MINMAX VIA DIFFERENTIAL INCLUSION

EHUD LEHRER AND SYLVAIN SORIN

Abstract. The asymptotic behavior of the solution of a differential inclusion provides a simple proof of a minmax theorem.

Date: April 2005.

1. Introduction

Consider a zero-sum game (I, J, A) where $I = \{1, ..., \ell\}$ (resp. $J = \{1, ..., m\}$) is the set of pure strategies of player 1 (the maximizer), (resp. of player 2 (the minimizer)) and A is the $\ell \times m$ payoff matrix. Denote by X and Y the sets of mixed strategies of player 1 and 2 (i.e., X and Y are the unit simplices in $\mathbb{R}^\ell$ and in $\mathbb{R}^m$, respectively). Hence, if $x \in X$ and $y \in Y$ are the players' mixed actions, the payoff is $xAy = \sum_{ij} x_i A_{ij} y_j$. Assume that $\max_X \min_Y xAy = 0$. We prove that $\min_Y \max_X xAy \le 0$.

The proof is based on an approximation of a discrete dynamics[1] related to Blackwell's approachability procedure (Blackwell, 1956; Sorin, 2002), by a continuous one, in the spirit of Benaim, Hofbauer and Sorin (2003). The goal of player 2 is to ensure that the payoff corresponding to any pure strategy of player 1 is less than or equal to zero. In other words, that the state (in the vector payoff space) will reach the negative orthant. At any point in time the current error, from player 2's point of view, is the difference between the time-average state and its projection on the negative orthant. As in approachability theory, the strategy of player 2 is a function of this error. In particular, the dynamics depends only on the behavior of player 2.

Other dynamical approaches have been used to prove the minmax theorem. Brown and von Neumann (1950) introduced a differential equation for two person symmetric zero-sum games. The variable is a symmetric mixed strategy that converges to the set of optimal strategies. Other procedures (e.g., fictitious play, Robinson (1951)) exhibit similar properties for non-symmetric games while the state variable is in the product space of mixed strategy. In these procedures the dynamics of each player's strategy depends, at each instant, on *both* players' strategies. In contrast, our dynamics is autonomous: it is defined over the vector payoff space of player 2 (i.e., the ℓ-dimensional vectors – a payoff for each pure strategy of player 1) and does not involve player's 1 behavior.

2. A discrete dynamics

Consider a sequence $\{y_n\}$ of player 2's mixed strategies and let $\bar{y}_n$ be the average of its n first elements. Thus,

$$\bar{y}_{n+1} - \bar{y}_n = \frac{1}{n+1}(y_{n+1} - \bar{y}_n). \tag{1}$$

The corresponding sequence of vector payoffs, $g_n = Ay_n \in \mathbb{R}^\ell$, satisfies:

$$\bar{g}_{n+1} - \bar{g}_n = \frac{1}{n+1}(g_{n+1} - \bar{g}_n). \tag{2}$$

For any $g \in \mathbb{R}^\ell$ define $g^+ = (g^{+1}, ..., g^{+n})$ by $g^{+k} = \max(g^k, 0)$. If $g^+ \neq 0$, then g^+ is proportional to some $x \in X$, and thus, by assumption, there exists $y \in Y$ that satisfies $xAy \le 0$, which implies $\langle g^+, Ay\rangle \le 0$.

Define a correspondence N from $\mathbb{R}^\ell$ to Y by:

$$N(g) = \{y \in Y; \langle g^+, Ay\rangle \le 0\},$$

[1] Similar discrete dynamics have been used by Lehrer and Sorin (2000) and Lehrer (2002).

for every $g \in I\!R^\ell$. The values of N are non-empty. Furthermore, N is an upper semi-continuous correspondence with compact convex values.

The discrete dynamics, respectively in the space of player 2's mixed strategies and vector payoffs are defined by

$$y_{n+1} \in N(A\bar{y}_n) \qquad \text{and} \qquad g_{n+1} \in AN(\bar{g}_n). \tag{3}$$

3. The theorem and its proof

We turn now to the formal presentation of the theorem and its proof.

