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Subjective multi-prior probability: A representation of a partial likelihood relation

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Abstract

This paper deals with an incomplete relation over events. Such a relation naturally arises when likelihood estimations are required within environments that involve ambiguity, and in situations which engage multiple assessments and disagreement among individuals' beliefs. The paper characterizes binary relations over events, interpreted as likelihood relations, that can be represented by a unanimity rule applied to a set of prior probabilities. According to this representation an event is at least as likely as another if and only if there is a consensus among all the priors that this is indeed the case. A key axiom employed is a cancellation condition, which is a simple extension of similar conditions that appear in the literature.

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1. Introduction

1.1. Motivation and background

Estimating the odds and comparing the likelihood of various events are essential parts of processes carried out by many organizations. For instance, the US intelligence community produces National Intelligence Estimates, in which the likelihood of various events is assessed. Questions such as, 'Is it more likely that democracy will prevail in Libya, or that a military regime will be established?', and the like, seem natural to ask. These questions and many others call for likelihood comparisons of different events. Other examples in which probabilistic estimates are used include forecasts published by central banks, that address issues such as the odds of inflation or recession, estimated likelihood of natural events such as global warming, that are based on individual opinions of scientists and on many experiments, and so forth. In all these instances, statements of the type 'event A is more likely than event B' seem fundamental to the respective context.

In many situations, assessments of the kind given above are a product of advisory entities (e.g. intelligence analysts, consulting firms, scientists), that output likelihood judgements and forward them to a decision maker, who in turn takes these into account when making a decision. Typically, such assessments rely on 'objective' data, such as reports of military movements, temperature measurements and the like, and are intended to be based as closely as possible on the data. Frequently, though, the events examined involve some degree of ambiguity. Knowledge or available information might be insufficient to determine which of two events under consideration is more likely, and a likelihood relation in such situations might therefore leave the comparison between some pairs of events unspecified.

Motivated by the above question, this paper characterizes a likelihood relation that may be incomplete. The question of representing an incomplete likelihood relation was already addressed is Nehring [17]. In his paper, Nehring considers an incomplete 'at least as likely as' relation over events, and proposes to represent it using a consensus rule over a set of prior probabilities.[2] His result matches the results of Giron and Rios [10], Bewley [2], and the purely subjective work of Ghirardato et al. [9] – all being characterizations of multi-prior expected utility representations – to the domain of comparisons between events. Our paper is closely related to Nehring [17]. Similarly, it formulates conditions on a binary relation over events, that are necessary and sufficient for the relation to be represented by a consensus rule over a set of prior probabilities.

Formally, let $\succsim$ denote a binary relation over events, where $A \succsim B$ for events A and B is interpreted as 'A is at least as likely as B'. The main theorem of the paper introduces necessary and sufficient conditions ('axioms') on $\succsim$ that guarantee the existence of a set of prior probabilities, $\mathcal{P}$, such that for any two events A and B,

$$A \succsim B \quad \Leftrightarrow \quad \mu(A) \geqslant \mu(B) \quad \text{for all } \mu \in \mathcal{P}. \tag{1}$$

The difference between the representation theorem in this paper and that in [17] lies in the axioms assumed, most notably a richness assumption used in [17] that we replace in the case of an infinite state space, and drop completely when finite state spaces are concerned. As a result, the paper contains what we consider to be a purely subjective treatment of partial likelihood

2 Nehring then pairs this likelihood relation with a preference relation and explores the compatibility between the two relations. This paper focuses on likelihood relations, and does not investigate preference relations.

relations, and includes a representation result for finite state spaces, which are not dealt with in [17].

In view of the similarity between the papers we do not elaborate further on motivation and background. The interested reader is referred to [17] for a survey of related literature and an extensive discussion that motivates the investigation of incomplete likelihood relations.

1.2. Axiomatization

Some notation is needed to facilitate the following discussion. Let S denote a nonempty state-space with a typical element s, and Σ an algebra of events over S. For an event E, $\mathbf{1}_E$ denotes the indicator function[3] of E. A binary relation $\succsim$, interpreted as an 'at least as likely as' relation, is defined over Σ. A probability measure P *agrees* with $\succsim$ if it represents it, in the sense that $A \succsim B \Leftrightarrow P(A) \geqslant P(B)$. A probability measure P *almost agrees* with $\succsim$ if the former equivalence is relaxed to $A \succsim B \Rightarrow P(A) \geqslant P(B)$.

De Finetti [4,5] introduced four basic postulates that must be satisfied by an 'at least as likely as' relation. The most substantial of the four is *Cancellation*,[4] which states that if A, B and C are events such that $A \cap C = B \cap C = \emptyset$, then $A \succsim B \Leftrightarrow A \cup C \succsim B \cup C$. Cancellation implies a form of separability over events in the sense that any event has its own likelihood weight, which is unrelated to other disjoint events. In other words, the marginal contribution of an event is the same, no matter what other disjoint events it is annexed to.[5]

The four de Finetti's assumptions are necessary for the relation to admit an agreeing probability. However as was demonstrated by an example in Kraft, Pratt and Seidenberg [13] they are insufficient, even to guarantee existence of an almost agreeing probability. The example implies that the de Finetti assumptions, completeness excluded, are also insufficient to obtain a *set* of representing probabilities in the sense of (1).

Kraft, Pratt and Seidenberg, and later on Scott [20], Krantz et al. [14] and Narens [16], suggested a strengthening of the Cancellation condition, *Finite Cancellation*. In case the state space is finite, this condition together with the other de Finetti assumptions were shown to suffice for an 'at least as likely as' relation to obtain an almost agreeing probability.

