

# Evaluating Information in Zero-Sum Repeated Games

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## Abstract

Two players play a zero-sum repeated game with incomplete information. Before the game starts one player receives a private signal determined by the information structure of the game. Different information structures induce different equilibrium payoffs. The value-of-information function of the game corresponds to the induced value to every information structure. We characterize those functions that may be value-of-information functions. In the case where each player can receive private information on the state we characterize the value-of-information functions of two classes of games.

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# 1 Introduction

Agents sometimes interact in an unknown environment but receive some information about the real situation. This scenario is modeled as a Bayesian game with information structure: Players have a prior distribution on the unknown state of the world and receive some partial information about the game actually played. This information affects the players' posterior distribution and thereby players' behavior. Thus, the information structure of the game affects its equilibrium payoffs.

An outside observer collects data about the Bayesian game. This data contains different information structures and the outcomes associated with them. An outside observer cannot observe players' prior distribution nor their actions. The question arises as to what observations can refute the possibility that the players play rationally in a Bayesian game. In other words, what conditions the data should satisfy in order to be consistent with a rational behavior of the agents in Bayesian games. This question is similar in spirit to a question answered by Afriat (1967). He asked: What conditions should a data that contains different prices and their corresponding consumption bundles be consistent with rational behavior of a consumer?

The issue of consistency of observed outcomes with a rational behavior of a Bayesian agent was first addressed by Gilboa and Lehrer (1991). They dealt with a decision maker who is informed by a partition of the state space (as in Aumann (1974)): the information he receives about the realized state is the cell of the partition containing that state.

Different partitions determine different optimal payoffs. The function that attaches the optimal payoff to any partition is called a value-of-information function. Gilboa and Lehrer (1991) found the conditions that the Bayesian model impose on such functions. In other words, Gilboa and Lehrer (1991) characterized those functions (defined on partitions) that can be realized as value-of-information functions.

As a first step toward a more comprehensive understanding of the role of information in interactive situations, we consider zero-sum repeated games. Folk theorems have already established that zero-sum games are not only important for their own sake but also for non-zero-sum repeated games; the value of a zero-sum game is involved in determining the equilibrium payoffs of non-zero-sum repeated games.

Zero-sum games have two advantages in examining the role of information. First,

they have a unique equilibrium payoff, the value. Second, receiving more information in zero-sum games can never be harmful. That is, the equilibrium payoff of a player cannot decrease as a result of obtaining more information. Hirshliefer (1971) was the first to notice that in general games players might prefer dropping payoff-relevant information, because their equilibrium payoff would then be higher (see also, Kamien et al. (1990), Bassan et al. (1999)).

The role of information in repeated games is different from that in one-shot games. In one-shot games it is always optimal to use all available information. In repeated games, by contrast, it might be that fully using the information is not optimal. This observation might lead one to expect that receiving more information is less beneficial in repeated games than in one-shot games. Therefore, one may think that the possible impacts of information structures on the value is different in repeated games from the possible impacts of information in one-shot games. This intuition turns out to be wrong.

We extend Gilboa and Lehrer (1991) to repeated interactive models. We consider two types of informational structures: one-sided and two-sided (independent). In both cases we characterize the functions defined over partition that can be realized as value-of-information functions.

The first result deals with the games where only one player receives information about the game actually played. It turns out that in such repeated games any monotonic function defined over partitions is a value-of-information function. In other words, for any monotonic function over partitions one can find a one-sided repeated game whose value coincides with the function. This means in particular that the Bayesian paradigm is rich enough to allow any monotonic function to be a value-of-information function for some game. That is, in order for a data about information structures and their corresponding outcomes to be consistent with the Bayesian paradigm, it is sufficient that the outcomes increase with the information. This result stands in sharp contrast with one-player decision problems. There monotonicity is not sufficient, and there is an additional non-trivial condition that a function should satisfy in order to be a value-of-information function.

The richness of the Bayesian model is also supported by our second finding which deals with two-sided information structures. The state space is a product space and each player receives information only about one component of the product. In this framework the value-of-information functions are defined on pairs of partitions; one for each player.

We introduce a new model where players can take an action called commitment before they receive any information about the real state. This commitment will not change the distribution over states nor will it change players' information. Commitments will affect only the payoffs. That is, commitments will change the outcomes of subsequent actions. It turns out that for a function defined on pairs of partitions, monotonicity in each argument is sufficient for it to be a value-of-information function for some repeated game with commitment.

An important tool used to investigate the issues described above is the value function of a one-shot zero-sum Bayesian game in which players receive no information beyond the prior. The value of such a Bayesian game can be viewed as a function from priors to numbers, called the value of the game with no information. These functions play a crucial role in the characterization of the values of repeated games with incomplete information (Aumann and Maschler (1995), Mertens et al. (1994)). The problem, which is of interest in its own right, is to characterize the real functions defined over the set of priors that are the values of some game with no information. This question can be fully answered when there are two states of nature, and remains open in the general case. In the general case we provide the properties needed for the rest of the paper.

Repeated games with incomplete information were introduced and studied by Aumann and Maschler (1995). Mertens and Zamir characterized the value of long repeated games with incomplete information on both sides. Gossner and Mertens (2001) compared different information structures in zero-sum repeated games with incomplete information. A companion paper (Lehrer and Rosenberg, 2003) considers value-of-information functions for one-shot zero-sum games.

This paper is organized as follows. In the next section we describe the model. We explain the notion of information structure and then introduce repeated games with one-sided information. The main issues and the main result is described in Section 3. A sketch of the proof is given in Section 4, and the paper ends with final comments and open problems.

## 2 The model

### 2.1 Information structures

We consider two-player zero-sum games with incomplete information. A state of nature is drawn from a finite set according to a known probability. Before the game starts the players receive signals which depend on the realized state. The information structure of the game specifies how signals depend on the state selected. As in Aumann (1976), we confine ourselves to partitional information structures.

#### *One-sided information structure*

A state  $k$  is drawn from a finite state space  $K$  according to a common prior  $p$ . The state space is partitioned into sets called cells, and when the state  $k$  is realized, player 1 is informed of the cell containing  $k$ . Formally, a *one-sided partitional information structure* (or simply *information structure*) is represented by a partition  $\mathcal{Q}$  of the set  $K$ . Denote by  $\mathcal{Q}(k)$  the cell of  $\mathcal{Q}$  that contains  $k$ . When  $k$  is realized, player 1 gets to know  $\mathcal{Q}(k)$ .

When  $\mathcal{Q} = \{K\}$ , for instance, player 1 cannot distinguish between any two states. In this case no player receives any information about  $k$ . However, if  $\mathcal{Q} = \{\{k\}; k \in K\}$ , then player 1 is fully informed of the realized state. Whatever the one-sided partitional information structure is, beyond the prior  $p$ , player 2 knows nothing about  $k$ . Note that any game has finitely many partitional information structures.

#### *Two-sided information structure*

A state  $(k, \ell)$  is drawn from a finite state space  $K \times L$  according to probability  $\pi$ .  $K$  and  $L$  are partitioned into  $\mathcal{P}$  and  $\mathcal{Q}$ , respectively. Player 1 is informed of  $\mathcal{P}(k)$  and player 2 of  $\mathcal{Q}(\ell)$ . Note that the probability  $\pi$  does not need to be a product of two independent probabilities, one on  $K$  and one on  $L$ . If  $\pi$  has full support and  $K$  and  $L$  are not singletons, then none of the players is fully informed of the realized state  $(k, \ell)$ . If  $K = L$  and  $k = \ell$  with probability 1, then the state space is equivalent to a regular space (not a product), where each player is informed of a different partition.

### 2.2 The repeated game

In this section we describe how the repeated game proceeds after the players receive their information. The game is an extension of the incomplete information game defined by Aumann and Maschler (1995).

### 2.2.1 The one-sided information game

The  $n$ -stage game with one-sided information, denoted by  $\Gamma_n(p, \mathcal{Q})$ , is defined by an integer  $n$ , an information structure  $\mathcal{Q}$ , a probability  $p$  over  $K$ , a finite set of actions for each player  $i \in \{1, 2\}$ ,  $A_i$ , and a payoff function  $g$  from  $K \times A_1 \times A_2$  to the reals. The payoff associated with  $(k, a_1, a_2)$  is denoted by  $g_k(a_1, a_2)$ .

