PARTIALLY SPECIFIED PROBABILITIES: DECISIONS AND GAMES

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December 27, 2005

A very preliminary draft. Comments are welcome.

Abstract. In Ellsberg paradox decision makers that are partially informed about the actual probability distribution violate expected utility paradigm. This paper develops a theory of decision making with a *partially specified probability.* The paper takes an axiomatic approach using Anscombe-Aumann (1963) setting and is based on a concave integral for capacities (see Lehrer, 2005).

The partially specified decision making is carried on to games in order to introduce *partially specified equilibrium.*

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[*] The author acknowledges Yaron Azrieli and David Schmeidler for their helpful suggestions.

1. Introduction

Ellsberg paradox demonstrates a situation where decision makers violate expected utility theory. This violation stems from the fact that the decision makers receive only partial information about the actual probability distribution. This paper develops a theory of decision making with a *partially specified probability.* The paper takes an axiomatic approach using Anscombe-Aumann setting (see Anscombe and Aumann, 1963) and is based on the concave integral for capacities (see Lehrer, 2005).

The orthodox Bayesian (finite) model assumes that a decision maker assigns a probability to every event. In variations of this model and in those that relax this assumption, the probability might be either distorted (Quiggin, 1982 and Yaari, 1987), or non-additive (Schmiedler, 1989) or multiple (Gilboa and Schmeidler, 1989). In this paper the decision maker obtains a partial information about the underlying regular (i.e., additive) probability. This information might include the probability of some events, but not of all, or of the expected value of some random variables, but not of all.

An *act* assigns to every state of nature a lottery. Like in Anscombe and Aumann (1963), a preference order is defined over the set of acts. The axiomatization provided here is based on five axioms, all of a rather standard form: completeness, continuity, independence, monotonicity and ambiguity aversion. The first three are as in von-Neumann and Morgenstern (1944), the forth was added by Anscombe and Aumann (1963) and the fifth originates from Schmeidler (1989) and is used also by Gilboa and Schmeidler (1989). The difference between the current axiomatization and previous ones is in the formulation of the independence axiom. The latter takes here two versions that, in turn, yield two versions of the decision making models with a partially specified probability.

The main idea is to apply the independence axiom to acts that are "fat-free" and to those related to them. A *fat-free* act is characterized by the property that if a lottery assigned to a state by this act is replaced by a worse one, the resulting act is strictly inferior to the original one. In a fat-free act there is no fat that can be cut while maintaining the same quality of the act. In contrast, in an act that contains fat there is at least one state whose assigned lottery can be replaced by a worse one, without affecting its quality: the modified act is equivalent to the original one.

An act g is *derived* from an act f if any two states that are assigned the same lottery under g are also assigned the same lottery under f. Any act induces a partition of the state space into events: those subsets of states over which the act is constant. An act g is derived from an act f if the partition induced by f is finer than that induced by g. Technically speaking, this means that g is measurable with respect to f.

The first version of the independence axiom applies to the acts derived from the same fat-free act. This version suggests that the significant factor of a fat-free act is the structure of events over which it is constant. The precise lotteries assigned by the fat-free act are insignificant, as far as this version of the independence axiom is concerned. It turns out that along with the other axioms, this one implies that the preference order is determined by evaluating acts using a regular (additive) probability specified only over a sub-algebra of events. In other words, the deviation from a standard expected utility maximization is due to a lack of information about the actual probability distribution.

A second version of the independence axiom is restricted to fat-free acts and to those that can be expressed as a weighted average of them. This version, together with the other four axioms, implies that the preference order is determined by evaluating acts based on a regular probability specified over some events (that do not necessarily form a sub-algebra) or on some random variables. The latter means that the expected values of some random variables are given.

Obtaining information about the expectation of random variables is particularly relevant in dynamic situations. Suppose, for instance, that at the incept of the process there are 30 red balls in Ellsberg urn and 60 green or blue. However, the balls multiply at a known rate. After a while the frequency of the red balls is not one third any more. As the process evolves, the only information available is about the expectation of some random variables (this is illustrated in Example 3 bellow). Subject to this restricted information, the decision maker ought to decide between several gambles whose prizes depend on drawing a random ball from the urn. This paper shows how this can be done.

Comparing between acts entails using available information. The question arises as to how one should treat available information and how one should treat unavailable information? The model presented here takes an extreme approach to this issue. The decision maker is using available information to its full extent while completely ignoring any unavailable information.

Any act is evaluated in terms of the "expected" utility that it yields. The latter is calculated by expressing the act under examination in terms of the information available. Using this information the decision maker decomposes this act to (sub-) acts that can be defined solely in terms of the information at hand. For instance, if the probability of event A is available, an act defined as: 'the lottery assigned to any state in A is, say ℓ', and to any other state the lottery assigned is ℓ ' is expressed only in terms of the information available.

Acts defined in such a way are easy to evaluate: the expected utility can be calculated since all the information needed for this task is available. The problem though concerns acts that cannot be expressed in terms of the information available. In this case, the decision maker is considering the best approximation possible that uses only what is known, and ignores what is unknown.

One might confuse partial preference ordering as in Bewley (2001) with partially specified probabilities. In order to avoid such a confusion it should be stressed that, as opposed to Bewley (2001), the completeness assumption is fully kept here: any two acts are comparable in the preference order. In other words, either one act is strictly preferred to the other or they are equivalent. In the current context, this complete preference order is determined by the evaluation of acts that is based on a partially specified probability.

The partially specified decision making is carried on to strategic interactions in order to introduce *partially specified equilibrium*. In a partially specified equilibrium players do not know precisely the mixed strategy played by each of the other players. Rather, players know only the probability of some subsets of strategies, without knowing the precise sub-division of probabilities within these subsets. In other words, the mixed strategies played (which are probability distributions over pure strategies) are partially specified. Moreover, different players may know different specifications of the mixed strategy employed by any individual. When the information of all players is complete, the partially specified equilibrium coincides with Nash equilibrium.

The assumption regarding ambiguity aversion implies that the best-response correspondence is convex valued. This in turn implies that for any information structure there exists a partially specified equilibrium.

2. Ellsberg paradox - a motivating example

Suppose that an urn contains 30 red balls and 60 other balls that are either green or blue. A ball is randomly drawn from the urn and a decision maker is given a choice between the two gambles.

Gamble **X**: to receive \$100 if a red ball is drawn.

Gamble **Y**: to receive \$100 if a green ball is drawn.

In addition, the decision maker is also given the choice between these two gambles:

Gamble **Z**: to receive \$100 if a red or blue ball is drawn.

Gamble **W**: to receive \$100 if a green or blue ball is drawn.

It is well documented that most people strongly prefer Gamble **X** to Gamble **Y** and Gamble **W** to Gamble **Z**. This is a violation of the expected utility theory.

There are three states of nature in this scenario: R, G and B, one for each color. Denote by S the set containing these states. Each of the gambles corresponds to a real function (a random variable) defined over S. For instance, Gamble **X** corresponds to the random variable X, defined as $X(R) = 100$ and $X(G) = X(B) = 0$.

The probability of four events are known: $p(\emptyset) = 0$, $p(S) = 1$, $p(\{R\}) = \frac{1}{3}$ and $p(\{G, B\}) = \frac{2}{3}$. The probability p is partially specified: it is defined only on a sub-collection of events and not on all events.

When the probability p is defined only over familiar events, the random variable X is allowed to be written as a positive linear combination of characteristic functions of *specified* events only. Using only the four specified events, X can be best decomposed as[1] $X = 100 \cdot \mathbb{1}_{\{R\}}$. This will be later the evaluation of X.

When doing the same for Y, one cannot obtain a precise decomposition of Y (the random variable that corresponds to Gamble **Y**). The maximal non-negative function which is lower than or equal to Y and can be written only in terms of the four familiar events is $0 \cdot \mathbb{1}_S$. The evaluation of Y will be therefore equal to 0. Since, $100 \cdot \frac{1}{3} > 0$, X is preferred to Y.