There are various formulations of Finite Cancellation in the literature.[6] One possible formulation, equivalent to those mentioned above when complete relations are concerned, states that for two finite sequences of events, $(A_i)_{i=1}^n$ and $(B_i)_{i=1}^n$, and a corresponding finite sequence of natural numbers, $(k_i)_{i=1}^n$,

$$\begin{aligned}
&\text{If } \sum_{i=1}^n k_i \mathbf{1}_{A_i}(s) = \sum_{i=1}^n k_i \mathbf{1}_{B_i}(s) \quad \text{for all } s \in S, \\
&\text{and } A_i \succsim B_i \quad \text{for } i = 1, \ldots, n-1, \\
&\text{then } B_n \succsim A_n.
\end{aligned} \tag{2}$$

[3] That is, $\mathbf{1}_E$ is the function that attains the value 1 on E and 0 otherwise.

[4] The other three are the basic Complete Order assumption, and Positivity and Non-Triviality, stated below in Section 2.1.

[5] This assumption is violated, for instance, when states are evaluated through a nonadditive probability v. Under such an evaluation, the marginal contribution of event C when added to event A, $v(A \cup C) - v(A)$, is not necessarily the same as its marginal contribution when added to event B, $v(B \cup C) - v(B)$.

[6] See Fishburn [7] for a thorough survey of Cancellation axioms and almost agreeing probabilities, and Wakker [22] for a discussion and related results.

We term this version *Generalized Finite Cancellation* (GFC for short). The idea behind (Generalized) Finite Cancellation is similar in essence to that lying at the basis of de Finetti's Cancellation condition. Like Cancellation, Finite Cancellation is based on the assumption that each state has always the same marginal contribution of likelihood, no matter to which other states it is added (see further elaboration in Section 2.1).

Our first result is that in case of a finite state space, the above generalized version of Finite Cancellation together with the basic de Finetti assumptions, completeness excluded, is both necessary and sufficient for the relation to admit a consensus multi-prior representation as in (1). We complement those conditions with a richness assumption to obtain the representation for an infinite state space.

While the solution we suggest is a cancellation-type axiom, supplemented by a richness condition for the case of infinite state spaces, Nehring [17] complements the assumptions of de Finetti with three axioms: a form of continuity, an axiom called *Splitting* and another called *Equidivisibility*. Splitting extends the incomplete 'at least as likely as' relation according to the rationale embedded in Cancellation, but only so as to restore an implication of Cancellation that is lost when completeness is not assumed. It is in fact a special case of Generalized Finite Cancellation. In order to obtain a representing set of probabilities Nehring [17] assumes two additional richness conditions. One is (a standard form of) continuity, and the other is Equidivisibility.

Equidivisibility hinges upon an explicit assumption that any event can be divided into two equally-likely events. This assumption is quite a strong one. First, it dictates an infinite state space. More importantly, from the behavioral point of view, it requires the individual to be able to exactly identify a 'half-event' for every given event. In some situations this assumption seems appropriate, for instance when a randomizing device, independent of the other events, is available and natural to the problem at hand. This is the case in the Anscombe–Aumann framework [1], where both subjective and objective (or at least, exogenous) probabilities are assumed. However, in other, 'purely subjective' situations, an individual considering an event might not perceive any sub-event as being half as likely, as required by Equidivisibility. It might be the case that such a 'half-event' is alien to the situation (imagine an intelligence analyst trying to pinpoint two equally-likely sub-events of the event 'revolution in Libya'). Moreover, even when an independent randomizing device is embedded properly in the state space, an individual (or individuals) need not perceive this device as being objective. Put differently, the probabilities involved in such a randomizing device may very well be subjective themselves.

This work aims to obtain a multiple priors consensus rule in a purely subjective setup. Equidivisibility is thus dropped, and GFC applied instead. As a result, finite state spaces, which are not treated in [17], are accommodated (Theorem 1). For infinite state spaces (Theorem 2) GFC is supplemented with a richness assumption.

Our richness assumption, which is an incomplete counterpart of Savage's P6 (see [19]), constrains the probabilities in the representing set to agree on null and universal events. By contrast, the representing set in [17] need not exhibit this kind of agreement. On the other hand, the set in [17] is explicitly assumed to have the property that, for any event A, there exists an event $B \subset A$ such that all priors μ agree that $\mu(B) = \frac{1}{2}\mu(A)$. It implies in particular that all the priors agree on a rich algebra of events: the one generated by dividing the entire space into 2^n equally likely events, for any integer n. In our work such agreement is implied only in special cases (in a similar fashion as Savage's P6 implies convex-rangedness of the representing probability), but need not hold in general (see Example 1 below).

The price paid for moving to a purely subjective axiomatization is uniqueness. In the representation theorems in this paper, as oppose to that in [17], the representing set of prior probabilities is not necessarily unique.[7] This is the reason why reference is made to the maximal, w.r.t. inclusion, representing set of priors.

1.3. Outline of the paper

Section 2 describes the essentials of the subjective multi-prior probability model. It details the setup and assumptions, and then formulates representation theorems, separately for the cases of a finite and an infinite spaces. Section 3 includes a few comments. All proofs appear in the last section.

2. The subjective multi-prior probability model

2.1. Setup and assumptions

Let S be a nonempty set, Σ an algebra over S, and $\succsim$ a binary relation over Σ. A statement $A \succsim B$ is to be interpreted as 'A is at least as likely as B'. For an event $E \in \Sigma$, $\mathbf{1}_E$ denotes the indicator function of E. In any place where a partition over S is mentioned, it is to be understood that all atoms of the partition belong to Σ.

The following assumptions are employed to derive a subjective multi-prior probability belief representation.

P1. Reflexivity. For all $A \in \Sigma$, $A \succsim A$.

P2. Positivity. For all $A \in \Sigma$, $A \succsim \emptyset$.

P3. Non-Triviality. $\neg(\emptyset \succsim S)$.