The game is played as follows: At stage 0 nature chooses an element  $k$  of  $K$  with probability  $p$  player 1 is then informed of  $\mathcal{Q}(k)$ . The game is played in  $n$  stages. At stage  $m = 1, \dots, n$  players 1 and 2 simultaneously choose actions according to probability distributions that may depend on the history of previous actions and signals. If the realized state is  $k$  and the pair of chosen actions is  $(a_1^m, a_2^m)$  the payoff at stage  $m$  is  $g_k(a_1^m, a_2^m)$ . The pair of chosen actions (and not the payoff) is then announced to both players and the game proceeds to its next stage. The payoff in  $\Gamma_n(p, \mathcal{Q})$  is the expected average of the  $n$ -stage payoffs received during the game.

A *behavior strategy* of player 1 is a sequence  $\tau_1 = (\tau_1^1, \tau_1^2, \dots, \tau_1^m, \dots)$ , where  $\tau_1^m$  is a function from his information at stage  $m$ ,  $\mathcal{Q} \times (A_1 \times A_2)^{m-1}$  to the set<sup>1</sup>  $\Delta(A_1)$  of probability distributions over his set of actions. A behavior strategy of player 2 is a sequence  $\tau_2 = (\tau_2^1, \tau_2^2, \dots, \tau_2^m, \dots)$ , where  $\tau_2^m$  is a function from his information at stage  $m$ ,  $(A_1 \times A_2)^{m-1}$  to the set  $\Delta(A_2)$ . When applied to the game  $\Gamma_n(p, \mathcal{Q})$ , all  $\tau_i^m$ ,  $m > n$ , are payoff irrelevant.

The probability distribution  $p$ , the partition  $\mathcal{Q}$  and the pair of strategies  $\tau_1, \tau_2$  induce a probability over the set of histories of length  $n$ ,  $H^n = K \times (A_1 \times A_2)^n$ . The expectation with respect to this probability will be denoted by  $\mathbf{E}_{\tau_1 \tau_2}^{p, \mathcal{Q}}$  or simply by  $\mathbf{E}$  when no confusion can arise. If the players use the strategies  $\tau_1, \tau_2$ , the associated payoff in the  $n$ -stage game is  $\gamma_n^{\mathcal{Q}}(\tau_1, \tau_2, p) = \mathbf{E}_{\tau_1 \tau_2}^{p, \mathcal{Q}} \left[ \frac{1}{n} \sum_{m=1}^n g_k(a_1^m, a_2^m) \right]$ . This is the expected average payoff received along the  $n$  stages of the game.

The game  $\Gamma_n(p, \mathcal{Q})$  is a finite game and therefore, by the minmax theorem, has a value denoted by  $v_n^{\mathcal{Q}}(p, (g_k)_{k \in K})$ . The following proposition whose proof is postponed to the appendix states that this sequence converges. It extends the same result proved in Aumann and Maschler (1995) that refers to the case in which player 1 gets full information on  $k$ .

**Proposition 1** *The sequence  $v_n^{\mathcal{Q}}(p, (g_k)_{k \in K})$  has a limit denoted by  $v^{\mathcal{Q}}(p, (g_k)_{k \in K})$ .*

The function  $v^{\mathcal{Q}}(p, (g_k)_{k \in K})$  will be referred to as the long-run value of the game. This is an approximation of the equilibrium payoff in long games. The long duration

<sup>1</sup>Throughout this paper  $\Delta(X)$  denotes the set of probability distributions over a set  $X$ .

of the games discussed implies in particular that player 1 might want to use only a part of the information he has. This is so because any use of information may increase his payoff in the short-run, but also reveal important information and thereby give a relative informational advantage to player 2. This might be harmful to player 1 in the long run. Obviously, all these arguments are imbedded in the long-run value.

**Remark 1** One could also define the infinitely repeated discounted version of this game. The strategy sets would be the same but the  $\lambda$ -discounted payoff would then be  $\gamma_\lambda^\mathcal{Q}(\tau_1, \tau_2, p) = \mathbf{E}_{\tau_1 \tau_2}^{p, \mathcal{Q}} [\sum_{m=1}^{\infty} \lambda(1-\lambda)^{m-1} g_k(a_1^m, a_2^m)]$ . By a standard minimax theorem this game has a value denoted  $v_\lambda^\mathcal{Q}(p, (g_k)_{k \in K})$ . Furthermore,  $\lim_{\lambda \rightarrow 0} v_\lambda^\mathcal{Q}(p, (g_k)_{k \in K})$  exists and is equal to  $v^\mathcal{Q}(p, (g_k)_{k \in K})$ . One could also define the uniform value of the game as in Aumann and Maschler (1995), again this value exists and is equal to the long-run value of the game.

### 2.2.2 Two-sided information games

In a game with incomplete information on both sides, both players get some information on the state of the world. This model follows the one introduced by Aumann and Maschler (1995). The  $n$ -stage game is defined by an integer  $n$ , two sets  $K$  and  $L$ , two information structure  $\mathcal{P}$  and  $\mathcal{Q}$ , a probability  $\pi$  over  $K \times L$ , a finite set of actions for each player  $i \in \{1, 2\}$ ,  $A_i$  and a payoff function  $g$  from  $K \times L \times A_1 \times A_2$  to the reals. The payoff associated with  $(k, \ell, a_1, a_2)$  is denoted by  $g_{k, \ell}(a_1, a_2)$ . The game is played like the one sided information game except that at the beginning of the game nature chooses  $(k, \ell)$  from  $K \times L$  with probability  $\pi$ . Player 1 is then informed of  $\mathcal{P}(k)$  and player 2 of  $\mathcal{Q}(\ell)$ .

A *behavior strategy* of player 1 is a sequence  $\tau_1 = (\tau_1^1, \tau_1^2, \dots, \tau_1^m, \dots)$ , where  $\tau_1^m$  is a function from  $\mathcal{P} \times (A_1 \times A_2)^{m-1}$  to  $\Delta(A_1)$ . A behavior strategy of player 2 is a sequence  $\tau_2 = (\tau_2^1, \tau_2^2, \dots, \tau_2^m, \dots)$ , where  $\tau_2^m$  is a function from his information at stage  $m$ ,  $\mathcal{Q} \times (A_1 \times A_2)^{m-1}$  to the set  $\Delta(A_2)$ . As before the game is played in  $n$  stages and we denote by  $v_n^{\mathcal{P}, \mathcal{Q}}(p, (g_{k, \ell})_{k \in K \ell \in L})$  its value.

Proposition 1 extends to this framework. The following proposition is an extension of a similar proposition proved in Mertens and Zamir (1971) for the case in which player 1 knows  $k$  and player 2 knows  $\ell$ .

**Proposition 2** *The sequence  $v_n^{\mathcal{P}, \mathcal{Q}}(\pi, (g_{k, \ell})_{k \in K \ell \in L})$  has a limit (denoted by  $v^{\mathcal{P}, \mathcal{Q}}(\pi, (g_{k, \ell})_{k \in K \ell \in L})$ ).*

### 2.2.3 Games with commitment

Suppose that before getting any information about the realized state the players can make a commitment. This commitment will affect the payoffs during the game and once taken is not amendable. To allow the possibility of making a commitment we extend the model of repeated games and introduce a commitment stage before the game starts.

Before receiving any information the players can choose actions from the sets  $C_i$ ,  $i = 1, 2$ . This pair of actions is observed and then a regular repeated game with incomplete information is played. The distribution  $\pi$  over the states does not depend on the actions chosen at the commitment stage, while the payoff functions may depend on them.

Formally, a *game with commitment* is defined as the regular repeated game with the addition of two sets,  $C_1$  and  $C_2$  (the sets of commitments). The payoff functions  $(g_{k,\ell})_{k \in K, \ell \in L}$  are defined now on the product  $C_1 \times C_2 \times A_1 \times A_2$ . Thus, the payoff at every stage depends not only on the pair of actions taken at that stage but also on the pair of commitments taken at the commitment stage (that takes place before the game starts).

In order to justify the name ‘game with commitment’ one may think of the set  $C_i$  as a set of actions. Had  $C_i$  been a regular set of actions, the action set of player  $i$  would have been  $A_i \times C_i$ . That is, player  $i$  would have to choose at every stage a pair from  $A_i \times C_i$ , where the second component (the one taken from  $C_i$ ) may vary across stages. However, in a game with commitment the choice of an action from  $C_i$  should be fixed along histories. In other words, player  $i$  commits to choosing the *same* action in  $C_i$  at each stage. Furthermore, this choice is independent of his information.

Note that there are more games with commitments than repeated games with incomplete information since any repeated game with incomplete information can be considered as a game with commitment in which both sets  $C_i$  are singletons.