A similar method applied to Z and W yields: $Z \geq 100 \cdot \mathbb{1}_{\{R\}}$ and the right-hand side is the greatest of its kind. Thus, the evaluation of Z is $100 \cdot \frac{1}{3}$, while W is optimally decomposed as $100 \cdot \mathbb{1}_{\{G,B\}}$. Therefore, the evaluation of W is $100 \cdot \frac{2}{3}$. Since $100 \cdot \frac{1}{3} < 100 \cdot \frac{2}{3}$, Gamble **W** is preferred to Gamble **Z**.

The intuition is that the decision maker bases her evaluation of random variables only on well-known figures: on the probability of the specified events. The best estimate is provided then by the maximal function which is not larger than the random variable and can be expressed using these events and the integral in turn is based on this estimate. ■

[1] $\mathbb{1}_A$ is the indicator of a set A, known also as the characteristic function of A.

3. The model and axioms

3.1. **The model.** Let N be a finite set of outcomes and $\Delta(N)$ be the set of distributions over N. S is a finite state space. Denote by L the set of all functions from S to $\Delta(N)$ and L_c the set of all constant functions in L. An element of L is called an *act*. The constant function that attains the value y will be denoted by $\mathbf{y}$ (i,e., $\mathbf{y}(s) = y$ for every $s \in S$).

L is a convex set: if $\alpha \in [0,1]$, $f, g \in L$, then $(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s)$.

A decision maker has a binary relation $\succsim$ over L. We say that $\succsim$ is *complete* if for every f and g in L either $f \succsim g$ or $g \succsim f$. It is *transitive* if for every f, g and h in L, $f \succsim g$ and $g \succsim h$ imply $f \succsim h$.

3.2. **Axioms.**

(i) Weak-Order: *The relation $\succsim$ is complete and transitive.*

The relation $\succsim$ defined over L induces a binary relation[2], $\succsim$, over $\Delta(N)$ as follows: $y \succsim z$ iff $\mathbf{y} \succsim \mathbf{z}$. The relation $\succsim$ induces the binary relations $\succ$ and $\sim$: $f \succ g$ iff $f \succsim g$ and not $g \succsim f$; $f \sim g$ iff $f \succsim g$ and $g \succsim f$.

Let f and g be two acts. We denote $f \geq g$ when $f(s) \succsim g(s)$ for every $s \in S$ and $f > g$ when $f \geq g$ and $f(s) \succ g(s)$ for at least one $s \in S$. For every $f \in L$, denote $W(f) = \{g;\ f \geq g\}$.

Definition 1. *(i) An act f is* fat-free *(denoted,* FaF*) if $f > g$ implies $f \succ g$.*
(ii) We say that an act g derives from *act f, if there is a function φ from $\Delta(N)$ to itself such that $g = \varphi \circ f$. The set of all acts that are derived from f is denoted $D(f)$.*

A fat-free act is such that if a lottery assigned to a state is replaced by a worse one, the resulting act is strictly inferior to the original one. In a fat-free act there is no fat that can be cut while maintaining the same quality of the act. In contrast, in an act that contains fat there is at least one state whose assigned lottery can be replaced by a worse one without affecting the quality of it: the modified act is equivalent to the original one.

An act g is derived from an act f if any two states assigned the same lottery under g are also assigned the same lottery under f. Any act induces a partition of the state space into events: those subsets of states over which the act is constant. An act g is

[2]At the minor risk of some confusions, although $\Delta(N)$ differs from L_c, the same notation is used to denote the preference orders defined over them.

derived from an act f if the partition induced by f is finer than that induced by g. Technically speaking, it means that g is measurable with respect to f.

Example 2. Consider Ellsberg paradox described in Section 2. Suppose that N is the set of integers between 0 and 100. The act X, which takes the values 100 on R and 0 on the rest is FaF because any reduction in any prize results in a worse act. For instance X' which takes the values 99 on R and 0 on the rest is less preferred than X. On the other hand, Y is not FaF since Y' which coincides with Y on R and B and is equal to 99 on G is equivalent to Y. Any act that takes one value on R and another value on $\{G, B\}$ is derived from X. ■

Note that for every act f, L_c is a subset of $D(f)$.

(ii^d) DERIVED FAT-FREE INDEPENDENCE: *Let f, g and h be acts derived from the same fat-free act. Then, for every $\alpha \in (0,1)$, $f \succ g$ implies that $\alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$.*

(iii) CONTINUITY: *For every f, g and h in L, (a) if $f \succ g$ and $g \succsim h$, then there is α in $(0,1)$ such that $\alpha f + (1-\alpha)h \succ g$; and (b) if $f \succsim g$ and $g \succ h$, then there is β in $(0,1)$ such that $g \succ \beta f + (1-\beta)h$.*

Since, L_c is a subset of $D(f)$, one can apply von Neumann-Morgenstern theorem to $\Delta(N)$: axioms (i), (ii^d) and (iii) imply that there is an affine function defined on L_c that represents $\succsim$ restricted to L_c, and therefore to $\Delta(N)$.

(iv) MONOTONICITY: *For every f and g in L, if $f \geq g$, then $f \succsim g$.*

Notation 1. *If $\boldsymbol{y}$ is constant, we let $[\boldsymbol{y}]$ be equal to $\boldsymbol{y}$. Let f be an act which is not constant. We denote by $[f]$ a FaF act which satisfies $f \geq [f]$ and $f \sim [f]$. The set that contains all the acts that are mixtures of FaF acts is denoted L_{FaF}. That is, $L_{FaF} = \mathrm{conv}\{[f]; f \in L\}$, where conv stands for the 'convex hull of'.*

Lemma 1. *Axioms (i), (ii^d), (iii) and (iv) imply that for every act f, there exists $[f]$.*

The proof is deferred to the Appendix.

(ii) FAT-FREE INDEPENDENCE: *Let f, g and h be in L_{FaF}. Then, for every $\alpha \in (0,1)$, $f \succ g$ implies that $\alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$.*

As before, axioms (i), (ii) and (iii) imply that there is an affine function defined on L_c that represents $\succsim$ (note that, by definition, $L_c \subseteq L_{FaF}$).

Lemma 2. *Axioms (i), (ii), (iii) and (iv) imply that for every act f, there exists $[f]$.*

The proof is deferred to the Appendix.

The following axiom originates from Schmeidler (1989) and is used also by Gilboa and Schmeidler (1989)

(v) UNCERTAINTY AVERSION: *For every f, g and h in L, if $f \succsim h$ and $g \succsim h$, then for every α in $(0,1)$, $\alpha f + (1-\alpha)g \succsim h$.*

4. PARTIALLY SPECIFIED PROBABILITIES

4.1. **Probabilities specified on a sub-algebra.** A *probability specified on a sub-algebra* over S is a pair $(P, \mathcal{A})$ such that $\mathcal{A}$ is an algebra[3] of subsets of S and P is a probability over S.

Integration w.r.t. probabilities specified on a sub-algebra:

Let ψ be a non-negative function defined over S and let $(P, \mathcal{A})$ be a probability specified on a sub-algebra over S.

Denote

$$(1) \quad \int \psi dP_{\mathcal{A}} = \max\Big\{ \sum_{E\in\mathcal{A}} \lambda_E P(E); \ \sum_{E\in\mathcal{A}} \lambda_E \mathbb{1}_E \le \psi \text{ and } \lambda_E \ge 0 \text{ for every } E \in \mathcal{A} \Big\},$$

where $\mathbb{1}_E$ is the indicator of E.

Note that if ψ is measurable with respect to $\mathcal{A}$, then this integral coincides with the regular expectation.

Definition 2. *Let I be a real function from $[0,1]^S$ and $X, Y \in [0,1]^S$.*
(i) We say that $X \ge Y$ if $X(s) \ge Y(s)$ for every $s \in S$ and $X > Y$ if $X \ge Y$ and $X(s) > Y(s)$ for at least one $s \in S$.

[3]A set $\mathcal{A}$ of subsets of S is called *algebra* if $S \in \mathcal{A}$ and for every $E_1, E_2 \in \mathcal{A}$ the intersection, $E_1 \cap E_2$, and the complement, $S \setminus E_1$, are also in $\mathcal{A}$.

(ii) A function X over S is fat-free *(FaF) wrt I if $X > Y$ implies $I(X) > I(Y)$.*
(iii) Y is derived from X if there is a function $\varphi : I\!R \to I\!R$ such that $Y = \varphi \circ X$.