These first three assumptions are standard. Positivity and Non-Triviality are two of de Finetti's suggested attributes. Since Completeness is not supposed, Reflexivity is added in order to identify the relation as a weak one. Transitivity is implied by the other axioms, hence it is not written explicitly.

Next we state Generalized Finite Cancellation, our central axiom (already introduced above).

P4. Generalized Finite Cancellation. Let $(A_i)_{i=1}^n$ and $(B_i)_{i=1}^n$ be two finite sequences of events from Σ, and $(k_i)_{i=1}^n$ a finite sequence of numbers from $\mathbb{N}$. Then,

$$\begin{aligned}
&\text{If } \sum_{i=1}^n k_i \mathbf{1}_{A_i}(s) = \sum_{i=1}^n k_i \mathbf{1}_{B_i}(s) \quad \text{for all } s \in S,\\
&\text{and } A_i \succsim B_i \quad \text{for } i = 1, \dots, n-1,\\
&\text{then } \ B_n \succsim A_n.
\end{aligned}$$

Consider the following two sequences of events: $(A_i)_{i=1}^n$ with each event A_i repeating k_i times (the A-sequence), and $(B_i)_{i=1}^n$ with each event B_i repeating k_i times (the B-sequence).

[7] Indeed there are likelihood relations that admit representation by two distinct (convex and closed) sets of priors – see Example 1 in [17].

The equality that appears in the axiom, between the two sums of indicators, means that each state appears the same number of times in the A-sequence as in the B-sequence. Following the rationale behind de Finetti's Cancellation, by which each state has its own marginal weight independent of other states, the equality between the two sums suggests that it cannot be that the A-sequence has an overall likelihood weight greater than that of the B-sequence. Generalized Finite Cancellation thus explicitly states that in order to balance the 'account', A_n and B_n *should be comparable*, and B_n should be at least as likely as A_n.[8]

When completeness is assumed, it is immaterial whether Finite Cancellation is formulated as P4 above, or, as is commonly the case (see e.g. Scott [20]), with the last pair of events, A_n and B_n, repeating only once. Obviously P4 implies the special case with $k_n = 1$. In the other direction, completeness renders $\neg(B_n \succsim A_n)$ impossible under 'standard' Finite Cancellation (formulated with one repetition). Without completeness, the above general version of Finite Cancellation is required, having a two fold role. First, it preserves consistency in the same manner as the standard Finite Cancellation. Furthermore, it allows for extensions of the relation to yet undecided pairs of events, by prescribing a completion that abides by the principle of each state having an invariant marginal contribution, unrelated to other states.

The next remark summarizes a few implications of P4.

Remark 1. Generalized Finite Cancellation implies that $\succsim$ satisfies Cancellation (de Finetti's condition; by letting $A_1 = A$, $B_1 = B$, $A_2 = B \cup C$ and $B_2 = A \cup C$). It also results in Transitivity (by letting $A_1 = B_3 = A$, $A_2 = B_1 = B$, and $A_3 = B_2 = C$), and together with Positivity yields that $A \succsim B$ whenever $A \supset B$. In particular,[9] $A \succsim B \Leftrightarrow B^c \succsim A^c$, hence $S \succsim B$ for all events B.

2.2. *Representation theorem: The case of a finite S*

The next theorem states that when S is finite assumptions P1–P4 are necessary and sufficient to obtain a multi-prior probability representation of $\succsim$. For simplicity, Σ is assumed to be the collection of all subsets of S (that is, $\Sigma = 2^S$).

Theorem 1. *Suppose that S is finite, and let $\succsim$ be a binary relation over events in S. Then statements* (i) *and* (ii) *below are equivalent:*

(i) $\succsim$ *satisfies axioms* P1 *through* P4.
(ii) *There exists a nonempty set $\mathcal{P}$ of additive probability measures over events in S, such that for every $A, B \subseteq S$,*

$$A \succsim B \quad \Leftrightarrow \quad \mu(A) \geqslant \mu(B) \quad \textit{for every } \mu \in \mathcal{P}.$$

[8] An analogue axiom to GFC, formulated on mappings from states to outcomes, appeared in Blume et al. [3] (under the name 'extended statewise cancellation'). In their paper, the axiom was applied in a different framework, and was used to obtain a representation that contains a *subjective* state space. Hence, the result of Blume et al. cannot be employed to characterize a likelihood relation over a given, primitive state space.

[9] To understand the following, note that $A = (A \cap B^c) \cup (A \cap B) \succsim (B \cap A^c) \cup (A \cap B) = B \Leftrightarrow A \cap B^c \succsim B \cap A^c \Leftrightarrow B^c = (A \cap B^c) \cup (A^c \cap B^c) \succsim (B \cap A^c) \cup (A^c \cap B^c) = A^c$.

2.3. Representation theorem: The case of an infinite S

To obtain a representation when S is infinite, an additional richness assumption is required. Without this assumption it is possible to obtain a set $\mathcal{P}$ of probability measures that only *almost agrees* with $\succsim$, in the sense that for all events A and B, $A \succsim B \Rightarrow \mu(A) \geqslant \mu(B)$ for every $\mu \in \mathcal{P}$, but not necessarily the other way around. The richness assumption requires a definition of strong preference for its formulation (the definition originates in Nehring [17]).

Definition 1. For two events $A, B \in \Sigma$, the notation $A \succ\succ B$ states that there exists a finite partition $\{G_1, \dots, G_r\}$ of S, such that $A \setminus G_i \succsim B \cup G_j$ for all i, j.

In the representation, having $A \succ\succ B$ is equivalent to the condition that there exists $\delta > 0$ for which $\mu(A) - \mu(B) > \delta > 0$ for every $\mu \in \mathcal{P}$.

P5. Non-Atomicity. If $\neg(A \succsim B)$ then there exists a finite partition of A^c, $\{A'_1, \dots, A'_m\}$, such that for all i, $A'_i \succ\succ \emptyset$ and $\neg(A \cup A'_i \succsim B)$.