Let  $v_n^{\mathcal{P}, \mathcal{Q}}(p, (g_{k,\ell})_{k \in K, \ell \in L})$  denote the value of the  $n$ -stage game. Then a consequence of Proposition 2 is:

**Proposition 3** *The sequence  $v_n^{\mathcal{P}, \mathcal{Q}}(p, (g_{k,\ell})_{k \in K, \ell \in L})$  has a limit (denoted by  $v^{\mathcal{P}, \mathcal{Q}}(p, (g_{k,\ell})_{k \in K, \ell \in L})$ ).*

## 2.3 The value-of-information functions

In this paper we focus on the long-run value of the game viewed as a function of the information structure. This will be called the *value of information* in this game. More precisely, in the one sided information case, let  $(g_k)_{k \in K}$  be a payoff function and let  $p$  be a distribution over  $K$ . In this fixed game the effect of the information  $\mathcal{Q}$  on the payoff is reflected by the value  $v^{\mathcal{Q}}(p, (g_k)_{k \in K})$ . As we are interested in the role of information we consider it as a function over partitions and call it the *value-of-information function* of the game. In general we define *value-of-information functions* as follows.

**Definition 1** 1. A function  $f$  defined over all partitions of  $K$  is the *value-of-information of an incomplete information game with one sided information and partitional information structure* if there is a distribution  $p$  over  $K$  and if there are payoff functions  $(g_k)_{k \in K}$  such that for any partitions  $\mathcal{Q}$  over  $K$ ,  $f(\mathcal{Q}) = v^{\mathcal{Q}}(p, (g_k)_{k \in K})$ .

2. A function  $f$  defined over all the pairs of partitions of  $K$  and  $L$  is the *value-of-information of an incomplete information game with partitional information structure* if there is a distribution  $\pi$  over  $K \times L$  and a repeated game with incomplete information defined by some payoff functions  $(g_{k,\ell})_{k \in K, \ell \in L} : A_1 \times A_2 \rightarrow R$  such that for any partitions  $\mathcal{P}, \mathcal{Q}$  over  $K$  and  $L$ ,  $f(\mathcal{P}, \mathcal{Q}) = v^{\mathcal{P}, \mathcal{Q}}(p, (g_{k,\ell})_{k \in K, \ell \in L})$ .

3. A function  $f$  defined over all the pairs of partitions of  $K$  and  $L$  is the *value-of-information function of an incomplete information game with partitional information structure and commitment* if there is a distribution  $\pi$  over  $K \times L$  and a repeated game with incomplete information and commitment defined by some payoff functions  $(g_{k,\ell})_{k \in K, \ell \in L} : C_1 \times C_2 \times A_1 \times A_2 \rightarrow R$  such that for any partitions  $\mathcal{P}, \mathcal{Q}$  over  $K$  and  $L$ ,  $f(\mathcal{P}, \mathcal{Q}) = v^{\mathcal{P}, \mathcal{Q}}(p, (g_{k,\ell})_{k \in K, \ell \in L})$ .

## 3 The results

### 3.1 One-sided information

The main issue in this paper is to study the properties that value-of-information functions must owe. Formally, let  $V(\mathcal{Q})$  be a function over partitions  $\mathcal{Q}$  of a fixed state space  $K$ . The question arises as to when there exist payoff functions  $(g_k)_{k \in K}$

and a distribution  $p$  over  $K$  such that for any  $\mathcal{Q}$ ,  $V(\mathcal{Q}) = v^{\mathcal{Q}}(p, (g_k)_{k \in K})$ . Answering this question amounts to giving an axiomatic characterization to value-of-information functions.

The same problem in the setting of a one-player decision problem has been analyzed by Gilboa and Lehrer (1991). Note that the one-player case is a particular case of a zero-sum game (with player 2 having only one action). Since there are more zero-sum games than one-player decision problems, there are more value-of-information functions of zero-sum games than of one-player decision problems. Thus, the conditions that characterize value-of-information functions of zero-sum games are less restrictive than those characterizing value-of-information functions of one-player decision problems.

The following monotonicity condition is clearly necessary for a value-of-information function. Indeed, in zero-sum games it is always beneficial for a player to have more information.

**Definition 2** *A function  $v$  from the set of partitions of a finite set  $K$  to the real numbers is said to be monotonic if for any two partitions  $\mathcal{Q}$  and  $\mathcal{Q}'$ , the fact that  $\mathcal{Q}$  refines  $\mathcal{Q}'$  (i.e., for any  $T \in \mathcal{Q}$  there is an  $T' \in \mathcal{Q}'$  such that  $T \subset T'$ ) implies  $v(\mathcal{Q}) \geq v(\mathcal{Q}')$ .*

The first theorem states that this condition is also sufficient.

**Theorem 1** *Let  $V$  be a function from the set of partitions of  $K$  to the real numbers. The function  $V$  is a value-of-information function of a repeated game with state space  $K$ , one-sided information, and partitional signaling if and only if it is monotonic.*

The sketch of the proof of this theorem is given in section 4 and a detailed proof is postponed to the Appendix.

The main implication of this theorem is that even when one takes into account the strategic use of information there is no restriction on value-of-information functions (except for the trivial one of monotonicity) imposed by the Bayesian paradigm. This means that the model of Bayesian rational players is rich enough to encompass a very wide range of possible effects of information. As long as the outcome of the game is monotonic with information, it is consistent with rational behavior of players in Bayesian games. Our main result implies in particular, that there is no restriction on the first derivative of the value as a function of information: the effect of a fixed addition information on the value can go down or up with the information already existing.

In Lehrer and Rosenberg (2003) it is proved that the same characterization holds for value-of-information functions of one-shot games. Therefore, repetition does not affect the *set* of value-of-information functions, although it might affect the value of a particular game.

### 3.2 Two-sided information

In this subsection we extend the previous result to games with two-sided information.

**Theorem 2** *Let  $V$  be a real function defined over pairs of partitions: one of  $K$  and one of  $L$ . The function  $V$  is a value-of-information function of a repeated game with incomplete information on both sides with state space  $K \times L$  and with commitment if and only if it is increasing in  $\mathcal{P}$  and it is decreasing in  $\mathcal{Q}$ .*

The conditions are necessary as a consequence of the Theorem 1. As in Theorem 1, this theorem states that these conditions are also sufficient. In other words, as soon as the players can commit to some action independently of the information they receive, the Bayesian model does not impose any restriction beyond monotonicity on the nature of the value as a function of information.

The set of value-of-information functions of games with one-sided information and commitment is larger than the set of value-of-information functions of games with one-sided information and no commitment. On the other hand any value-of-information function of a game with commitment is increasing. Thus, Theorem 1 implies that in the one-sided information case the set of value-of-information functions of games with and without commitment coincide.

A close examination of the proof of the theorem reveals that the probability  $\pi$  used to define the game can be taken to be a product of independent probabilities. Moreover, the payoff function of the repeated game is the sum of a payoff that depends only on  $k$  and on the actions and a payoff that depends only on  $\ell$  and the actions.

A natural question to ask is which functions are value-of-information functions of repeated games with no commitment. The following result gives a partial answer in the case in which  $k$  and  $\ell$  are independent and payoffs satisfy the following condition of separability.

**Definition 3** *Let  $K, L, A_1, A_2$  be finite sets. A payoff function  $g : K \times L \times A_1 \times A_2 \rightarrow R$  is separable, if (i) each set  $A_i$  can be written as a product  $A_i = A'_i \times A''_i$ , (ii) there exist two functions  $g_1$  and  $g_2$  such that  $g_1 : K \times A'_1 \times A'_2 \rightarrow R$  and  $g_2 : L \times A''_1 \times A''_2 \rightarrow R$*

such that for any  $(k, l, a'_1, a'_2, a''_1, a''_2) \in K \times L \times A'_1 \times A'_2 \times A''_1 \times A''_2$ ,  $g(k, l, a'_1, a'_2, a''_1, a''_2) = g_1(k, a'_1, a'_2) + g_2(l, a''_1, a''_2)$ .

Let  $\mathcal{T}$  denote the trivial partition. The following definition is needed for the stating the result.

**Definition 4** *A real function  $V$  defined over pairs of partitions, one of  $K$  and one of  $L$ , satisfies the independence condition if for any partition  $\mathcal{Q}$  of  $L$ ,  $V(\mathcal{P}, \mathcal{Q}) - V(\mathcal{P}, \mathcal{T})$  is independent of  $\mathcal{P}$ .*

$V(\mathcal{P}, \mathcal{Q}) - V(\mathcal{P}, \mathcal{T})$  is the value of player 2's partition,  $\mathcal{Q}$ , when player 1 is informed through  $\mathcal{P}$ . When the distribution over states is a product probability and the payoff function is separable this value does not depend on the information player 1 has on  $k$  and therefore the value-of-information satisfies the independence condition.

