq V'(x, s, y_2) \geq 0$. \square

Theorem 4. *Let $\beta_2 > \beta_1$ be two discount factors, then $g(x, s, \beta_2) \geq g(x, s, \beta_1)$ for all $(x, s) \in Z$. Furthermore, if $g(x, s, \beta_1) \in (0, x)$ then $g(x, s, \beta_2) > g(x, s, \beta_1)$.*

Proof. Assume that $V'(x, s, \beta)$ is increasing in β . Let $(x, s_i) \in Z$ and $b \in C(x)$. Since V' is increasing in β for $\beta_2 \geq \beta_1$ we have

$$\sum_{j=1}^n P_{ij} \rho(s_j) V'(\rho(s_j)b + y(s_j), s_j, \beta_2) \geq \sum_{j=1}^n P_{ij} \rho(s_j) V'(\rho(s_j)b + y(s_j), s_j, \beta_1).$$

Multiplying the LHS of the last inequality by β_2 and the RHS of the last inequality by β_1 preserves the last inequality. Adding $-u'(x-b)$ to each side of the last inequality yields

$$(7) \quad h'(x, b, s_i, \beta_2, V(x, s, \beta_2)) \geq h'(x, b, s_i, \beta_1, V(x, s, \beta_1)),$$

which from Theorem 1 proves that $g(x, s, \beta_2) \geq g(x, s, \beta_1)$ for all $(x, s) \in Z$. Inequality (7) is strict if $\beta_2 > \beta_1$. Thus, Lemma 2 implies that if $g(x, s, \beta_1) \in (0, x)$, then $g(x, s, \beta_2) > g(x, s, \beta_1)$. \square

Lemma 3. *Assume that the transition probabilities are inertial and $\rho(s) = r$ for all $s \in S$. Then, $V'(x, s)$ is decreasing in s for all $x \in X$ and the consumption policy $\sigma(x, s)$ is increasing in s for each $x \in X$.*

Proof. Assume that $V'(x, s)$ is decreasing in s for all $x \in X$. Let $i > k$, $x \in X$, and $b \in C(x)$. Since V' is decreasing in s we have

$$V'(rb + y(s_k), s_k) \geq V'(rb + y(s_i), s_k) \geq V'(rb + y(s_i), s_i),$$

where the first inequality follows from the concavity of V . Thus, $V'(rb + y(s), s)$ is decreasing in s . Since the probabilities are inertial we have

$$\sum_{j=1}^n P_{kj} V'(rb + y(s_j), s_j) \geq \sum_{j=1}^n P_{ij} V'(rb + y(s_j), s_j).$$

Multiplying by βr and adding $-u'(x - b)$ to each side of the last inequality yields

$$h'(x, b, s_k, V) \geq h'(x, b, s_i, V).$$

Together with Theorem 1 this proves the Lemma. \square

Theorem 5. *Assume that the transition probabilities P are inertial and $\rho(s) = r$ for all $s \in S$. If $P \geq Q$ then $\sigma(x, s, P) \geq \sigma(x, s, Q)$ for all $(x, s) \in Z$.*

Proof. From Theorem 1 we need to show only that the fact that $V'(x, s, Q) \geq V'(x, s, P)$ for every $(x, s) \in Z$ implies that $h'(x, b, s, Q, V(x, s, Q)) \geq h'(x, b, s, P, V(x, s, P))$.

Let $(x, s_i) \in Z$ and $b \in C(x)$. Lemma 3 implies that $V'(rb + y(s), s)$ is decreasing in s . Since P first order stochastically dominates Q we have

$$\begin{aligned} \sum_{j=1}^n Q_{ij} V'(rb + y(s_j), s_j, Q) &\geq \sum_{j=1}^n Q_{ij} V'(rb + y(s_j), s_j, P) \\ &\geq \sum_{j=1}^n P_{ij} V'(rb + y(s_j), s_j, P). \end{aligned}$$

Multiplying by βr and adding $-u'(x - b)$ to each side of the last inequality yields

$$h'(x, b, s_i, Q, V(x, s, Q)) \geq h'(x, b, s_i, P, V(x, s, P)),$$

which proves the theorem. \square

Theorem 6. *Assume that $P_{ij} = \theta_j$ for all $s_i \in S$ and that the interest rate is r . In this case the savings policy function does not depend on the state s . Let θ, θ' be two distributions. Assume that θ is riskier than θ' and u' is convex. Then, for all $x \in X$, $g(x, \theta) \geq g(x, \theta')$.*

Proof. Lemma 1 in Miller (1976) shows that if $u'(c)$ is convex then $V'(x, \theta)$ is convex in x for a fixed distribution θ . Assume $V'(x, \theta) \geq V'(x, \theta')$ for all $x \in X$. Let $x \in X$ and $b \in C(x)$. We have

$$\begin{aligned} \sum_{j=1}^n \theta_j V'(rb + y(s_j), \theta) &\geq \sum_{j=1}^n \theta'_j V'(rb + y(s_j), \theta) \\ &\geq \sum_{j=1}^n \theta'_j V'(rb + y(s_j), \theta'), \end{aligned}$$

where the first inequality follows from the fact that the function $f(a) := V'(rb+a, \theta)$ is convex and non-increasing in a and from θ second order stochastically dominating θ' . Multiplying by βr and adding $-u'(x-b)$ to each side of the last inequality yields

$$h'(x, b, \theta, V(x, \theta)) \geq h'(x, b, \theta', V(x, \theta')),$$

which together with Theorem 1 proves that $g(x, \theta) \geq g(x, \theta')$ for all $x \in X$. □