Remark 2. As $\neg(A \succsim B) \Leftrightarrow \neg(B^c \succsim A^c)$, Non-Atomicity is equivalent to:
If $\neg(A \succsim B)$ then there exists a finite partition of B, $\{B_1, \dots, B_m\}$, such that for all i, $B_i \succ\succ \emptyset$ and $\neg(A \succsim B \setminus B_i)$.

Non-Atomicity is the incomplete-relation version of Savage's richness assumption P6 [19]. Adding Completeness makes P5 (along with the definition of strict preference) identical to Savage's P6, as negation of preference simply reduces to strict preference in the other direction. The setup used here is somewhat weaker than that in Savage, as Σ is assumed to be an algebra and not necessarily a σ-algebra. Still, adding Completeness yields a unique probability that represents the 'at least as likely as' relation $\succsim$ (see Kopylov [11] for this result for an even more general structure of Σ).

Before the representation result for infinite state spaces is stated, another definition is required.

Definition 2. A set $\mathcal{P}$ of probability measures is *uniformly strongly continuous* if:

(a) For any event B, $\mu(B) > 0$ if and only if $\mu'(B) > 0$, for every pair of probabilities $\mu, \mu' \in \mathcal{P}$.
(b) For every $\varepsilon > 0$, there exists a finite partition $\{G_1, \dots, G_r\}$ of S, such that for all j, $\mu(G_j) < \varepsilon$ for all $\mu \in \mathcal{P}$.

Theorem 2. *Let $\succsim$ be a binary relation over Σ. Then statements* (i) *and* (ii) *below are equivalent:*

(i) *$\succsim$ satisfies axioms* P1–P5.
(ii) *There exists a nonempty, compact*[10] *and uniformly strongly continuous set $\mathcal{P}$ of additive probability measures over Σ, such that for every $A, B \in \Sigma$,*

$$A \succsim B \quad \Leftrightarrow \quad \mu(A) \geqslant \mu(B) \quad \textit{for every } \mu \in \mathcal{P}. \tag{3}$$

[10] In the weak* topology, in which convergence of measures corresponds to event-wise convergence.

The proof appears in Section 4. In essence, it relies on a separation argument in the vector space generated by linear combinations of indicator functions for events in Σ. The closed convex cone generated by indicator differences that indicate ranking is considered, and the crucial step is to prove that it cannot contain an indicator difference that does *not* correspond to a ranking. Convexity is handled using GFC. The fact that the closedness operation does not add vectors that do not indicate ranking is the result of Non-Atomicity.

The next example demonstrates that under the assumptions of the paper it can be the case that there are no events, other than trivial ones, to which the probabilities in the representing set assign the same probability. The example imitates Example 1 in [17].

Example 1. Denote by λ the Lebesgue measure. Let $S = [0, 1)$ and Σ the algebra generated by all intervals $[a, b)$ contained in $[0, 1)$. For $A \in \Sigma$ such that $\lambda(A) = \frac{1}{2}$, let π_A be the probability measure defined by the density:

$$f_A(s) = \begin{cases} \frac{1}{2} & s \in A \\ \frac{3}{2} & s \notin A \end{cases}$$

Let $\mathcal{P}$ be the convex and closed set generated by all probability measures π_A. All measures in the set are mutually absolutely continuous with λ. For $\varepsilon > 0$, letting $\{E_1, \dots, E_n\}$ be a partition with $\lambda(E_i) < \frac{2}{3}\varepsilon$ obtains $\pi_A(E_i) < \varepsilon$ for all measures π_A, hence for all measures in $\mathcal{P}$. The set $\mathcal{P}$ is therefore uniformly strongly continuous.

The resulting subjective multi-prior probability representation satisfies assumptions P1–P5. Note that for any event $B \in \Sigma$ for which $0 < \lambda(B) < 1$,

$$\begin{aligned} \max_{\pi \in \mathcal{P}} \pi(B) &= \frac{3}{2} \min\left(\lambda(B), \frac{1}{2}\right) + \frac{1}{2} \max\left(0, \lambda(B) - \frac{1}{2}\right) \\ &> \frac{1}{2} \min\left(\lambda(B), \frac{1}{2}\right) + \frac{3}{2} \max\left(0, \lambda(B) - \frac{1}{2}\right) = \min_{\pi \in \mathcal{P}} \pi(B), \end{aligned}$$

and by the construction of the measures, the derived likelihood relation is: $A \succsim B$ if and only if $\lambda(A \setminus B) \geqslant 3\lambda(B \setminus A)$. Importantly, the measures in $\mathcal{P}$ do not agree on any event which is non-null and non-universal. Moreover, there are even no events with probability ε-close to a fixed value $0 < p < 1$. Note also that for events A and B, it cannot be that all measures in $\mathcal{P}$ agree that A and B have equal probabilities, therefore events cannot be partitioned into equally likely events.

3. Comments

3.1. On the uniqueness of the set of prior probabilities

The set of prior probabilities obtained in the above theorems need not be unique. For an extreme example imagine a simple coin toss, where the individual believes that 'Heads' is at least as likely as 'Tails'. In this case the likelihood relation is complete, and yet, any probability measure that lends 'Heads' a probability of at least half would represent the relation. Such a phenomenon will occur in every finite case. However,[11] considering that the likelihood relation may later be

[11] Recall the advisory entity interpretation from the Introduction, and see the extensive discussion in [17].

used as basis for decision making, a cautious choice would be to represent this relation by the largest possible set, even at the expense of potentially replacing a singleton representing set with a much larger set, as in the coin toss example.