**Theorem 3** *Let  $V$  be a real function defined over pairs of partitions, one of  $K$  and one of  $L$ . The function  $V$  is a value-of-information function of a repeated game with incomplete information on both sides with state space  $K \times L$ , in which  $k$  and  $\ell$  are chosen independently and the payoff function is separable if and only if it is increasing in  $\mathcal{P}$  and it is decreasing in  $\mathcal{Q}$  and satisfies the independence condition.*

The set of games with commitment includes the set of games with no commitment. Therefore, the set of value-of-information functions of games with commitment contains the set of value-of-information functions of games with no commitment. The set of value-of-information functions in games with no commitment is unknown. However, in the case of independent information and a separable payoff we provide a full characterization. Theorem 3 states in particular that any increasing in  $\mathcal{P}$ , decreasing in  $\mathcal{Q}$  function that satisfies the independence condition is value-of-information function of a game with a separable payoff and independent information.

## 4 The proofs

In this section we provide a sketch of Theorem 1's proof and the proofs of Theorems 2 and 3. An elaborate proof of Theorem 1 is deferred to the Appendix. The proofs employ a few tools that will be described in the following subsections.

## 4.1 The one-shot game

The first tool we need is the value of the one-shot Bayesian game.

**Notation 1** Suppose that  $p = (p_k) \in \Delta(K)$ .  $u(p)$  is the value of the game defined by the action sets  $A_1, A_2$  and the payoff  $\sum_{k \in K} p(k)g_k(a_1, a_2)$ , when the pair of actions  $(a_1, a_2)$  is played.

The value of the one-shot Bayesian game with null information (i.e., no player obtains additional information about the realized state beyond the prior  $p$ ) is denoted  $u(p)$ . This function plays also a central role in the theory of repeated games with incomplete information (see Aumann and Maschler, 1995).

**Definition 5** A function  $u$  defined on  $\Delta(K)$  is realizable if there are games  $G_k$ ,  $k \in K$  with the same action sets, such that the value of  $\bar{G}(p) = \sum_{k \in K} p(k)G_k$  is  $u(p)$  for every  $p$  in  $\Delta(K)$ .

The proof of Theorem 1 relies on the fact that the set of realizable functions is large. Mertens and Zamir (1971) proved that the set of realizable functions is dense in the set of continuous functions. This is not sufficient for our purposes since we need a precise realization and not merely an approximated one. The following proposition is what we need.

### Proposition 4

- (i) Given a finite number of pairs  $(x_\ell, y_\ell) \in \Delta(K) \times \mathbb{R}$ ,  $\ell = 1, \dots, L$ , there is a realizable function  $u$  such that  $u(x_\ell) = y_\ell$ ,  $\ell = 1, \dots, L$ .
- (ii) If  $C_1$  is finite,  $C_2$  is a union of closed polygons,  $C_1 \cap C_2 = \emptyset$  and  $c_1$  and  $c_2$  are two numbers such that  $c_1 > c_2$ , then there is a realizable function  $u$  that satisfies  $u(x) \leq c_2$  when  $x \in C_2$ ,  $u(x) = c_1$  when  $x \in C_1$ , and  $u(x) \leq c_1$  otherwise.
- (iii) Let  $C_1$  and  $C_2$  be two disjoint closed semi-algebraic sets, and  $f_1$  and  $f_2$  two realizable functions. Then, there is a realizable function  $u$  that satisfies  $u(x) = f_1(x)$  when  $x \in C_1$ , and  $u(x) = f_2(x)$  when  $x \in C_2$ .

## 4.2 The long-run value

The main results of the paper are based on a well-known result by Aumann and Maschler (1995). It characterizes the long-run value of the game. In order to describe this result we need the following notation.

**Notation 2** Let  $f$  be a real-valued function defined on  $\Delta(K)$ .  $\text{cav}(f)$  denotes the minimal concave function which is greater than or equal to  $f$ .

**Theorem 4** (Aumann and Maschler, 1995) In a repeated one-sided information game where player 1 is fully informed of the state (i.e., when  $\mathcal{Q} = \{\{k\}, k \in K\}$ ), the value of information,  $v^{\mathcal{Q}}(p, (g_k)_{k \in K})$ , is equal to  $\text{cav}(u)(p)$ .

Our proofs resort to an extension of this characterization.

For a partition  $\mathcal{Q}$  and  $B \in \mathcal{Q}$  denote by  $p(B)$  the probability  $\sum_{k \in B} p(k)$ . Denote by  $\pi_p(\cdot|B)$  the conditional probability over  $K$ , given the set  $B \subset K$ ; that is,  $\pi_p(k|B) = \frac{p(k)}{p(B)}$ .

**Notation 3** For a partition  $\mathcal{Q}$  denote by  $\mathcal{M}(\mathcal{Q})$  the set of matrices  $M = (m_{iB})_{\substack{i \leq D \\ B \in \mathcal{Q}}}$  with  $D$  lines ( $D$  can be any positive integer) and  $|\mathcal{Q}|$  columns such that for any  $i, B$ ,  $m_{iB} \geq 0$ , and for any  $B$ ,  $\sum_{i \leq D} m_{iB} = p(B)$ . For such a matrix we denote by  $m_i$  the quantity  $\sum_{B \in \mathcal{Q}} m_{iB}$ , and by  $p_i(M)$  the probability distribution over  $K$  defined by,

$$p_i(M)(k) = \frac{\sum_{B \in \mathcal{Q}} m_{iB} \pi_p(k|B)}{m_i}.$$

The following proposition extends Theorem 4.

**Proposition 5** The value of the game with one-sided information is

$$v^{\mathcal{Q}}(p, (g_k)_{k \in K}) = \max \left\{ \sum_{i \leq D} m_i u(p_i(M)) \mid M \in \mathcal{M}(\mathcal{Q}) \right\}. \quad (1)$$

The concavification of  $u$  can be defined also as  $\text{cav}(u)(p) = \max \left\{ \sum_{i \leq D} m_i u(p_i) \mid m_i \geq 0, \sum_i m_i = 1, \sum_i m_i p_i = p \right\}$ . Proposition 5 states that the long-run value is the maximum over a smaller class of possible  $(m_i, p_i)$ ,  $i \leq D$ . In this sense  $v^{\mathcal{Q}}(p, (g_k)_{k \in K})$  is a local concavification of the function  $u$ .

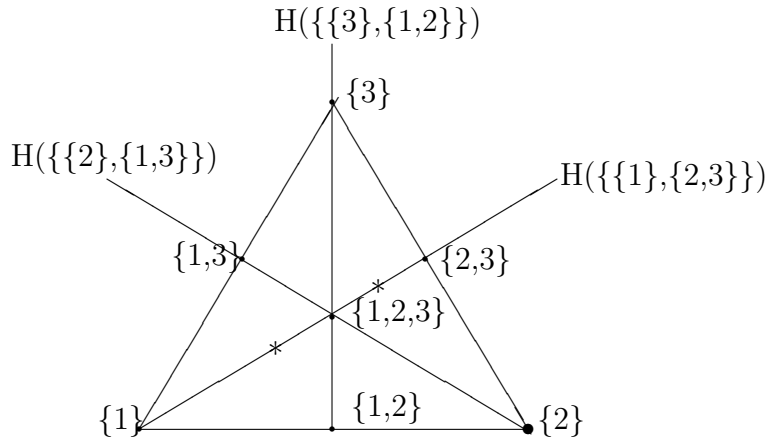
### 4.3 A sketch of the proof of Theorem 1

Let  $V$  be a monotonic function of partitions. Our goal is to construct a game for which the long-run value associated with the partition  $\mathcal{Q}$  is equal to  $V(\mathcal{Q})$ . To this end we construct a realizable function  $u$  such that  $V(\mathcal{Q}) = v^{\mathcal{Q}}(p, (g_k)_{k \in K})$  as defined by equation (1).