Proposition 1. *Let I be a real function from $[0,1]^S$. There is a probability specified on a sub-algebra, $(P, \mathcal{A})$, over S such that $I(X) = \int X dP_{\mathcal{A}}$ for every $X \in [0,1]^S$ iff*

(1) *I is monotonic wrt to $\geq$;*
(2) *For every X, there is a FaF function wrt I, $[X]$, such that $X \geq [X]$ and $I(X) = I([X])$;*
(3) *If X and Y are derived from the same FaF function and $\alpha \in (0,1)$, then $I(\alpha X) + I((1-\alpha)Y) = I(\alpha X + (1-\alpha)Y)$;*
(4) *If X is FaF wrt I, then for every positive c, if $cX \in [0,1]^S$, then $I(cX) = cI(X)$;*
(5) *For every X and Y such that $I(X) = I(Y)$ and $\alpha \in (0,1)$, $I(\alpha X) + I((1-\alpha)Y) \leq I(\alpha X + (1-\alpha)Y)$; and*
(6) *$I(\mathbb{1}_S) = 1$.*

The proof is deferred to the Appendix.

4.2. **Partially specified probability.**

Example 3 (Dynamic Ellsberg Urn): Suppose that, like in Ellsberg paradox, an urn contains balls of three colors. It is known that yesterday there were 30 red balls and 60 other balls that were either green or blue. However, the green balls are actually one-cell organisms that split once a day. Today, the decision maker is required to choose among a few gambles, like in Section 2.

It turns out that the information available to the decision-maker does not include the probability of any event. The decision-maker may infer the expectation of some random variables. To illustrate this point, suppose that the current number of red, green and blue balls are denoted by n_r, n_g and n_b, respectively. Yesterday there were $\frac{n_g}{2}$ green balls. It is known that $\frac{n_r}{n_r + \frac{n_g}{2} + n_b} = \frac{1}{3}$. By rearranging terms one obtains, $\frac{n_r + \frac{1}{6} n_g}{n_r + n_g + n_b} = \frac{1}{3}$. This is the expectation of the random variable which attains the value 1 on Red, $\frac{1}{6}$ on Green and 0 on Blue. That is, according to the information available the expectation of this random variable is $\frac{1}{3}$.

There is a regular probability underlying this decision problem: the actual distribution of balls. However, the decision maker is only partially informed. Beyond the obvious information concerning the probability of the whole space and the empty set, the decision-maker is informed only of the expectation of the random variable $(1, \frac{1}{6}, 0)$. Clearly, he may deduce the expectation of any random variable of the form $c(1, \frac{1}{6}, 0)$, $c > 0$. ■

The following model of decision making with partially specified probability captures also this example.

A *partially specified probability* over S is a pair $(P, \mathcal{Y})$, where $\mathcal{Y}$ is a closed set of real functions over S and P is a probability over S and $\mathbb{1}_S \in \mathcal{Y}$. Note that any probability specified on a sub-algebra, $(P, \mathcal{A})$, can be identified with a partially specified probability, $(P, \mathcal{Y})$: $\mathcal{Y} = \{\mathbb{1}_A;\ A \in \mathcal{A}\}$.

Let ψ be a non-negative function defined over S and let $(P, \mathcal{Y})$ be a partially specified probability. Denote,

$$(2) \quad \int \psi dP_{\mathcal{Y}} = \max\Big\{\sum_{Y\in\mathcal{Y}} \lambda_Y E_P(Y);\ \sum_{Y\in\mathcal{Y}} \lambda_Y Y \le \psi \text{ and } \lambda_Y \ge 0 \text{ for every } Y \in \mathcal{Y}\Big\}.$$

The following lemma states that without loss of generality, the set $\mathcal{Y}$ can be assumed to be convex, or otherwise, the set of extreme points of a convex set.

Lemma 3. *Let $(P, \mathcal{Y})$ be a partially specified probability.*
(i) Denote the convex hull of $\mathcal{Y}$ by $\mathcal{Y}'$, then $\int \psi dP_{\mathcal{Y}} = \int \psi dP_{\mathcal{Y}'}$ for every ψ.
(ii) Denote the set of extreme points of the convex hull of conv$\mathcal{Y}$ *by $\mathcal{Y}''$, then $\int \psi dP_{\mathcal{Y}} = \int \psi dP_{\mathcal{Y}''}$ for every ψ.*

The proof is deferred to the Appendix.

Notation 2. *Let I be a real function from $[0,1]^S$. For every X, if X is constant, denote $[X] = X$. Otherwise, denote by $[X]$ a function which satisfies $X \ge [X]$, $I(X) = I([X])$ and if $[X] > Y$, then $I([X]) > I(Y)$.*

Proposition 2. *Let I be a real function from $[0,1]^S$. There is a partially specified probability, $(P, \mathcal{Y})$, such that $I(X) = \int X dP_{\mathcal{Y}}$ iff*

(1) *I is monotonic wrt to $\ge$;*
(2) *For every X, there is a FaF function wrt I, $[X]$ such that $X \ge [X]$ and $I(X) = I([X])$;*
(3) *If X and Y are FaF functions and $\alpha \in (0,1)$, then $I(\alpha X) + I((1-\alpha)Y) = I(\alpha X + (1-\alpha)Y)$;*
(4) *Let X be FaF wrt I. For every positive c, if $cX \in [0,1]^S$, then $I(cX) = cI(X)$;*

(5) *For every* X *and* Y *such that* $I(X) = I(Y)$ *and* $\alpha \in (0,1)$, $I(\alpha X) + I((1-\alpha)Y) \leq I(\alpha X + (1-\alpha)Y)$*; and*

(6) $I(c\mathbb{1}_S) = c$ *for every* $c \in [0,1]$.

The proof is deferred to the Appendix.

5. Theorems

5.1. Decision making with a probability specified on sub-algebra.

Theorem 1. *Let* $\succsim$ *be a binary relation over* L. *This satisfies (i), (ii*^{d}*),(iii)-(v) if and only if there is a probability specified on a sub-algebra* $(P, \mathcal{A})$ *and an affine function* u *on* $\Delta(N)$ *such that for every* $f, g \in L$,

$$f \succsim g \quad \text{iff} \quad \int u(f(s))dP_{\mathcal{A}} \geq \int u(g(s))dP_{\mathcal{A}}. \tag{3}$$

Remark 1. A probability specified on a sub-algebra, $(P, \mathcal{A})$, induces a capacity v, defined over S: $v(E) = \int \mathbb{1}_E dP_{\mathcal{A}} = \max_{F \subseteq E,\ F \in \mathcal{A}} P(F)$. This v is convex[4]. According to Lehrer (2005) the Choquet integral wrt to v coincides with the integral in eq. (1). Thus, a Decision-maker whose preference order satisfies axioms (i), (ii^d),(iii)-(v) is in particular a Choquet expected utility maximizer. In other words, the theory proposed by Theorem 1 is a subset of that of Schmeidler (1989). Moreover, as a convex game, v has a large core[5] and by Azriely and Lehrer (2005), axioms (i), (ii^d), and (iii)-(v) imply that the preference over acts is determined by the minimum of the expectations of many probability distributions, as in Gilboa and Schmeidler (1989).

Theorem 1 suggests that the probability underlying the decision making is not just a general non-additive one. Rather, the actual probability is indeed additive, but the decision maker is only informed about the probabilities of some of the events (that form a sub-algebra) and not about all of them. ■

5.2. Decision making with a partially specified probability.

Theorem 2. *Let* $\succsim$ *be a binary relation over* L. *This satisfies (i)-(v) if and only if there is a partially specified probability* $(P, \mathcal{Y})$ *and an affine function* u *on* $\Delta(N)$ *such that for every* $f, g \in L$,

[4] A capacity v is *convex* if for every $E, F \subseteq S$, $v(E) + v(F) \leq v(E \cup F) + v(E \cap F)$.