We thus suggest to consider the union of all representing sets as the representing set of probabilities. This union is itself a representing set, and is maximal w.r.t. inclusion. It contains all the probability measures that almost agree with the relation. When analyzing judgements made under ambiguity, the maximal set w.r.t. inclusion seems to be a natural choice to express belief, as it takes into consideration all priors that may be relevant to the case at hand. If an incomplete relation (for a finite or infinite state spaces) is to be later completed (e.g., according to a minimum probability rule), the completion may yield different results under different representing sets. Taking the maximal set guarantees that the completion does not ignore possible likelihood assessments.

3.2. Partial observations

When discussing an incomplete relation, the set of axioms serves both as a consistency tool and as a means to derive additional relationships between events. Given a subset of comparative likelihood observations, this subset can be matched with a representing set of priors just as long as it does not contradict any of the axioms. Specifically, all relationships implied by GFC can be added to the subset of comparisons if needed. Technically speaking, the full set of relationships that emerges from the observations can be identified by computing the closed convex cone of the set $\{\mathbf{1}_A - \mathbf{1}_B \mid A \succsim B \text{ was observed}\}$. If an indicator difference $\mathbf{1}_E - \mathbf{1}_F$ happens to be a member in the closed convex cone computed, then $E \succsim F$ should hold (and hence added even if not directly observed). The obtained set of observations admits a multi-prior representation as in the above theorems.

3.3. Equidivisibility and Lyapunov Theorem

As a result of Lyapunov Theorem (a version thereof, see Theorem 1.4 in Rao [18]), if Σ is a σ-algebra, and the representing set of probabilities is finite (or, equivalently, is the convex hull of a finite number of probabilities), then uniform strong continuity implies Nehring's [17] axiom of Equidivisibility. As a result, there is a unique convex and closed representing set of probabilities.

3.4. Countably additive priors

In case one wishes to guarantee that all priors are countably additive, the following additional axiom is required.

P6. Monotone Continuity. If $E_1 \supseteq E_2 \supseteq \cdots$ is a sequence of events converging to the empty set (i.e., $\bigcap_n E_n = \emptyset$), and F is an event such that $F \succ\succ \emptyset$, then there exists n_0 such that for all $n > n_0$, $F \succsim E_n$.

The axiom is a version of monotone continuity conditions due to Villegas [21], Kopylov [12] and others.

Proposition 3. *Let Σ be a σ-algebra over S, and suppose that $\succsim$ is a binary relation over Σ, admitting the multi-prior representation as in (ii) of Theorem 2. Then all prior probabilities in $\mathcal{P}$ are countably additive if and only if $\succsim$ satisfies* P6.

4. Proofs

Let $B_0(S, \Sigma)$ denote the vector space generated by linear combinations of indicator functions $\mathbf{1}_A$ for $A \in \Sigma$, endowed with the supremum norm. The next claims show that under assumptions P1 through P4, preference is preserved under convex combinations (the proof is joint for the finite and infinite cases). In some of the claims, conclusions are more easily understood considering the following alternative formulation of Generalized Finite Cancellation:

Let A and B be two events, and $(A_i)_{i=1}^n$ and $(B_i)_{i=1}^n$ two sequences of events from Σ, that satisfy:

$$A_i \succsim B_i \quad \text{for all } i, \text{ and for some } k \in \mathbb{N},$$
$$\sum_{i=1}^{n} \big[\mathbf{1}_{A_i}(s) - \mathbf{1}_{B_i}(s)\big] = k\big[\mathbf{1}_A(s) - \mathbf{1}_B(s)\big] \quad \text{for all } s \in S.$$

Then $A \succsim B$.

Claim 1. *Suppose that*[12] *$r_i \in \mathbb{Q}_{++}$, and $A_i \succsim B_i$ for $i = 1, \ldots, n$. If $\mathbf{1}_A - \mathbf{1}_B = \sum_{i=1}^n r_i(\mathbf{1}_{A_i} - \mathbf{1}_{B_i})$, then $A \succsim B$.*

Proof. Let k denote the common denominator of $r_1, \ldots, r_n$, and write $r_i = \frac{m_i}{k}$ for $m_i \in \mathbb{N}$ and $i = 1, \ldots, n$. It follows that $k(\mathbf{1}_A - \mathbf{1}_B) = \sum_{i=1}^n m_i(\mathbf{1}_{A_i} - \mathbf{1}_{B_i})$ for all $s \in S$. By GFC applied to sequences $(A_i)_{i=1}^N$ and $(B_i)_{i=1}^N$, where each A_i and each B_i repeats m_i times ($N = m_1 + \cdots + m_n$), it follows that $A \succsim B$. □

Claim 2. *Suppose that $A_i \succsim B_i$ for $i = 1, \ldots, n$. If there are $\alpha_i > 0$, $i = 1, \ldots, n$, such that $\mathbf{1}_A - \mathbf{1}_B = \sum_{i=1}^n \alpha_i(\mathbf{1}_{A_i} - \mathbf{1}_{B_i})$, then $A \succsim B$.*

Proof. Suppose there are $\alpha_i > 0, i = 1, \ldots, n$, such that $\mathbf{1}_A - \mathbf{1}_B = \sum_{i=1}^n \alpha_i(\mathbf{1}_{A_i} - \mathbf{1}_{B_i})$. Consider the partition induced by $A_1, \ldots, A_n, B_1, \ldots, B_n, A, B$, and denote it by $\mathcal{A}$, with atoms denoted by a. The assumed indicators identity for all $s \in S$ translates to the following finite system of linear equations, with the variables $\alpha_1, \ldots, \alpha_n$:

$$\sum_{i=1}^{n} \delta_i(a)\alpha_i = \delta(a), \quad a \in \mathcal{A},$$
$$\delta_i(a) = \mathbf{1}_{A_i}(s) - \mathbf{1}_{B_i}(s), \quad s \in a, \qquad \delta(a) = \mathbf{1}_A(s) - \mathbf{1}_B(s), \quad s \in a.$$

Since all coefficients in the above equations, $\delta(a)$ and $\delta_i(a)$, are integers, it follows that if there is a solution then there is a rational solution. Moreover, due to the denseness of the rationals within the reals there are rational solutions that are arbitrarily close to the original solution, therefore if this one is positive, then there exists a positive and rational solution. By the previous claim it follows that $A \succsim B$. □

[12] $\mathbb{Q}_{++}$ is the set of strictly positive rational numbers.