Let  $p$  be the uniform probability  $(\frac{1}{|K|}, \dots, \frac{1}{|K|})$ . A probability distribution over the state space is represented by a point in the simplex. A subset  $B$  of  $K$  is represented by the point in the simplex associated with the conditional probability  $\pi(\cdot|B)$ . A partition is represented by a linear subspace, denoted by  $H(\mathcal{Q})$ , spanned by  $\pi(\cdot|B)$ , where  $B$  runs over all the cells of the partition. Note that  $H(\mathcal{Q}) \subset H(\mathcal{Q}')$  if and only if  $\mathcal{Q}'$  refines  $\mathcal{Q}$ . Note moreover that for every partition  $\mathcal{Q}$ ,  $p \in H(\mathcal{Q})$  (see the figure below).

To satisfy equation (1) it is sufficient to find  $u$  that satisfies the following two conditions: (i) If  $\mathcal{Q}'$  refines  $\mathcal{Q}$ , then  $u$  is less than or equal to  $V(\mathcal{Q}')$  on  $H(\mathcal{Q})$ ; and (ii) For each partition  $\mathcal{Q}$  there exist points  $p_1, \dots, p_{|\mathcal{Q}|}$  in  $H(\mathcal{Q})$ , such that  $u(p_\ell) = V(\mathcal{Q})$  and  $p$  can be written as a convex combination of  $p_1, \dots, p_{|\mathcal{Q}|}$ . Such a realizable function exists by the Proposition 4.

The following figure illustrates the case of  $K = \{1, 2, 3\}$ . The partition  $\mathcal{Q} = \{1, 2, 3\}$  is represented by the 0-dimensional space  $H(\{1, 2, 3\})$ , which is the center of the triangle, denoted in the figure as  $\{1, 2, 3\}$ ; the partitions of  $K$  into two sets are represented by lines and  $H(\{\{1\}, \{2, 3\}\})$  is the whole simplex. The points marked  $*$  are on  $H(\{\{1\}, \{2, 3\}\})$  and the center of the triangle is a convex combination of them.



The detailed proof, provided in the appendix, shows that if  $V$  is monotonic, then there is a realizable function  $u$  such that the value  $u$  attains at the center is less than or equal to  $V(\{\{1\}, \{2, 3\}\})$  while the values  $u$  attains at the points marked  $*$  is equal to it.

## 4.4 Proof of Theorem 2

Let  $p$  be the product of two independent probability distributions:  $\mu$  over  $K$  and  $\nu$  over  $L$ , both with full support. Suppose that  $V$  is an increasing function in  $\mathcal{P}$  and decreasing in  $\mathcal{Q}$ .

### 4.4.1 A duality relation

In this section we define a notion of duality that will be used in the sequel. For each function  $V$  which is increasing in  $\mathcal{P}$  and decreasing in  $\mathcal{Q}$ , one can define a dual function  $W$  as follows:  $W$  is a function of a partition of  $K$  and an increasing cost function  $c$  defined over partitions of  $L$ .

Set,

$$W(\mathcal{P}, c) = \min_{\mathcal{Q}} [V(\mathcal{P}, \mathcal{Q}) + c(\mathcal{Q})].$$

$W$  is a minimum of linear functions in  $c$  and it is therefore concave in  $c$ . This function is reminiscent of the Fenchel conjugate of a convex function (see, Rockefeller (1970)). The introduction of such functions into the area of repeated games is due to De Meyer (1996).

Assuming that  $V(\mathcal{P}, \mathcal{Q})$  indeed represents the value of a game  $G(\mathcal{P}, \mathcal{Q})$ , we interpret the function  $W(\mathcal{P}, c)$  as the value of the dual game. In the dual game, player 2 first buys an information structure  $\mathcal{Q}$  at cost  $c(\mathcal{Q})$ , and then the game  $G(\mathcal{P}, \mathcal{Q})$  is played. It turns out that when the cost is  $c$ , the value of the dual game is  $W(\mathcal{P}, c)$ .

**Remark 2**  $V$  is increasing in its first argument. Thus,  $W$  is also increasing in its first argument.

The following lemma, proved in the Appendix, states that there is a duality relation between  $V$  and  $W$ .

**Lemma 1** *If  $V$  is increasing in  $\mathcal{P}$  and decreasing in  $\mathcal{Q}$ , then*

$$V(\mathcal{P}, \mathcal{Q}) = \max_c [W(\mathcal{P}, c) - c(\mathcal{Q})],$$

*where the maximum is taken over increasing cost functions  $c(\cdot)$ .*

#### 4.4.2 Two auxiliary games

We employ Theorem 1 and define two auxiliary games. The repeated game needed for the proof of Theorem 2 will combine these two games.

Suppose that player 2 is the informed player in a one-sided game. For each increasing cost function  $c$ , Theorem 1 ensures that  $-c$  is a value-of-information function. Thus, there is a one-sided information game (player 2 is informed of  $\mathcal{Q}$ ),  $\Gamma_c$  with state space  $L$ , whose long-run value is  $-c(\mathcal{Q})$ .

Fix a cost function  $c$ . By Remark 2,  $W$  is increasing in  $\mathcal{P}$ . Therefore, by Theorem 1 there is a one-sided information game,  $G_c$ , with state space  $K$  whose long-run value is  $W(\mathcal{P}, c)$  when player 1 gets the information  $\mathcal{P}$ .

#### 4.4.3 The game

Define the following repeated game with commitment: Let  $C_1$  be the set of all increasing cost functions  $c$  and let  $C_2$  be a singleton. At the commitment stage, which takes place at the beginning of the game, player 1 commits to a cost function  $c$ . The state  $(k, \ell)$  is then drawn from  $K \times L$  according to the commonly known product probability  $p$ ; player 1 is informed by  $\mathcal{P}$  and player 2 by  $\mathcal{Q}$ . In all subsequent stages, the players play simultaneously in the stage games of  $\Gamma_c$  and  $G_c$ . The payoff is the sum of the payoffs.

#### 4.4.4 Analysis

Given the commitment  $c$ , the players play  $\Gamma_c$  and  $G_c$  simultaneously. Since  $k$  and  $\ell$  are independent, the optimal strategy of player 2 in  $G_c$  does not depend on and is unaffected by information that he may have about  $k$ . Likewise, the optimal strategy of player 1 in  $\Gamma_c$  does not depend on and is unaffected by information that he may have about  $\ell$ . Therefore, the value of the repeated game after the commitment  $c$  has been made is the sum of  $W(\mathcal{P}, c)$  and  $-c(\mathcal{Q})$ . The optimal commitment of player 1 is therefore to take  $c$  that maximizes this quantity. Thus, the value is  $\max_{c \in C} W(\mathcal{P}, c) - c(\mathcal{Q}) = V(\mathcal{P}, \mathcal{Q})$ , by Lemma 1.

### 4.5 Proof of theorem 3

#### 4.5.1 The ‘If’ side

We first prove that if  $V$  satisfies the conditions of Theorem 3, then it is a value-of-information function of a game with independent information and separable payoffs.

Suppose that  $V$  satisfies the conditions of Theorem 3. We define a dual function  $W$  as in subsection 4.4.1 and two auxiliary games as in subsection 4.4.2. Note that each game depends on either  $k$  or on  $\ell$  but not on both, and that they have different action sets. We now define a game similar to the game defined in subsection 4.4.3.

For each cost function  $c$ , we define the following repeated game with incomplete information, denoted  $\tilde{\Gamma}(c)$ .  $(k, \ell)$  is drawn from  $K \times L$  according to a product probability, player 1 is informed by  $\mathcal{P}$  and player 2 by  $\mathcal{Q}$ . Then, the players simultaneously play  $G_c$  and  $\Gamma_c$ .

The independence condition (recall Definition 4) implies the following lemma whose proof appears in the appendix.

**Lemma 2** *Assume  $V$  is increasing in  $\mathcal{P}$ , decreasing in  $\mathcal{Q}$  and satisfies independence. Then, there exists an increasing cost function  $c^*$  such that for any partitions  $\mathcal{P}$  and  $\mathcal{Q}$ ,*

$$W(\mathcal{P}, c^*) = V(\mathcal{P}, \mathcal{Q}) + c^*(\mathcal{Q}).$$

As above, since  $p$  is a product probability, the value of  $\tilde{\Gamma}(c)$  is  $W(\mathcal{P}, c) - c(\mathcal{Q})$ . Therefore, the value of the repeated game  $\tilde{\Gamma}(c^*)$  is  $W(\mathcal{P}, c^*) - c^*(\mathcal{Q}) = V(\mathcal{P}, \mathcal{Q})$ . Since  $\tilde{\Gamma}(c^*)$  is a game with separable payoffs, it completes the proof of the ‘If’ direction.