[5] A capacity v *has a large core* if for every vector $Q = (Q_i)_{i \in N}$, $\sum_{i \in T} Q_i \geq v(T)$ for every $T \subseteq S$, implies that there is R in the core of v such that $Q \geq R$.

$$(4) \qquad f \succsim g \quad \text{iff} \quad \int u(f(s))dP_{\mathcal{Y}} \geq \int u(g(s))dP_{\mathcal{Y}}.$$

Remark 2. A partially specified probability $(P, \mathcal{Y})$ induces a capacity v, defined over S: $v(E) = \int 1\!\!1_E dP_{\mathcal{Y}}$ for every $E \subsetneqq S$. This v is neither convex nor does it have a large core. Thus, by Azriely and Lehrer (2005), $\int$constant$+XdP_{\mathcal{Y}}$ is typically **not** equal to constant$+\int XdP_{\mathcal{Y}}$. Equality holds only if X is either constant or FaF. This implies that a decision-maker whose preferences are determined by a partially specified probability $(P, \mathcal{Y})$ is typically not a Choquet expected utility maximizer nor does he follow the model of decision making based on the minimum over a set of priors (Gilboa and Schmeidler, 1989)

To illustrate this point, let $X = (\frac{1}{6}, \frac{2}{6}, \frac{3}{6})$ (that is, X is a random variable defined on $S = \{1, 2, 3\}$) and suppose that $\mathcal{Y} = \{1\!\!1_S, X\}$. Moreover, it is given that $E(X) = \frac{5}{12}$. In this case $\int(0, 1, 1)dP_{\mathcal{Y}} = 0$, and there is no probability distribution Q that satisfies $E_Q(X) \geq \frac{5}{12}$ and the last equation. Thus, there is no set of priors such that the minimum of the respective expected values of any ψ coincides with $\int \psi dP_{\mathcal{Y}}$.

A sophisticated agent would be able to calculate the expected value of any random variable that can be expressed as a linear combination (not necessarily with positive coefficients) of elements in $\mathcal{Y}$, denoted as span$\mathcal{Y}$. In the previous example, $(0, \frac{1}{2}, 1)$ and $(1, \frac{1}{2}, 0)$ are both in span$\mathcal{Y}$ (e.g., $(0, \frac{1}{2}, 1) = 3(\frac{1}{6}, \frac{2}{6}, \frac{3}{6}) - \frac{1}{2}1\!\!1_S$). Thus, $\int(0, 1, 1)dP_{\mathcal{Y}} = \int(0, \frac{1}{2}, 1)dP_{\mathcal{Y}} = 3 \cdot \frac{5}{12} - \frac{1}{2} = \frac{3}{4}$.

It must be noted though that a preference order determined by $(P, \text{span}\mathcal{Y})$ might also be out of the multiple prior model. For instance, assume that $S = \{1, 2, 3\}$, $\mathcal{Y} = \{(\frac{1}{4}, 0, \frac{3}{4}), 1\!\!1_S\}$ and P is the uniform distribution. Thus, $E_P(\frac{1}{4}, 0, \frac{3}{4}) = \frac{1}{3}$, but $\int 1\!\!1_{\{3\}} dP_{\mathcal{Y}} = 0$. Since there is no probability distribution Q such that $E_Q(\frac{1}{4}, 0, \frac{3}{4}) \geq \frac{1}{3}$ and $E_Q(1\!\!1_{\{3\}}) = 0$, the preference orders induced by $(P, \mathcal{Y})$ and by $(P, \text{span}\mathcal{Y})$ are not equivalent to any order determined by the minimum over a set of probability distributions.

Another way to demonstrate that the preference order induced by $(P, \mathcal{Y})$ is not determined by the minimum over a set of probability distributions is to show that the integral is not co-variant with adding a constant. Consider the previous example. On one hand $\int \frac{3}{4}1\!\!1_{\{3\}} dP_{\mathcal{Y}} + \frac{1}{4} = \frac{1}{4}$ and on the other, $\int \frac{3}{4}1\!\!1_{\{3\}} + \frac{1}{4}1\!\!1_S dP_{\mathcal{Y}} = \int(\frac{1}{4}, \frac{1}{4}, 1)dP_{\mathcal{Y}}$. Since, $(\frac{1}{4}, \frac{1}{4}, 1) \geq (\frac{1}{4}, 0, \frac{3}{4})$, and $(\frac{1}{4}, 0, \frac{3}{4}) \in \mathcal{Y}$, one obtains, $\int(\frac{1}{4}, \frac{1}{4}, 1)dP_{\mathcal{Y}} \geq \frac{1}{3} > \frac{1}{4}$. Thus, $\int \frac{3}{4}1\!\!1_{\{3\}} dP_{\mathcal{Y}} + \frac{1}{4} \neq \int \frac{3}{4}1\!\!1_{\{3\}} + \frac{1}{4}1\!\!1_S dP_{\mathcal{Y}}$. ■

6. Proofs of the theorems

Proof of Theorem 1. If for all f and g in L, $f \sim g$, then $u \equiv 0$ satisfies eq. (3) with any partially specified probability. We can therefore assume that there are f and g in L such that $f \succ g$.

Let f be FaF. Consider the set $D(f)$. The order $\succsim$ restricted to this set satisfies the axioms of von Neumann-Morgenstern. By von Neumann-Morgenstern theorem (1944), there is an affine function, say U_f over $D(f)$, that represents $\succsim$. That is, for every g and h in $D(f)$, $g \succsim h$ iff $U_f(g) \geq U_f(h)$. Note that $L_c \subseteq D(f)$. As a von Neumann-Morgenstern utility function, U_f is unique up to a positive affine transformation. Therefore, there is a unique von Neumann-Morgenstern utility function over $D(f)$ that represents $\succsim$ and satisfies[6] $U_f(\mathbf{m}) = 0$ and $U_f(\mathbf{M}) = 1$. Thus, for every $\mathbf{k} \in L_c$ and every two FaF acts g and f, $U_g(\mathbf{k}) = U_f(\mathbf{k})$.

According to (iv), $0 \leq U_f(f) \leq 1$. Moreover, there is $\mathbf{k(f)} \in L_c$ such that $U_f(\mathbf{k}(\mathbf{f}))= U_f(f)$, meaning that $\mathbf{k}(\mathbf{f}) \sim \mathbf{f}$. Define for every act f, $U(f) = U_{[f]}([f])$.

$f \succsim g$ if and only if $[f] \succsim [g]$ if and only if $\mathbf{k([f])} \succsim \mathbf{k([g])}$ if and only if $U_{[f]}(\mathbf{k([f])}) \geq U_{[f]}(\mathbf{k([g])})$ if and only if $U_{[f]}(\mathbf{k([f])}) \geq U_{[g]}(\mathbf{k([g])})$ if and only if $U_{[f]}([f]) \geq U_{[g]}([g])$. Thus U represents $\succsim$.

Define $u(y) = U(\mathbf{y})$ for every $y \in \Delta(N)$. For every $X \in [0,1]^S$ there is an act f_X such that $X(s) = U(\mathbf{f_X(s)})$ (recall that $\mathbf{f_X(s)}$ is the act that takes constantly the value $f_X(s)$) and for every $X \in [0,1]^S$ there is an act f_X such that $X(s) = u(f_X(s))$. Note also that $X_{f_X} = X$. Moreover, if X is FaF, so is f_X. Define $I(X) = U(f_X)$, for every $X \in [0,1]^S$. Thus, $I(X_f) = U(f)$ for every $f \in L$.

In order to use Proposition 1, it will be shown that I satisfies properties (1)-(6). (1) follows from (iv) and from the fact that if $X \geq Y$, then $f_X \geq f_Y$ and $[f_X] \geq [f_Y]$. (2) follows from Lemma 1. As for (3), assume that X and Y are derived from the same FaF Z. Then, f_Z is FaF and furthermore, f_X and f_Y are in $D(f_Z)$. Therefore,

$$U_{f_Z}(\alpha f_X + (1-\alpha) f_Y) = \alpha U_{f_Z}(f_X) + (1-\alpha) U_{f_Z}(f_Y). \tag{5}$$

However, for every $h \in D(f_Z)$, $U_{f_Z}(h) = U_{f_Z}(\mathbf{k(h)}) = U_{[h]}(\mathbf{k(h)}) = U_{[h]}([h])$. Thus, $U_{f_Z}(\alpha f_X + (1-\alpha) f_Y) = U_{[\alpha f_X + (1-\alpha) f_Y]}([\alpha f_X + (1-\alpha) f_Y]) = I(\alpha f_X + (1-\alpha) f_Y)$. On the other hand, $U_{f_Z}(f_X) = U_{[f_X]}([f_X]) = I(X)$, and $U_{f_Z}(f_Y) = U_{[f_Y]}([f_Y]) = I(Y)$. Due to eq. (5) one obtains (3).