4.1. Proof of Theorem 1

First it is proved that under assumptions P1–P4, the consensus multiple-priors representation follows (direction (i) $\Rightarrow$ (ii)).

When S is finite, there are finitely many pairs of events. Let $A_i, B_i, i = 1, \ldots, n$ denote all pairs of events such that $A_i \succsim B_i$, and define $y_i = \mathbf{1}_{A_i} - \mathbf{1}_{B_i}$. Consider $cone\{y_1, \ldots, y_n\}$, the convex cone generated by the vectors y_i. By Positivity and Reflexivity it is nonempty and contains zero. For each pair of events A and B, $\mathbf{1}_A - \mathbf{1}_B$ is in $cone\{y_1, \ldots, y_n\}$ if and only if $A \succsim B$: By definition, if $A \succsim B$ then $\mathbf{1}_A - \mathbf{1}_B$ is included in $cone\{y_1, \ldots, y_n\}$, and according to Claim 2 the opposite is also true.

Define[13] $\mathcal{V} = \{v \in \mathbb{R}^S \mid v \cdot y \geqslant 0 \text{ for all } y \in cone\{y_1, \ldots, y_n\}\}$. The set $\mathcal{V}$ is a closed convex cone, and contains the zero function.

Claim 3. *$y \in cone\{y_1, \ldots, y_n\} \Leftrightarrow v \cdot y \geqslant 0$ for every $v \in \mathcal{V}$.*

Proof. By definition of $\mathcal{V}$, if $y \in cone\{y_1, \ldots, y_n\}$ then $v \cdot y \geqslant 0$ for every $v \in \mathcal{V}$. Now suppose that $x \notin cone\{y_1, \ldots, y_n\}$. Since $cone\{y_1, \ldots, y_n\}$ is a closed convex cone then by a standard separation theorem there exists a nonzero vector $w \in \mathbb{R}^S$ separating it from x. As $cone\{y_1, \ldots, y_n\}$ contains the zero vector and $\alpha x \notin cone\{y_1, \ldots, y_n\}$ for all $\alpha > 0$, it must be that $w \cdot y \geqslant 0 > w \cdot x$, for every $y \in cone\{y_1, \ldots, y_n\}$. It follows that $w \in \mathcal{V}$, and the proof is completed. □

Conclusion 1. *$A \succsim B \Leftrightarrow v \cdot (\mathbf{1}_A - \mathbf{1}_B) \geqslant 0$ for every $v \in \mathcal{V}$.*

For every event $A \in \Sigma$ and vector $v \in \mathcal{V}$, denote $v(A) = v \cdot \mathbf{1}_A = \sum_{s \in A} v(s)$. According to the Non-Triviality assumption, $\mathcal{V} \neq \{0\}$. By Positivity, $\mathbf{1}_A \in cone\{y_1, \ldots, y_n\}$ for all $A \in \Sigma$, therefore $v(A) \geqslant 0$ for every $v \in \mathcal{V}$. It follows that the set $\mathcal{P} = \{\pi = v/v(S) \mid v \in \mathcal{V} \setminus \{0\}\}$ is a nonempty set of additive probability measures over Σ, such that:

$$A \succsim B \quad \Leftrightarrow \quad \pi(A) \geqslant \pi(B) \quad \text{for every } \pi \in \mathcal{P}.$$

By its definition, $\mathcal{P}$ is the maximal set w.r.t. inclusion that represents the relation.

The other direction, from the representation to the axioms, is trivially implied from properties of probability measures (GFC follows easily by taking expectation on both sides).

4.2. Proof of Theorem 2

4.2.1. Proof of the direction (i) $\Rightarrow$ (ii)

Define a subset $D_{\succsim}$ of $B_0(S, \Sigma)$,

$$D_{\succsim} = closure\left\{ \sum_{i=1}^{n} \alpha_i [\mathbf{1}_{A_i} - \mathbf{1}_{B_i}] \,\middle|\, A_i \succsim B_i,\ \alpha_i \geqslant 0,\ n \in \mathbb{N} \right\}.$$

That is, $D_{\succsim}$ is the closed convex cone generated by indicator differences $\mathbf{1}_A - \mathbf{1}_B$, for $A \succsim B$. By Reflexivity, Positivity and Nontriviality, $D_{\succsim}$ has vertex at zero, it is not the entire space $B_0(S, \Sigma)$, and it contains every nonnegative vector $\psi \in B_0(S, \Sigma)$. According to Claim 2, if an

13 For $x = (x_1, \ldots, x_{|S|})$, $y = (y_1, \ldots, y_{|S|}) \in \mathbb{R}^S$, $x \cdot y$ denotes the inner product of x and y. That is, $x \cdot y = \sum_{i=1}^{|S|} x_i y_i$.

indicator difference $\mathbf{1}_A - \mathbf{1}_B$ is obtained as a convex combination of indicator differences that correspond to preference, then $A \succsim B$. In order to show that $D_{\succsim}$ contains exactly those indicator differences which correspond to preference, it should further be proved that preference is preserved under the closure operation. This is done in the next claims.