#### 4.5.2 The ‘Only If’ side

We now prove that for any repeated game  $G$  with independent information and separable payoffs the value of information satisfies the independence condition. Assume that  $g = g_1 + g_2$  where  $g_1$  depends only on  $k$  and the actions  $(a'_1, a'_2)$ , and  $g_2$  depends only on  $\ell$  and the actions  $(a''_1, a''_2)$ . Let  $V_1$  (resp.  $V_2$ ) be the value of information function associated with the repeated game  $G_1$  with one-sided information, state space  $K$  and payoff function  $g_1$  (resp.  $G_2$  with state space  $L$  and payoff function  $g_2$ ).

Since  $k$  and  $\ell$  are independent, the fact that player 2 receives information about  $\ell$  (resp. player 1 about  $k$ ) does not affect his behavior in  $G_1$  (resp in  $G_2$ ). Therefore, the value of this game is  $V_1(\mathcal{P}) + V_2(\mathcal{Q}) = V(\mathcal{P}, \mathcal{Q})$ . Thus, for any  $\mathcal{P}$  and  $\mathcal{Q}$ ,

$$\begin{aligned} V(\mathcal{P}, \mathcal{Q}) - V(\mathcal{P}, \mathcal{T}) &= V_1(\mathcal{P}) + V_2(\mathcal{Q}) - V_1(\mathcal{P}) - V_2(\mathcal{T}) \\ &= V_2(\mathcal{Q}) - V_2(\mathcal{T}), \end{aligned}$$

Since  $V_2(\mathcal{Q}) - V_2(\mathcal{T})$  does not depend of  $\mathcal{P}$ , the proof is complete.

## 5 Comments and open problems

### 5.1 General information structures

The main theorem focuses on partitional structures. It would be interesting to characterize the value-of-information functions defined over more general signaling structures.

### 5.2 The value of information in repeated games with incomplete information on both sides

We extended theorem 1 to the case in which both players get some information and the state space is  $S = K \times L$  and in which commitment is allowed at least to one player. We do not know how to characterize the set of value of information functions of games without commitment. We conjecture that any function that is increasing in  $\mathcal{P}$  and decreasing in  $\mathcal{Q}$  is a value of information function.

Moreover, even in games with commitment, we constructed a game in which the probability over the state space is a product of two independent probabilities. This result does not extend to the case in which the players are informed through distinct partitions of the *same* space  $S$ . Characterizing the value-of-information functions for games with incomplete information on both sides and a general state space is an open problem.

### 5.3 The value of one-shot games with one-sided information

In the proofs, we relied on some properties of the function  $u(p)$ ; the value of the one-shot Bayesian game with null information. A similar analysis of a one-sided game is of interest. More elaborately, suppose that a game  $G_k$  is chosen with probability  $p(k)$ , is informed to the maximizing player and is played once. Denote the value of this game by  $w(p)$ .

It is clear that every such function  $w$  is a function  $u$  of some game with null information. Moreover, any such function is concave. The question arises as to whether every concave function which is a value-of-information function of a game with null information is a value-of-information function of a one-shot one-sided information game.

## 5.4 The general null information case

One can fully characterize the value of games with null information and only two states. Indeed, along the same lines as the one provided in the appendix, one can prove that when there are two states, if  $f$  is a real polynomial that does not vanish, then  $1/f$  is realizable. This in turn implies that a function is realizable if and only if it is piecewise rational.

In the case where there are more than two states, we have only partial results. We could prove that every polynomial is realizable, and that for every continuous piecewise rational function  $q$  and every  $\varepsilon > 0$ , there is a realizable function  $u$  that coincides with  $q$  over a set whose Lebesgue measure is  $1 - \varepsilon$  (where the entire  $\Delta(K)$  has measure 1).

We conjecture that every continuous piecewise rational function over  $\Delta(K)$ , as in the two-state case, is realizable.

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## 7 Appendix

### Proof of Proposition 1.

We rewrite the game as a zero-sum game with incomplete information on one side, as defined in Aumann and Maschler (1995). Denote by  $\mu$  the probability on  $\mathcal{Q}$  defined by  $\mu(B) = p(B)$ . Consider a zero-sum game with incomplete information with state space  $\mathcal{Q}$ . Player 1 is informed of  $B \in \mathcal{Q}$  and player 2 is not, the set of actions of the players is respectively  $A_1$  and  $A_2$ , and the stage payoff is  $\bar{g}_B(a_1, a_2) = \sum_{k \in B} \pi(k|B)g_k(a_1, a_2)$ .

The corresponding value of the  $n$ -th stage game is denoted by  $w_n(\mu)$ . It can be seen that  $w_n(\mu) = v_n^{\mathcal{Q}}(p, (g_k)_{k \in K})$ . By Aumann and Maschler (1995),  $w_n(\mu)$  converges to  $w(\mu)$ , as  $n$  goes to infinity. Therefore,  $v_n^{\mathcal{Q}}(p, (g_k)_{k \in K})$  also converges to  $w(\mu)$ . ■

The proof of Proposition 2 goes along the same lines.

### Proof of Proposition 4.

The proof is broken into three parts, one for each item of the proposition.

#### Proof of Proposition 4 (i)

**Lemma 3** *If  $u$  is realizable, then so is  $u^2$ .*

**Proof.** If  $u$  is realizable by the game  $G_1, \dots, G_{|K|}$ , then the game whose  $i$ -th matrix game is  $\begin{pmatrix} G_i & 0 \\ 2G_i - 1 & G_i \end{pmatrix}$  realizes  $u^2$ . ■

**Lemma 4** *Any polynomial is realizable.*

**Proof.** Mertens and Zamir (1971) showed that if  $u$  and  $v$  are realizable then so is  $u + v$ . By Lemma 3,  $(u + v)^2$  is also realizable as are  $u^2$  and  $v^2$ . Therefore,  $uv = \frac{1}{2}((u + v)^2 - u^2 - v^2)$  is also realizable.

Moreover, any constant function is realizable. The game  $G_1, \dots, G_{|K|}$  all of whose matrices are identically 0, except for the  $i$ -th one which is identically 1, realizes the polynomial  $f_i(p) = p_i$ . Any polynomial is therefore realizable by iteratively adding and multiplying constants and the polynomials  $f_i$ . ■

Proposition 4 (i) follows from the fact that for every  $(x_\ell, y_\ell) \in \Delta(K) \times \mathbb{R}$ ,  $\ell = 1, \dots, L$ , there is a polynomial  $u$  such that  $u(x_\ell) = y_\ell$ ,  $\ell = 1, \dots, L$ .

**Proof of Proposition 4 (ii)**

**Lemma 5** *Let  $C$  be a closed semi-algebraic set. Then there is a realizable function  $u$  that satisfies  $u(x) \leq 0$  when  $x \in C$ , and  $u(x) > 0$ , otherwise.*

**Proof.** Let  $D$  be a set of the form

$$\left\{x \in \mathbb{R}^k; f_1(x) = f_2(x) = \dots = f_\ell(x) = 0, r_1(x) \geq 0, r_2(x) \geq 0, \dots, r_m(x) \geq 0\right\},$$

where  $f_1, \dots, f_\ell, r_1, \dots, r_m$  are polynomials. By Proposition 4,  $f_1, \dots, f_\ell, r_1, \dots, r_m$  are realizable and by Mertens and Zamir (1971),  $u_D = \min\{f_1, \dots, f_\ell, -f_1, \dots, -f_\ell, r_1, \dots, r_m\}$  is realizable. Clearly,  $u_D(x) \geq 0$  when  $x \in C$  and  $u_D(x) < 0$ , otherwise. Since any closed semi-algebraic set is a finite union of such  $D$ 's (see Bochnak et al. (1988), p. 46), the desired  $u$  is  $u = -\max\{u_D\}_D$ , which is also realizable. ■

A union of closed polygons is a closed semi-algebraic set. By Lemma 5, there is a realizable function  $u$  which is less than or equal to 0 on  $C_2$  and greater than 0 otherwise. Let  $c$  be the minimum of  $u$  over the set  $C_1$ . By multiplying  $u$  with the constant  $\frac{c_1 - c_2}{c}$  and adding  $c_2$  one obtains a realizable function  $u'$  that satisfies  $u'(x) \leq c_2$  when  $x \in C_2$  and  $u'(x) \geq c_1$  when  $x \in C_1$ . Taking the minimum of  $u'$  and  $c_1$  would yield the realizable function needed for Proposition 4 (ii).