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[6] $\mathbf{m}$ and $\mathbf{M}$ are the minimum and the maximum over L_c wrt $\succsim$, resp. These exist since $\succsim$ has a von Neumann-Morgenstern utility representation.

Let X be FaF. Thus, f_X is FaF and $\mathbf{m}, f_{cX} \in D(f_X)$ whenever $cX \in [0,1]^S$. Therefore, if $c < 1$, then $f_{cX} \sim cf_X + (1-c)\mathbf{m}$ and $I(cX) = U_{[f_{cX}]}([f_{cX}]) = U_{f_X}(f_{cX}) = cU_{f_X}(f_X) + (1-c)U_{f_X}(\mathbf{m}) = cU_{f_X}(f_X) = cI(X)$. For $c > 1$ the proof method is similar, and is therefore omitted. Thus, (4) is proven.

(5) follows from (v) and (6) from construction. One can now use Proposition 1 to obtaine I that satisfies properties (1)-(6) of Proposition 2 and therefore, there exists a probability specified on a sub-algebra, $(P, \mathcal{A})$ such that $I(X) = \int X dP_{\mathcal{A}}$ for every $X \in [0,1]^S$. Thus, $U(f) = I(X_f) = \int X_f dP_{\mathcal{A}} = \int u \circ f dP_{\mathcal{A}}$, as desired. ■

Proof of Theorem 2 . Axioms (i)-(iii) guarantee that there is a vN-M representation over L_{FaF}. That is, there is an affine U defined over L_{FaF} that satisfies $U(f) \geq U(g)$ if and only if $f \succsim g$ for every $f, g \in L_{FaF}$. Moreover, U is normalized to take the values 0 and 1 on $\mathbf{m}$ and $\mathbf{M}$, respectively. Define $u(y) = U(\mathbf{y})$ for every $y \in \Delta(N)$.

Extend the domain of U to L by defining $U(f) = U([f])$, for every $f \in L \setminus L_{FaF}$. U represents $\succsim$ over all L. For every $f \in L$ define $X_f \in [0,1]^S$ as follows: $X_f(s) = u(f(s))$ for every $s \in S$.

Like in the previous proof, define I on $[0,1]^S$ as follows. For every $X \in [0,1]^S$ there is an act f_X such that $X(s) = u(f_X(s))$. Note that $X_{f_X} = X$. Moreover, if X is FaF, so is f_X. Define $I(X) = U(f_X)$. Thus, $I(X_f) = U(f)$ for every $f \in L$.

I satisfies properties (1)-(6) of Proposition 2 and therefore, there is a partially specified probability $(P, \mathcal{Y})$ such that $I(X) = \int X dP_{\mathcal{Y}}$ for every $X \in [0,1]^S$. Thus, $U(f) = I(X_f) = \int X_f dP_{\mathcal{Y}} = \int u \circ f dP_{\mathcal{Y}}$, as desired. ■

7. Partially specified equilibrium

Let $G = (M, \{B_i\}_{i \in M}, \{u_i\}_{i \in M})$ be a game: M is a finite set of players, B_i is player i's set of strategies (finite), and $u_i : B \to I\!R$ is player i's payoff function, where $B = \times_{i \in M} B_i$. Denote by $\Delta(B_i)$ the set of mixed strategies of player i.

Suppose that player i knows about player j only the probabilities of the events in a sub-algebra $\mathcal{A}_i^j$ (of subsets of B_j). That is, when player j plays the mixed strategy p_j, player i knows only the probabilities $p_j(A)$, $A \in \mathcal{A}_i^j$. Denote for every i, $B_{-i} = \times_{j \neq i} B_j$. Let $\mathcal{A}_i$ be the algebra of subsets of B_{-i} generated by $\mathcal{A}_i^j$, $j \neq i$. $\mathcal{A}_i$ is a product algebra: any atom of it is a product of subsets of B_j, $j \neq i$.

For every $b_{-i} \in B_{-i}$ and player i's mixed strategy $p_i \in \Delta(B_i)$, define $u_i(p_i, b_{-i}) = \sum_{b_i \in B_i} p_i(b_i) u_i(b_i, b_{-i})$. This is the linear extension of u_i to $\Delta(B_i) \times B_{-i}$.

Let $\{p_i\}_{i\in M} \in \times_{i\in M}\Delta(B_i)$ be a profile of mixed strategies. For every $i \in M$ denote by P^{-i} the product distribution induced by $\{p_j\}_{j\neq i}$ on B_{-i}.

Definition 3. *Let $\mathcal{A}_i$ be the algebra of events known to player i. A profile $\{p_i\}_{i\in M} \in \times_{i\in M}\Delta(B_i)$ of mixed strategies is a* partially specified equilibrium *wrt $\mathcal{A}_1, ..., \mathcal{A}_n$, if for every player i the mixed strategy p_i maximizes his payoff. In other words, for every $p_i' \in \Delta(B_i)$,*

$$\int u_i(p_i, b_{-i})dP^{-i}_{\mathcal{A}_i} \geq \int u_i(p_i', b_{-i})dP^{-i}_{\mathcal{A}_i}. \tag{6}$$

When $\mathcal{A}_i$ is the discrete algebra for every i, then a partially specified equilibrium wrt $\mathcal{A}_1, ..., \mathcal{A}_n$ is Nash equilibrium. Unlike Nash equilibrium, in a partially specified equilibrium when the algebra $\mathcal{A}_i$ is not discrete, the pure strategies in the support of player i's strategy are not necessarily best responses. To illustrate this point consider the following example.

Example 4: (i) Consider the following two-player coordination game.

	L	R
T	1,0	0,1
B	0,3	2,0

Suppose that each player knows nothing about his opponent's strategy. The row player chooses a distribution $(p, 1-p)$ over his set of action $\{T, B\}$. When playing $(p, 1-p)$, the expected payoff of the row player is $\min(p, 2(1-p))$. The maximum over p is achieved when $p = \frac{2}{3}$. The same reasoning applies to player 2: she plays $(\frac{1}{4}, \frac{3}{4})$ and the only partially specified equilibrium when both players' information is trivial, is $\big((\frac{2}{3}, \frac{1}{3}), (\frac{1}{4}, \frac{3}{4})\big)$. Note that R obtains a positive probability, although it not a best response to $(\frac{2}{3}, \frac{1}{3})$. It is not a best response to the mixed strategy played by player 1 (neither as player 2 who knows nothing about this strategy nor as player 2 who knows it).

(ii) Let a two-player game be given by

	L	R
T	2,2	2,0
B	0,2	3,3

Suppose, as in the previous example, that each player knows nothing about his opponent's strategy. When playing $(p, 1-p)$, the expected payoff is $\min(2p, 2p+3(1-p))$. The maximum is achieved when $p = 1$. The same applies to player 2, and the only partially specified equilibrium when both players' information is trivial, is (T, L).

In case player 1 has a full information about player 2's strategy, still (T, L) is the only partially specified equilibrium.

(iii) Let a two-player game be given by

	L	M	R
T	3,3	0,0	0,0
C	0,0	2,2	0,0
B	0,0	0,0	1,1

Consider the case where player 1 knows the probability assigned by player 2 to R, and player 2 knows the probability assigned by player 1 to B. In other words, $\mathcal{A}_1$ is generated by the partition $\{\{L, M\}, \{R\}\}$ and $\mathcal{A}_2$ is generated by the partition $\{\{T, C\}, \{B\}\}$. In equilibrium player 1 plays (p_1, p_2, p_3) and player 2 plays (q_1, q_2, q_3). When $p_1 + p_2 > 0$, as in example (i), $\frac{p_1}{p_2} = \frac{q_1}{q_2} = \frac{2}{3}$.

There are three equilibria: a. $p = (\frac{2}{5}, \frac{3}{5}, 0)$, $q = (\frac{2}{5}, \frac{3}{5}, 0)$; b. $p = (0, 0, 1)$, $q = (0, 0, 1)$; and c. $p = (\frac{2}{11}, \frac{3}{11}, \frac{6}{11})$, $q = (\frac{2}{11}, \frac{3}{11}, \frac{6}{11})$. ■

The following claim can be proven by a standard fixed-point technique. It is based on the fact that the integral is concave and therefore the best-response correspondence has convex values.