Claim 4. *If $A \succ\succ B$, then $\mathbf{1}_A - \mathbf{1}_B$ is an interior point of $D_{\succsim}$.*

Proof. By definition, $A \succ\succ B$ implies that there exists a partition $\{A_1, \ldots, A_k\}$ of A and a partition $\{B'_1, \ldots, B'_l\}$ of B^c, such that for all i, j, $A \setminus A_i \succsim B \cup B_j$. First observe that it cannot be that $A = \emptyset$, for it would imply, on the one hand, that $A = \emptyset \succsim B$, by definition of strong preference, and on the other hand, by Generalized Finite Cancellation, $\emptyset \succsim B^c \Leftrightarrow B \succsim S$, contradicting Non-Triviality. Similarly $B = S$ is impossible. Hence, $k, l \geqslant 1$. Using the definition of strong preference, monotonicity of $\succsim$ w.r.t. set inclusion and the structure of $D_{\succsim}$ obtains:

$$\begin{aligned}
&\mathbf{1}_A - \mathbf{1}_B - \mathbf{1}_{A_i} \in D_{\succsim}, \quad i = 1, \ldots, k, \quad \text{and} \\
&\mathbf{1}_A - \mathbf{1}_B - \mathbf{1}_{B_j} \in D_{\succsim}, \quad j = 1, \ldots, l, \quad \text{therefore} \\
&(k+l)(\mathbf{1}_A - \mathbf{1}_B) - \mathbf{1}_A - \mathbf{1}_{B^c} = (k+l-1)(\mathbf{1}_A - \mathbf{1}_B) - \mathbf{1}_S \in D_{\succsim} \quad \Rightarrow \\
&(\mathbf{1}_A - \mathbf{1}_B) - \frac{1}{k+l-1}\mathbf{1}_S \in D_{\succsim}.
\end{aligned}$$

It is next shown that there exists a neighborhood of $\mathbf{1}_A - \mathbf{1}_B$ in $D_{\succsim}$. Let $\varepsilon < \frac{1}{2(k+l-1)}$ and let $\varphi \in B_0(S, \Sigma)$ be such that $\|\mathbf{1}_A - \mathbf{1}_B - \varphi\| < \varepsilon$. For all $s \in S$, $\varphi(s) > \mathbf{1}_A(s) - \mathbf{1}_B(s) - \frac{1}{2(k+l-1)}$, therefore φ dominates $\mathbf{1}_A - \mathbf{1}_B - \frac{1}{k+l-1}\mathbf{1}_S$. It follows that $\varphi = \mathbf{1}_A - \mathbf{1}_B - \frac{1}{k+l-1}\mathbf{1}_S + \psi$ for $\psi \in D_{\succsim}$ (since ψ is nonnegative), hence $\varphi \in D_{\succsim}$ and $\mathbf{1}_A - \mathbf{1}_B$ is an internal point of $D_{\succsim}$. □

Claim 5. *If $\mathbf{1}_A - \mathbf{1}_B$ is on the boundary of $D_{\succsim}$, then $A \succsim B$.*

Proof. Suppose on the contrary that for some events A and B, $\mathbf{1}_A - \mathbf{1}_B$ is on the boundary of $D_{\succsim}$, yet $\neg(A \succsim B)$. As $\mathbf{1}_A - \mathbf{1}_B$ is on the boundary of $D_{\succsim}$, there exists $\varepsilon' > 0$ and $\varphi \in D_{\succsim}$ such that $\mathbf{1}_A - \mathbf{1}_B + \delta\varphi \in D_{\succsim}$ for every $0 < \delta < \varepsilon'$.

On the other hand, employing Non-Atomicity, there exists an event $F \subseteq A^c$ such that $F \succ\succ \emptyset$ and $\mathbf{1}_A - \mathbf{1}_B + \mathbf{1}_F \notin D_{\succsim}$. The previous claim entails that $\mathbf{1}_F$ is an interior point of $D_{\succsim}$, hence there exists $\varepsilon_\varphi > 0$ such that $\mathbf{1}_F + \delta\varphi \in D_{\succsim}$ for all $|\delta| < \varepsilon_\varphi$. Let $0 < \delta < \min(\varepsilon_\varphi, \varepsilon')$, then $\mathbf{1}_A - \mathbf{1}_B + \delta\varphi + \mathbf{1}_F - \delta\varphi = \mathbf{1}_A - \mathbf{1}_B + \mathbf{1}_F$ is in $D_{\succsim}$, since it is a sum of two vectors in $D_{\succsim}$. Contradiction. □

Conclusion 2. $A \succsim B \Leftrightarrow \mathbf{1}_A - \mathbf{1}_B \in D_{\succsim}$.

Proof. The set $D_{\succsim}$ contains, by its definition, all indicator differences $\mathbf{1}_{A'} - \mathbf{1}_{B'}$ for $A' \succsim B'$ (thus also the zero vector), and their positive linear combinations. However, by the previous claims, if $\mathbf{1}_A - \mathbf{1}_B$ may be represented as a positive linear combination of indicator differences $\mathbf{1}_{A'} - \mathbf{1}_{B'}$ for which $A' \succsim B'$, or if $\mathbf{1}_A - \mathbf{1}_B$ is on the boundary of $D_{\succsim}$, then $A \succsim B$. That is, every indicator difference $\mathbf{1}_A - \mathbf{1}_B$ in the closed convex cone generated by indicator differences indicating preference also satisfies $A \succsim B$. □

Denote by $B(S, \Sigma)$ the space of all Σ-measurable, bounded real functions over S, endowed with the supremum norm. Denote by $ba(\Sigma)$ the space of all bounded, additive functions from Σ

to $\mathbb{R}$, endowed with the total variation norm. The space $ba(\Sigma)$ is isometrically isomorphic to the conjugate space of $B(S,\Sigma)$. Since $B_0(S,\Sigma)$ is dense in $B(S,\Sigma)$, $ba(\Sigma)$ is also isometrically isomorphic to the conjugate space of $B_0(S,\Sigma)$.