### Proof of Proposition 4 (iii)

By Lemma 5, for  $i = 1, 2$ , there is  $u'_i$  which is at least 0 on  $C_i$  and less than 0 otherwise. Consider  $u_i = \max\{\min\{u'_i + 1, 1\}, 0\}$ .  $u_i$  is realizable, bounded between 0 and 1, and equal to 1 on  $C_i$ , and less than 1 otherwise.

For any integer  $\ell_i$ ,  $u_i^{\ell_i}$  is also realizable. It is bounded between 0 and 1, and is equal to 1 on  $C_i$  and less than 1 otherwise. By adding a large positive number, say  $M$ , we may assume that the functions  $f_1$  and  $f_2$  are positive. There are sufficiently large  $\ell_i$ 's such that  $f_j > f_i u_i^{\ell_i}$ ,  $i \neq j$ , on  $C_j$ . Thus,  $\max\{f_1 u_1^{\ell_1}, f_2 u_2^{\ell_2}\}$  is realizable and (subtracting  $M$ , if necessary) satisfies Proposition 4 (iii). ■

### Proof of Proposition 5.

We first define an auxiliary game with one-sided information in which player 1 is fully informed of the state of nature. The set of states of nature in this game is  $\mathcal{P}$ , the sets of actions of the players remain unchanged, and the payoff for a pair of actions  $a_1, a_2$  in state  $B$  is  $\bar{g}_B(a_1, a_2) = \sum_{k \in K} \pi_p(k|B) g_k(a_1, a_2)$ . The probability of the state  $B$  is  $p(B)$ . It will be convenient to denote this probability on  $\mathcal{P}$  by  $r$ .

The original game is now viewed as a game with incomplete information as defined in Aumann and Maschler (1995), where the states are player 1's signals. The payoff function in the auxiliary game is naturally defined as the appropriate expected payoff of the original game. It is easily seen that the values of the  $n$ -stage repetitions of both games, the original and the auxiliary, are the same.

Thus,  $r$  is the distribution over the new set of states. By Theorem 4 the long run value of this game is  $\text{cav}(\bar{u})(r)$ , where  $\bar{u}$  denotes the value of the auxiliary game in which player 1 gets no information about  $\mathcal{Q}$ .

For any  $q$  in  $\Delta(\mathcal{Q})$ , let  $p_q \in \Delta(K)$  satisfy  $p_q(k) = \sum_{B \in \mathcal{Q}} q(B) \pi_p(k|B)$ . We prove first that  $\bar{u}(q) = u(p_q)$ . Indeed,  $\bar{u}(q)$  is the value of the game with the following payoff:

$$\begin{aligned} \sum_{B \in \mathcal{Q}} q(B) \bar{g}_B(a_1, a_2) &= \sum_{B \in \mathcal{Q}} \left( \sum_{k \in K} \pi_p(k|B) g_k(a_1, a_2) \right) q(B) \\ &= \sum_{k \in K} \left( \sum_{B \in \mathcal{Q}} q(B) \pi_p(k|B) \right) g_k(a_1, a_2) = \sum_{k \in K} p_q(k) g_k(a_1, a_2). \end{aligned}$$

Therefore, the value of this game is indeed  $u(p_q)$ . We now compute

$$\text{cav}(\bar{u})(r) = \max \left\{ \sum_i m_i \bar{u}(q_i) \mid \begin{array}{l} m_i \geq 0, \sum_i m_i = 1 \\ q_i \in \Delta(\mathcal{P}), \sum_i m_i q_i = r \end{array} \right\}.$$

For any  $q_i$  in  $\Delta(\mathcal{Q})$ , denote  $m_{iB} = m_i q_i(B)$  and define the matrix  $M = (m_{iB})$ . Since,  $\sum_{i,B} m_{iB} = \sum_i m_i = 1$  and  $\sum_i m_{iB} = r(B)$ , we conclude that  $M \in \mathcal{M}(\mathcal{Q})$ . Moreover,

$$p_i(M)(k) = \frac{\sum_{B \in \mathcal{Q}} m_{iB} \pi_p(k|B)}{m_i} = \sum_{B \in \mathcal{Q}} q_i(B) \pi_p(k|B) = p_{q_i}(k). \quad (2)$$

Hence,  $\bar{u}(q_i) = u(p_i(M))$  and

$$\text{cav}(\bar{u})(r) \leq \max \left\{ \sum_{i \leq D} m_i u(p_i(M)) \mid M \in \mathcal{M}(\mathcal{P}) \right\}. \quad (3)$$

On the other hand, for any  $M \in \mathcal{M}(\mathcal{Q})$ , due to (2),  $\sum_i m_i u(p_i(M)) = \sum_i m_i \bar{u}(q_i)$ , with  $q_i(B) = \frac{m_{iB}}{m_i}$ . Therefore,  $u(p_i(M)) = \bar{u}(q_i)$  and

$$\max \left\{ \sum_{i \leq D} m_i u(p_i(M)) \mid M \in \mathcal{M}(\mathcal{Q}) \right\} \leq \text{cav}(\bar{u})(r). \quad (4)$$

Equations (3) and (4) yield the desired result.

## A detailed Proof of Theorem 1.

Let  $V$  be a monotonic function over partitions. Let  $p$  be  $(\frac{1}{|K|}, \dots, \frac{1}{|K|})$ . Order all partitions of  $K$ :  $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ , so that if  $j < i$  then  $V(\mathcal{P}_j) \leq V(\mathcal{P}_i)$ , and therefore  $\mathcal{P}_j$  does not refine  $\mathcal{P}_i$ .

**Notation 4** Let  $\mathcal{P}$  be a partition. Denote by  $H(\mathcal{P})$  the space in  $\mathbb{R}^{|K|}$  spanned by  $\{\mathbb{1}_T; T \in \mathcal{P}\}$ , where  $\mathbb{1}_T$  denotes the characteristic vector of the set  $T$ .

Note that the partition  $\mathcal{P}$  refines  $\mathcal{P}'$  if and only if  $H(\mathcal{P}')$  is a subspace of  $H(\mathcal{P})$ . This implies in particular that if  $i > j$ , then<sup>2</sup>  $\dim[H(\mathcal{P}_j) \cap H(\mathcal{P}_i)] < \dim H(\mathcal{P}_i)$ .

We prove that for any  $i \leq \ell$  the following property, denoted by  $E_i$ , holds: there is a realizable function  $u$  such that for any  $j \leq i$  and every  $p \in H(\mathcal{P}_j)$ ,  $u(p) \leq V(\mathcal{P}_j)$ . Furthermore, for any  $j \leq i$  there is a  $D_j \times \ell$  matrix<sup>3</sup>  $M_j \in \mathcal{M}(\mathcal{P}_j)$  such that for any  $r \leq D_j$ ,  $u(p_r(M_j)) = V(\mathcal{P}_j)$ .

We proceed by induction over  $i$ . Let  $i = 1$ . The partition  $\mathcal{P}_1$  is the trivial partition and  $u$  can be taken to be the constant function  $V(\mathcal{P}_1)$ . Now assume that  $E_{i-1}$  holds and denote by  $u_{i-1}$  the corresponding realizable function.

<sup>2</sup> $\dim(H)$  denotes the dimension of  $H$ .

<sup>3</sup>Recall Notation 3.  $\mathcal{P}_j$  in  $\mathcal{M}(\mathcal{P}_j)$  stands for the information structure that corresponds to the partition  $\mathcal{P}_j$ .

**Step 1: Definition of a class of matrices.**

Consider the following square matrix,  $M_1$ , with  $|\mathcal{P}_i|$  columns, all of whose entries except for those on the diagonal are zero. The diagonal entry corresponding to  $T \in \mathcal{P}_i$  is  $p(T)$ . Note that for every row  $r$  of  $M_1$ ,  $p_r(M_1) \neq p$ . Define  $M_2$  as a matrix of the same dimension whose entries in the column corresponding to the cell  $T \in \mathcal{P}_i$  are all equal to  $\frac{p(T)}{D}$ . Obviously,  $M_1, M_2 \in \mathcal{M}(\mathcal{P}_i)$ . Define  $M = \alpha M_1 + (1 - \alpha)M_2$ . If  $\alpha$  is positive and sufficiently small, then  $M \in \mathcal{M}(\mathcal{P}_i)$  and for every row  $r$  of  $M$ ,  $p_r(M) \neq p$ . Furthermore, all entries of  $M$  are strictly positive.