Proposition 3. *Let $\mathcal{A}_i$ be the (product) algebra known to player i, $i \in M$. Then, there exists a partially specified equilibrium wrt $\mathcal{A}_1, ..., \mathcal{A}_n$.*

8. Discussion

8.1. **On ambiguity aversion.** This analysis is meant to introduce a first order approximation to the way by which people take decisions in the presence of a partially specified probability. In Ellsberg original decision problem, Gamble **X** is weakly dominated by gamble **W**. Nevertheless, the theory of decision making with a partially specified probability would predict that X and W are equivalent. To make things even worse, suppose that X is modified a bit and instead of \$100, the prize for drawing a red ball is \$101. In this case, **X** is predicted to be strictly preferred to **W**.

The reason for this difficulty is that the theory takes an extremal ambiguity aversion approach: any information provided is taken without any sense of skepticism and anything else is ignored.

Similar difficulties arise also within the expected utility theory. It might happen that one act weakly dominates another while they are actually equivalent. This happens when big prizes are ascribed probability 0 and are therefore not counted in the expected utility.

To improve upon the current theory, one might think of discriminating between different information sources according to their reliability. More reliable sources will be getting a greater weight than less reliable ones. In this case, wild guesses are also reliable to a certain extent, and should be therefore taken into account with a weight determined according to their reliability level.

Recall eq. (2). To make the previous paragraph more formal, let $\mathcal{Y}$ be a set of random variables and v be real function defined on $\mathcal{Y}$. $v(Y)$ is interpreted as 'the expectation of Y is claimed to be $v(Y)$' for every $Y \in \mathcal{Y}$. However, the function v may summarize the information received from various sources. The reliability of these sources might vary.

One may think of a reliability coefficient r_Y attached to every $Y \in \mathcal{Y}$. This coefficient r_Y is meant to indicate the extent to which the information about Y is reliable. This coefficient could be then factored in the evaluation of a function ψ, as follows:

$$\int \psi dP_{\mathcal{Y}} = \max\Big\{ \sum_{Y\in\mathcal{Y}} r_Y \lambda_Y v(Y);\ \sum_{Y\in\mathcal{Y}} \lambda_Y Y \leq \psi \text{ and } \lambda_Y \geq 0 \text{ for every } Y \in \mathcal{Y} \Big\}.$$

In this figure the number $\lambda_Y v(Y)$ is discounted by the coefficient r_Y to obtain $r_Y \lambda_Y v(Y)$.

8.2. **Partially specified probability and framing effects.** In order to evaluate an act a decision maker uses $\mathcal{Y}$, which captures available data. Anything in $\mathcal{Y}$ is considered to its full extent, while anything out of $\mathcal{Y}$ is totally ignored. Theorem 2 imposes no structural restriction on $\mathcal{Y}$ beyond closeness (which could be dropped at the expense of replacing minimum with infimum) and $\mathbb{1}_S \in \mathcal{Y}$, which basically means that the decision maker knows that the probability of the entire space is 1.

Consider a state space $S = \{1, 2, 3\}$. It might be that $\mathcal{Y} = \{\mathbb{1}_S, \mathbb{1}_{\{1,2\}}\}$. Notice that in this case the decision maker knows the probability of the event $\{1, 2\}$, but ignores the probability of the event $\{3\}$ that can be easily calculated. Such a decision maker trusts only explicit information that he obtains and ignores anything else. While obtaining the probability of $\{1, 2\}$ is equivalent to obtaining the probability of $\{3\}$, the specific framing of information might therefore affect the entire decision making process. To put it more formally, denote $\mathcal{Y}' = \{\mathbb{1}_S, \mathbb{1}_{\{3\}}\}$. The partially specified probabilities $(P, \mathcal{Y})$ and $(P, \mathcal{Y}')$ are not equivalent.

In order to avoid such framing effects, the decision maker can extend the scope of his available data to span$\mathcal{Y}$ and can employ then the partially specified probability $(P, \text{span}\mathcal{Y})$ (see Remark 2).

8.3. **On the continuity axiom.** The continuity axiom used above is a bit stronger than that used by von Neumann and Morgenstern (1944). It has been required in (iii)(a) (the first part of the continuity axiom) that if $f \succ g$ and $g \succsim h$, then there is α in $(0, 1)$ such that $\alpha f + (1 - \alpha)h \succ g$. The vN-M axiom requires that the same conclusion holds under a weaker condition: $f \succ g$ and $g \succ h$.

The reason for which a stronger axiom is needed is that the independence axiom holds only for fat-free acts (or their derivatives). A-priori, there is only one fat-free act: $\mathbf{m}$. Axioms (i), (ii) and a weaker version of (iii) would ensure that there is a vN-M utility representation of $\succsim$ over the constant acts. However, to ensure a broader scope of the independence axiom, it is essential that every act has an equivalent act which is fat-free (as in Lemma 1). A weaker version of the continuity axiom, would not be sufficient for this purpose.

8.4. **On the fat-free independence axiom.** The FaF independence axiom states that if f, g and h are in L_{FaF}, then for every $\alpha \in (0, 1)$, $f \succ g$ implies that $\alpha f + (1 - \alpha)h \succ \alpha g + (1 - \alpha)h$. Theorem 2 could have proven using a weaker version of this axiom: Let f, g and h be acts of the form $\alpha z + (1 - \alpha)\mathbf{y}$, with z is being FaF, $\mathbf{y}$ being a constant and $\alpha \in (0, 1)$. Then, for every $\alpha \in (0, 1)$, $f \succ g$ implies that $\alpha f + (1 - \alpha)h \succ \alpha g + (1 - \alpha)h$. In other words the independence property is required to hold within every $L_z = \text{conv} L_c \cup \{z\}$.

8.5. **Additivity and fat-free acts.** It turns out that when the probability is specified on a sub-algebra, the integral in eq. (1) is additive over the set of functions that are measurable with respect to this sub-algebra. This means that the implied expected utility is additive when restricted to fat-free acts derived from them.

When the probability is partially specified (not over a sub-algebra), additivity is preserved over the acts that are expressed as a convex combination of fat-free acts. This does not mean that a convex combination of fat-free acts is necessarily fat-free.

9. Final comments

9.1. **Convex capacities and partially specified probabilities.** Let v be a convex capacity (i.e, v is a real function defined on the power set of S such that $v(\emptyset) = 0$ and $v(S) = 1$.) An event E in S is called *fat-free* if $F \subsetneqq E$ implies $v(F) < v(E)$. Let $(P, \mathcal{A})$ be a probability specified on a sub-algebra. Define the capacity $v_{P,\mathcal{A}}$ as follows. $v_{P,\mathcal{A}}(E) = \int 1\!\!1_E dP_{\mathcal{A}} = \max_{B \subseteq E;\ B \in \mathcal{A}} P(B)$. It is clear that $v_{P,\mathcal{A}}$ is convex (see footnote 4). The following proposition characterizes those convex capacities that are of the form $v_{P,\mathcal{A}}(E)$.

Proposition 4. *Let v be convex non-additive probability. There exists a probability specified on a sub-algebra $(P, \mathcal{A})$ such that $v = v_{P,\mathcal{A}}$ if and only if for every fat-free event E and for every F, $v(E) + v(F) = v(E \cup F) + v(E \cap F)$.*

The proof appears in the Appendix.

Let $T \subseteq S$. A *unanimity capacity* u_T is defined as $u_T(E) = 1$ if $T \subseteq E$, and $u_T(E) = 0$, otherwise. It turns out that a unanimity capacity is also of the form $v_{P,\mathcal{A}}$: $\mathcal{A} = \{T, S \setminus T\}$, $P(T) = 1$ and $P(S \setminus T) = 0$. Moreover, a capacity $v_{P,\mathcal{A}}$ is a convex combination of unanimity games of a special kind, as demonstrated by the following lemma.

The lemma uses this notation: a sub-algebra $\mathcal{A}$ is generated by a partition, denoted $\mathcal{Q}(\mathcal{A})$.