Theorem 1. *If* $\max_X \min_Y xAy = 0$*, then there exists a point* $y^* \in Y$ *such that all the coordinates of the vector* Ay^* *are less than or equal to* 0.

Proof. Let D denote the negative orthant, $I\!R^\ell_-$, and let $C = AY$, which is a compact convex image of Y.

Consider the continuous dynamics in the vector payoffs space defined by the following differential inclusion (note the similarity to equation (3)):

$$\dot{g} \in AN(g) - g. \tag{4}$$

Since N is an upper semi-continuous correspondence with non-empty compact convex values, there exists a solution $g(t)$ of (4) (see e.g., Theorem 3, p. 98 in Aubin and Cellina (1984)). When $g(t)$ is on the boundary of C, $\dot{g}(t)$ points inside C. Thus, if $g(0) \in C$, then $g(t) \in C$ for all $t \geq 0$ (see e.g., Theorem 5.7, p. 129 in Smirnov (2002)).

Let $Z(g) = \|g^+\|^2$ be the square of the distance from a point $g \in I\!R^\ell$ to D. Hence, $\nabla Z(g) = 2g^+$. Let $z(t) = Z(g(t))$, then

$$\begin{aligned} \dot{z}(t) &= \langle \nabla Z(g(t)), \dot{g}(t)\rangle = 2\langle g^+(t), \dot{g}(t)\rangle \leq -2\langle g^+(t), g(t)\rangle \\ &= -2\langle g^+(t), g^+(t)\rangle = -2Z(g(t)) = -2z(t), \end{aligned}$$

where the inequality is due $\dot{g}(t) + g(t) \in AN(g)$ and $\langle g^+, Ay\rangle \leq 0$ for any $y \in N(g)$. Thus, $z(t)$ (hence the distance between $g(t)$ and D) decreases exponentially to 0 and if it reaches 0 it keeps this value.[2]

Any accumulation point, g^*, of $g(t)$ (as $t \to \infty$) satisfies $Z(g^*) = 0$, which means that $g^* \in D$. Since $g^* \in C$, there exists y^* with $Ay^* \in D$, as desired. □

Remark: The proof immediately extends to the case where X is a finite dimensional simplex, Y is a convex compact subset of a topological vector space and the payoff is a real bilinear function on $X \times Y$, continuous on Y for each $x \in X$.

References

[1] Aubin J.-P. and A. Cellina, (1984) *Differential Inclusions*, Springer.

[2] Benaim, M.,J. Hofbauer and S. Sorin, (2003) "Approximations and differential inclusions," *SIAM J. Opt. Control*, forthcoming.

[3] Blackwell, D. (1956) "An analog of the minmax theorem for vector payoffs," *Pacific Journal of Mathematics*, **6**, 1-8.

[4] Brown, G.W. and J. von Neumann (1950) "Solutions of games by dierential equations," *Annals of Mathematical Studies* **24**, 73-79.

[2] This is in opposition with the discrete dynamics whose distance to D does not converge monotonically to D.

[5] Lehrer, E. (2002) "Allocation Processes in Cooperative Games," *International Journal of Game Theory*, **31**, 341-351.
[6] Lehrer, E. and S. Sorin, (2000) "Approachability and applications," manuscript.
[7] Robinson, J. (1951) An iterative method of solving a game, *Annals of Mathematics*, **54**, 296-301.
[8] Smirnov, G., V. (2002) *Introduction to the Theory of Differential Inclusions*, Graduate Studies in Mathematics 41, American Mathematical Society, Providence.
[9] Sorin, S. (2002) *A First Course on Zero-Sum Repeated Games*, Springer.

School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel
E-mail address: lehrer@post.tau.ac.il

Laboratoire d'Econométrie, Ecole Polytechnique, 1 rue Descartes, 75005 Paris and Equipe Combinatoire et Optimisation, UFR 929, Université P. et M. Curie - Paris 6, 175 Rue du Chevaleret, 75013 Paris, France
E-mail address: sorin@math.jussieu.fr