Define  $H(\mathcal{P}_i)^0 = \{v; v = \sum \varepsilon_T \pi_p(\cdot|T), \sum_{T \in \mathcal{P}_i} \varepsilon_T = 0\}$ . For every  $v \in H(\mathcal{P}_i)^0$  let  $a_{rT} = m_r \varepsilon_T \langle p_r(M) - p, v \rangle$ ,  $r \leq D$ ,  $T \in \mathcal{P}_i$ . Consider the matrix  $M^v$  whose  $(r, T)$ -th entry is  $m_{rT} + a_{rT}$ . If the  $\varepsilon_T$ 's that define  $v$  are small enough, then the entries of  $M^v$  are positive.

We show now that  $M^v$  is in  $\mathcal{M}(\mathcal{P}_i)$ . This is so since  $\sum_r m_{rT} + a_{rT} = p(T) + \sum_r m_r \varepsilon_T \langle p_r(M) - p, v \rangle = p(T) + \varepsilon_T \langle \sum_r m_r p_r(M) - p, v \rangle = p(T)$ . The last equality is due to the fact that  $\sum_r m_r p_r(M) = p$ .

**Step 2: Proof of the claim that there is  $v \in H(\mathcal{P}_i)^0$  such that for any row  $r$ ,  $p_r(M^v) \notin \cup_{j < i} H(\mathcal{P}_j)$ .**

Note that  $\sum_{T \in \mathcal{P}_i} a_{rT} = \sum_{T \in \mathcal{P}_i} m_r \varepsilon_T \langle p_r(M) - p, v \rangle = m_r \langle p_r(M) - p, v \rangle \sum_{T \in \mathcal{P}_i} \varepsilon_T = 0$ . Therefore,  $p_r(M^v) = p_r(M) + \langle p_r(M) - p, v \rangle \sum_T \varepsilon_T \pi_p(\cdot|T) = p_r(M) + \langle p_r(M) - p, v \rangle v$ . Assume by contradiction that the claim is incorrect. Then there is a neighborhood of  $H(\mathcal{P}_i)^0$  around the origin, denoted  $W$ , such that for every  $v \in W$  there is  $j < i$  and a row  $r$  such that  $p_r(M^v) = p_r(M) + \langle p_r(M) - p, v \rangle v \in H(\mathcal{P}_j)$ . Define the set  $F_{rj}$  to be the set containing  $v \in \overline{W}$  such that  $p_r(M) + \langle p_r(M) - p, v \rangle v \in H(\mathcal{P}_j)$ , where  $\overline{W}$  is the closure (the relative one in  $H(\mathcal{P}_i)^0$ ) of  $W$ .  $F_{rj}$  is a closed set for every  $r$  and  $j$ .

By assumption, the union of the closed sets  $F_{rj}$  contains  $\overline{W}$ . Since  $\overline{W}$ , as a complete space, is of category II, at least one of the  $F_{rj}$ 's contains an open set. Thus, there are  $j$  and  $r$  so that  $p_r(M) + \langle p_r(M) - p, v \rangle v \in H(\mathcal{P}_j)$  for  $v$ 's in an open (in  $\overline{W}$ ) set.

Note that for every  $v \in W$ ,  $p_r(M) + \langle p_r(M) - p, v \rangle v \in H(\mathcal{P}_i) \cap \Delta(K)$ . Furthermore, since  $p_r(M) - p \neq 0$ , the map  $v \mapsto p_r(M) + \langle p_r(M) - p, v \rangle v$  is an open map. Thus,  $H(\mathcal{P}_j) \cap \Delta(K)$  contains an open set of  $H(\mathcal{P}_i) \cap \Delta(K)$ . Since both are intersections of linear spaces whose spanning vectors are in  $\Delta(K)$  with  $\Delta(K)$ , it implies that  $H(\mathcal{P}_j) \cap \Delta(K)$  contains  $H(\mathcal{P}_i) \cap \Delta(K)$ . However, when  $j < i$ ,  $\dim[H(\mathcal{P}_j) \cap H(\mathcal{P}_i)] < \dim H(\mathcal{P}_i)$ . This implies that  $H(\mathcal{P}_i) \cap \Delta(K)$  is not included in

$H(\mathcal{P}_j) \cap \Delta(K)$ . We therefore conclude that there exists a matrix, which we denote now by  $M_i$ , in  $\mathcal{M}(\mathcal{P}_i)$  that satisfies  $p_r(M_i) \notin \cup_{j < i} H(\mathcal{P}_j)$  for every row  $r$  of  $M_i$ .

**Step 3: Conclusion of the proof.**

By Proposition 4 (ii), there is a realizable function  $f$  that satisfies:

- for every row  $r$  of  $M_i$ ,  $f$  attains its maximum,  $V(\mathcal{P}_i)$ , on  $p_r(M_i)$ ; and
- $f$  is smaller than or equal to  $\min_{1 \leq j \leq i-1} V(\mathcal{P}_j)$  on  $\cup_{1 \leq j \leq i-1} H(\mathcal{P}_j) \cap \Delta(K)$ .

By taking the maximum of  $f$  and the function  $u_{i-1}$  we get a realizable function  $u_i$  that satisfies:

- (a) for any  $j \leq i$ ,  $u_i$  is smaller than or equal to  $V(\mathcal{P}_j)$  on  $H(\mathcal{P}_j) \cap \Delta(K)$ ; and
- (b) for any  $j \leq i$ , there is a  $D_j \times l$  matrix  $M_j \in \mathcal{M}(\mathcal{P}_j)$  such that for any  $1 \leq r \leq D_j$ ,  $u(p_r(M_j)) = V(\mathcal{P}_j)$ .

Property  $E_l$  is therefore proven by induction. Let  $(g_k)_{k \in K}$  be the payoff functions that realize  $u_l$ . (a) implies that  $v^{\mathcal{P}_i \mathcal{O}}(p, (g_k)_{k \in K}) \leq V(\mathcal{P}_i)$  for every  $i \leq l$ . Proposition 5 and (b) imply that  $v^{\mathcal{P}_i \mathcal{O}}(p, (g_k)_{k \in K}) = V(\mathcal{P}_j)$ , which completes the proof. ■

**Proof of Lemma 1.**

By definition of  $W$  it is clear that for all  $c$  and all  $\mathcal{Q}$ ,

$$V(\mathcal{P}, \mathcal{Q}) \geq W(\mathcal{P}, c) - c(\mathcal{Q}).$$

Therefore, what needs to be proved is that for all  $\mathcal{P}$ , there exists an increasing cost function  $c_{\mathcal{P}}$  such that

$$V(\mathcal{P}, \mathcal{Q}) \leq W(\mathcal{P}, c_{\mathcal{P}}) - c_{\mathcal{P}}(\mathcal{Q}).$$

Let  $c_{\mathcal{P}} := V(\mathcal{P}, \mathcal{T}) - V(\mathcal{P}, \mathcal{Q})$ , where  $\mathcal{T}$  is the trivial partition. This function is indeed increasing in  $\mathcal{Q}$ . Moreover

$$\begin{aligned} W(\mathcal{P}, c_{\mathcal{P}}) - c_{\mathcal{P}}(\mathcal{Q}) &= \min_{\mathcal{Q}'} V(\mathcal{P}, \mathcal{Q}') + c_{\mathcal{P}}(\mathcal{Q}') - c_{\mathcal{P}}(\mathcal{Q}) \\ &= V(\mathcal{P}, \mathcal{Q}') + V(\mathcal{P}, \mathcal{T}) - V(\mathcal{P}, \mathcal{Q}') - V(\mathcal{P}, \mathcal{T}) + V(\mathcal{P}, \mathcal{Q}) \\ &= V(\mathcal{P}, \mathcal{T}) + V(\mathcal{P}, \mathcal{Q}), \end{aligned}$$

where  $\mathcal{Q}^*$  is the partition that attains the minimum. This is the desired result. ■

**Proof of Lemma 2.**

The proof of Lemma 1 shows that for each  $\mathcal{P}$  there is a cost function  $c_{\mathcal{P}}$  such that  $W(\mathcal{P}, c_{\mathcal{P}}) = V(\mathcal{P}, \mathcal{Q}) + c_{\mathcal{P}}(\mathcal{Q})$ . We now prove that  $c_{\mathcal{P}}$  is independent of  $\mathcal{P}$ . Indeed, by the proof of Lemma 1,  $c_{\mathcal{P}}(\mathcal{Q}) = V(\mathcal{P}, \mathcal{T}) - V(\mathcal{P}, \mathcal{Q})$ , which means that the right-hand side does not depend on  $\mathcal{P}$ . Thus,  $V$  satisfies the independence condition, as desired. ■