Lemma 4. *For every probability specified on a sub-algebra $(P, \mathcal{A})$,*

$$v_{P,\mathcal{A}} = \sum_{T \in \mathcal{Q}(\mathcal{A})} P(T) u_T.$$

The proof is simple and is therefore omitted.

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10. Appendix

Proof of Lemma 1. First, note that f itself is a member of the set $A(f) = W(f) \cap \{g;\ f \sim g\}$. The independence axiom (ii^d) applies in particular to L_c, and along with (i) and (iii) implies that $\succsim$ over L_c is continuous. Using a a diagonalization method one can find an infimum of $A(f)$, say h. The act h satisfies: (a) $g \geq h$ for every $g \in A(f)$; and (b) if h' is such that $h'(s) \succ h(s)$ whenever $f(s) \succ h(s)$ ($s \in S$), then there exists $g' \in A(f)$ such that $h' > g'$.

Now suppose to the contrary that $f \succ h$. Fix $g \in A(f)$. Then, by (iii), there is β in $(0,1)$ such that $g \succ \beta f + (1-\beta)h$. However, $\beta f + (1-\beta)h > h$. This is a contradiction to h being an infimum, because if $g' \in A(f)$ such that $\beta f + (1-\beta)h > g'$ exists, then by (iv), $\beta f + (1-\beta)h \succsim g' \sim g \succ \beta f + (1-\beta)h$.

We obtained that $f \sim h$, and h is FaF, as desired. ∎

Proof of Lemma 2. A closer look at the previous proof reveals that it hinges upon (iii), (iv) and upon the fact that $\succsim$ is continuous over L_c. Since the independence axiom (ii) applies to L_c, the latter is true under the assumptions of this lemma and thus the proof is complete. ∎

Proof of Lemma 3. (i) Note that since $\mathcal{Y} \subseteq \mathcal{Y}'$, $\int \psi dP_{\mathcal{Y}} \geq \int \psi dP_{\mathcal{Y}'}$ for every ψ. By definition, $\int \psi dP_{\mathcal{Y}'} = \sum_{Y \in \mathcal{Y}'} \lambda_Y E_P(Y)$, where $\sum_{Y \in \mathcal{Y}'} \lambda_Y Y \leq \psi$ and $\lambda_Y \geq 0$ for every $Y \in \mathcal{Y}$. Since $Y \in \mathcal{Y}'$, there are non-negative coefficients θ_Y^Z, $Z \in \mathcal{Y}$ that add up to 1 and $Y = \sum_{Z \in \mathcal{Y}} \theta_Y^Z Z$. Since, E_P is additive, $\sum_{Y \in \mathcal{Y}'} \lambda_Y E_P(Y) = \sum_{Y \in \mathcal{Y}'} \sum_{Z \in \mathcal{Y}} \theta_Y^Z E_P(Z)$. This means that $\int \psi dP_{\mathcal{Y}} \leq \int \psi dP_{\mathcal{Y}'}$ and equality is established.

(ii) Since $\text{conv}\mathcal{Y} = \text{conv}\mathcal{Y}''$, and due to (i), the proof is done. ∎

Proof of Proposition 1. **Step 1: For every Y and $c > 0$, $I(cY) = cI(Y)$.** Fix Y. By (4), $I(c[Y]) = cI([Y])$. By the definition of $[Y]$, $cY \geq c[Y]$. Thus, by (1), $I(cY) \geq I(c[Y])$. It remains to show that it cannot be that $I(cY) > I(c[Y])$. If $I(cY) = I([cY]) > I(c[Y])$, then, since $[Y]$ is FaF, and due to (4), $I([cY]) > cI([Y])$. Thus, $\frac{1}{c}I([cY]) > I([Y])$. By (4) again, $I(\frac{1}{c}[cY]) > I([Y])$. However, $Y \geq \frac{1}{c}[cY])$. Thus, by (1) $I(Y) \geq I(\frac{1}{c}[cY]) > I([Y])$, which is a contradiction.

Step 2: For every X, Y and $0 < \alpha < 1$, $I(\alpha X + (1-\alpha)Y) \geq \alpha I(X) + (1-\alpha)I(Y)$. If $I(X) = 0$ or $I(Y) = 0$ this claim is implied by the previous step and (1). Otherwise, $I(X) > 0$ and $I(Y) > 0$. Note that by the previous step, $I(\frac{X}{I(X)}) = I(\frac{Y}{I(Y)})$. Denote,

$d = \alpha I(X) + (1-\alpha)I(Y)$. By (5), $I(\frac{\alpha I(X)}{d}\frac{X}{I(X)} + \frac{(1-\alpha)I(Y)}{d}\frac{Y}{I(Y)}) \geq \frac{\alpha I(X)}{d}I(\frac{X}{I(X)}) + \frac{(1-\alpha)I(Y)}{d}I(\frac{Y}{I(Y)})$. By Step 1, $\frac{1}{d}I(\alpha X + (1-\alpha)Y) \geq \frac{1}{d}(\alpha I(X) + (1-\alpha)I(Y)$, as desired.

Step 3: If X_i, is derived from FaF, $\alpha_i \geq 0$, $i = 1, ..., \ell$, and $\sum \alpha_i = 1$, then $I(\sum \alpha_i X_i) = \sum \alpha_i I(X_i)$. Note that if X_i, is derived from FaF, then $\mathbb{1}_S - X_i$ is derived from the same FaF. By (3),

$$I(\mathbb{1}_S - X_i) + I(X_i) = I(\mathbb{1}_S). \tag{7}$$

By Step 2, $I(\sum \alpha_i X_i) \geq \sum \alpha_i I(X_i)$ and $I(\sum \alpha_i(\mathbb{1}_S - X_i)) \geq \sum \alpha_i I(\mathbb{1}_S - X_i)$. Summing these two inequality up, one obtains from eq. (7), $I(\sum \alpha_i X_i) + I(\sum \alpha_i(\mathbb{1}_S - X_i)) \geq I(\mathbb{1}_S)$. However, by Step 2, the left hand side is less than or equal to $I(\sum \alpha_i(X_i + \mathbb{1}_S - X_i)) = I(\sum \alpha_i(\mathbb{1}_S)) = I(\mathbb{1}_S)$. We obtained that all the inequalities are actually equalities, as desired.

Step 4: If $X >> Y$ (i.e., $X(s) > Y(s)$ for every $s \in S$), then $I(X) > I(Y)$. There is $C > 0$ such that $X - Y \geq c\mathbb{1}_S$, or $X \geq c\mathbb{1}_S + Y$. Thus, $X \geq c[\mathbb{1}_S] + [Y]$. (1) implies, $I(X) \geq I(c[\mathbb{1}_S] + [Y])$ and Step 3 implies $I(c[\mathbb{1}_S] + [Y]) \geq I(c[\mathbb{1}_S]) + I([Y])$. (4) implies $I(c[\mathbb{1}_S]) + I([Y]) = cI([\mathbb{1}_S]) + I([Y])$. Thus, $I(X) \geq cI([\mathbb{1}_S]) + I([Y]) = c + I([Y]) > I(Y)$. (The last equality is by (6)).

Step 5: Separation. Define $D = \text{conv}\{X - I(X)\mathbb{1}_S;\ X \text{ is derived from FaF}\}$. D is a convex set in $I\!R^S$. D does not intersect the open negative orthant, because if $\sum \alpha_i(X_i) - I(X_i)\mathbb{1}_S << 0$, then Step 3 implies $I(\sum \alpha_i X_i) < I(\sum \alpha_i I(X_i)\mathbb{1}_S)$. However, Step 3 implies that both sides are equal, which is a contraction.

Thus, D and the open negative orthant can be separated by a non-zero vector[7] P: $P \cdot a \geq P \cdot b$ for every $a \in D$ and $b << 0$. Fix $a \in D$. Since for every $b << 0$, $P \cdot a \geq P \cdot b$, $P \cdot a \geq 0$. This means that $P \cdot X \geq I(X)$ for every X derived from FaF.

Note that $[\mathbb{1}_S] - I(\mathbb{1}_S)\mathbb{1}_S = [\mathbb{1}_S] - \mathbb{1}_S \in D$. Thus D intersects the closed negative orthant. Thus, $0 \geq P \cdot b$ for every $b << 0$ and thus, $P \geq 0$. We can therefore assume (one can multiply P by a positive constant if necessary) that the coordinates of P sum up to 1, meaning that P is actually a probability vector.

Step 6: Definition of $\mathcal{A}$. We obtained that $P \cdot X \geq I(X)$ for every X derived from FaF and therefore, $P \cdot (\mathbb{1}_S - X) \geq I(\mathbb{1}_S - X)$. Summing up the two inequalities one obtains, $P \cdot \mathbb{1}_S \geq I(\mathbb{1}_S)$, which is actually equality. Thus, $P \cdot X = I(X)$ for every X derived from FaF. For such X, $X = \sum \lambda_i \mathbb{1}_{R_i}$, where $\mathbb{1}_{R_i}$ are derived from X. Thus, $P \cdot X = I(X) = \sum \lambda_i I(\mathbb{1}_{R_i}) = \sum \lambda_i P \cdot \mathbb{1}_{R_i} = \sum \lambda_i P(R_i)$. Define $\mathcal{A}$ to be the algebra generated by all such R_i's.

[7]Here, '$\cdot$' denotes the inner product between two vectors.

If $X \geq \sum \lambda_i \mathbb{1}_{R_i}$, then by (1), $I(X) \geq I(\sum \lambda_i \mathbb{1}_{R_i})$. If $R_i \in \mathcal{A}$, for every i, then by Step 3, $I(\sum \alpha_i \mathbb{1}_{R_i}) = \sum \lambda_i I(\mathbb{1}_{R_i})$. Thus, $I(X) \geq \sum \alpha_i P(R_i)$. On the other hand, (2) ensures that every X has $[X]$. $[X]$ can be written as $\sum \lambda_i \mathbb{1}_{R_i}$. The functions of the sort $\mathbb{1}_{R_i}$ are derived from FaF (i.e, $[X]$). Thus, $I(X) = I([X]) = \sum \lambda_i P(R_i)$. By the definition of $[X]$, $I([X]) = \max\Big\{ \sum_i \lambda_i P(R_i);\ \sum_{R_i \in \mathcal{A}} \lambda_i \mathbb{1}_{R_i} \leq X \text{ and } \lambda_i \geq 0 \text{ for every } i\Big\}$, which is the desired claim (see eq. (1)). ■

Proof of Proposition 2. **Step 1: $I(\alpha[X] + \beta[Y]) = \alpha I([X] + \beta I([Y])$.** By Step 1 of the previous proof (which uses (1) and (4) that are common to both propositions) shows that for every function Y and a constant $c > 0$, $I(cY) = cI(Y)$.

It is claimed that if $\alpha > 0$, then $\alpha[X] = [\alpha X]$. By the definition of $[X]$, $\alpha X \geq \alpha[X]$. If, to the contrary, there is Y such that $\alpha[X] > Y$ such that $I(\alpha X) = I(Y)$, then $\alpha I(X) = I(Y)$ and $I(X) = \frac{1}{\alpha} I(Y) = I(\frac{Y}{\alpha})$. Since $[X] > \frac{Y}{\alpha}$, this means that $[X]$ is not FaF, contradicting its definition.

We obtained, $I(\alpha[X] + \beta[Y]) = I([\alpha X] + [\beta Y])$ the right hand side is equal by (3) to $I([\alpha X]) + I([\beta Y]) = I(\alpha[X]) + I(\beta[Y]) = \alpha I([X]) + \beta I([Y])$.

Step 2: If $X >> Y$ then $I(X) > I(Y)$. There is a positive c such that, $X \geq Y + c\mathbb{1}_S \geq [Y] + c\mathbb{1}_S$. From (1), $I(X) \geq I([Y]) + cI(\mathbb{1}_S) = I([Y]) + c$ (the last equality is by (6)).

Step 3: Define $D = \text{conv}\{[X] - I([X])\mathbb{1}_S\}$. D is a convex set in $\mathbb{R}^S$. Note that $\mathbb{1}_S - \mathbb{1}_S = 0 \in D$. D does not intersect the open negative orthant. Indeed, if $\sum_X \alpha_X([X] - I([X])\mathbb{1}_S) << 0$, then $\sum_X \alpha_X[X] << \sum_X \alpha_X I([X])\mathbb{1}_S$. By Step 2, $I(\sum_X \alpha_X[X]) < I(\sum_X \alpha_X I([X])\mathbb{1}_S)$ and by Step 1, $\sum_X \alpha_X I([X]) < \sum_X \alpha_X I([X]) \cdot I(\mathbb{1}_S) = \sum_X \alpha_X I([X])$, which is a contradiction.

The proof proceeds now like in the previous proof: there is distribution vector p (a separating vector) that along with the set $\mathcal{Y}$, defined as the closure of $\{[X];\ X \text{ is in } [0,1]^S\}$ represents I, as required in the proposition.

Taking the closure is needed in order to make sure that $\mathcal{Y}$ is closed, as required. Taking the closure does not hurt because if Z is the limit of a sequence $([X]_j)$, then by extracting a subsequence and by multiplying its elements by a close-to-1 constants, one can assume that Z is a limit of weakly-increasing (wrt $\geq$) sequence. If there is W such that $W \geq Z$ and $P \cdot Z > P \cdot [W]$, then it mean that there is already some $[X]_j$ in the sequence that satisfies $W \geq [X]_j$ and $P \cdot [X]_j > P \cdot [W]$. This contradicts the fact that $[W]$ is FaF. Thus, taking the closure does not make any damage.

Moreover, $\mathbb{1}_S \in \mathcal{Y}$. Thus, $\mathcal{Y}$ is both, closed and contains $\mathbb{1}_S$, as required. ■

Proof of Proposition 4. The 'only if' direction is easy to verify. As for the 'if' direction, note first that if E is FaF and if F does not intersect E, then $v(E)+v(F) = v(E \cup F)$. A set E is *minimal positive FaF* if $v(E) > 0$ and if $F \subsetneqq E$ implies $v(F) = 0$. Since S is finite, every set E contains at least one minimal positive FaF.

Secondly, let E and F be two minimal positive FaF, then $E \cap F = \emptyset$. Indeed, $v(E \cup F) = v(E \cup (F \setminus E)) = v(E) + v(F \setminus E)$. If $E \cap F \neq \emptyset$, then $F \setminus E \subsetneqq F$ and therefore, $v(F \setminus E) = 0$. Thus, $v(E \cup F) = v(E)$. By a similar argument, $v(E \cup F) = v(F)$. Due to minimality, $E \cap F$ is a strict subset of both, E and F. Thus, $v(E \cap F) = 0$, and therefore, $v(E) + v(F) = v(E \cup F) + 0$. This implies that $2v(E) = v(E)$, which contradicts the fact that $v(E) > 0$.

Let $\mathcal{A}' = \{A_1, ..., A_\ell\}$ be the set of all minimal positive FaF. The intersection of any two sets in $\mathcal{A}'$ is empty. It has been shown that v is additive over the sets in $\mathcal{A}'$. Denote by S' the union of $\mathcal{A}'$. It remains to show that $v(S) = v(S')$. Note that $v(S \setminus S') = 0$, because otherwise a minimal positive FaF would have included in $S \setminus S'$ (and therefore a part of $\mathcal{A}'$). Since each of the A_i's is FaF, $v(A_1 \cup (S \setminus S')) = v(A_1) + v(S \setminus S') = v(A_1)$, then $v(A_2 \cup (A_1 \cup (S \setminus S'))) = v(A_2) + v(A_1)$ and so forth until one gets $v(S) = v(S')$, as desired.

Finally, let $\mathcal{A}$ be the sub-algebra generated by $\mathcal{A}' \cup \{S \setminus S'\}$, define $P(A) = v(A)$ for every $A \in \mathcal{A}'$, extend P in a linear manner to $\mathcal{A}$, and for every $A \notin \mathcal{A}$ set $P(A) = \max_{B \subseteq A;\ B \in \mathcal{A}} P(B)$. It is clear that for every $A \subseteq S$, $P(A) \leq v(A)$, with equality for every $A \in \mathcal{A}$. To show a universal equality one may apply the sequential argument used in the previous paragraph. ■