Consider an additional topology on $ba(\Sigma)$. For $\varphi \in B_0(S,\Sigma)$ and $m \in ba(S,\Sigma)$, let $\varphi(m) = \int_S \varphi\, dm$. Every φ defines a linear functional over $ba(S,\Sigma)$, and $B_0(S,\Sigma)$ is a total space of functionals on $ba(S,\Sigma)$.[14] The $B_0(S,\Sigma)$ topology of $ba(S,\Sigma)$, by its definition, makes a locally convex linear topological space, and the linear functionals on $ba(S,\Sigma)$ which are continuous in this topology are exactly the functionals defined by $\varphi \in B_0(S,\Sigma)$. Event-wise convergence of a bounded generalized sequence μ_α in $ba(S,\Sigma)$ to μ is identical to its convergence to μ in the following topologies: the $B_0(S,\Sigma)$ topology, the $B(S,\Sigma)$ topology, and the weak* topology (see Maccheroni and Marinacci [15]). Hence, the notion of closedness of bounded subsets of $ba(S,\Sigma)$ is identical in all three topologies.

Let $\mathcal{M} = \{m \in ba(\Sigma) \mid \int_S \varphi\, dm \geqslant 0 \text{ for all } \varphi \in D_{\succsim}\}$. The set $\mathcal{M}$ is a convex cone, and contains the zero function. For a generalized sequence $\{m_\tau\}$ in $\mathcal{M}$, which converges to m in the $B_0(S,\Sigma)$ topology, $m_\tau(\xi) \to m(\xi)$ for every $\xi \in B_0(S,\Sigma)$. Therefore, having $m_\tau(\varphi) \geqslant 0$ for every $\varphi \in D_{\succsim}$ and every τ, yields that $m \in \mathcal{M}$. The set $\mathcal{M}$ is thus closed in the $B_0(S,\Sigma)$ topology.[15]

Claim 6. *$\varphi \in D_{\succsim} \Leftrightarrow \int_S \varphi\, dm \geqslant 0$ for every $m \in \mathcal{M}$.*

Proof. According to the definition of $\mathcal{M}$, it follows that if $\varphi \in D_{\succsim}$ then $\int_S \varphi\, dm \geqslant 0$ for every $m \in \mathcal{M}$. Now suppose that $\psi \notin D_{\succsim}$. Since $D_{\succsim}$ is a closed convex cone, and $B_0(S,\Sigma)$, endowed with the supnorm, is locally convex, then by a Separation Theorem (see Dunford and Schwartz [6, Corollary V.2.12]) there exists a non-zero, continuous linear functional separating $D_{\succsim}$ and ψ. Hence, since $0 \in D_{\succsim}$ and $a\psi \notin D_{\succsim}$ for all $a > 0$, there exists $m' \in ba(\Sigma)$ such that $\int_S \varphi\, dm' \geqslant 0 > \int_S \psi\, dm'$, for every $\varphi \in D_{\succsim}$. It follows that $m' \in \mathcal{M}$, and the proof is completed. □

Conclusion 3. *$A \succsim B \Leftrightarrow \int_S (\mathbf{1}_A - \mathbf{1}_B)\, dm \geqslant 0$ for every $m \in \mathcal{M}$.*

Proof. Follows from Conclusion 2 and the previous claim. □

According to the Non-Triviality assumption, $\mathcal{M} \neq \{0\}$. By Positivity, $\mathbf{1}_A \in D_{\succsim}$ for all $A \in \Sigma$, therefore $\int_S \mathbf{1}_A\, dm \geqslant 0$ for every $m \in \mathcal{M}$. It follows that the set $\mathcal{P} = \{\pi = m/m(S) \mid m \in \mathcal{M} \setminus \{0\}\}$ is a nonempty, $B_0(S,\Sigma)$-closed and convex set of additive probability measures over Σ, such that:

$$A \succsim B \quad \Leftrightarrow \quad \pi(A) \geqslant \pi(B) \quad \text{for every } \pi \in \mathcal{P}.$$

Observation 1. The set $\mathcal{P}$ is bounded (in the total variation norm), hence it is compact in the $B(S,\Sigma)$ topology, thus in the $B_0(S,\Sigma)$ topology (see Corollary V.4.3 in Dunford and Schwartz [6]). As $\mathcal{P}$ is weak* closed and bounded, it is weak* compact (by Alaoglu's Theorem). In addition, by its definition $\mathcal{P}$ is maximal w.r.t. inclusion (any $\pi' \notin \mathcal{P}$ yields $\int_S \varphi\, d\pi' < 0$ for some

[14] That is, $\varphi(m) = 0$ for every $\varphi \in B_0(S,\Sigma)$ implies that $m = 0$.

[15] This part of the proof is similar to a proof found in 'Ambiguity from the differential viewpoint', a previous version of Ghirardato, Maccheroni and Marinacci [8].

$\varphi \in D_{\succsim}$, hence $\pi'(A) < \pi'(B)$ for some pair of events that satisfy $A \succsim B$), $B_0(S, \Sigma)$-closed and convex.

Claim 7. *If $A \succ\succ B$ then there exists $\delta > 0$ such that $\pi(A) - \pi(B) > \delta$ for every $\pi \in \mathcal{P}$.*

Proof. By definition of strong preference, $A \succ\succ B$ if and only if there exists a partition $\{G_1, \dots, G_r\}$ of S, such that $A \setminus G_i \succsim B \cup G_j$ for all i, j. This means that there are partitions $\{A_1, \dots, A_k\}$ of A, and $\{B'_1, \dots, B'_l\}$ of B^c, such that $\pi(A) - \pi(A_i) \geqslant \pi(B) + \pi(B'_j)$ for all $\pi \in \mathcal{P}$ and all i, j. It cannot be that $\pi(A) - \pi(A_i) = 0$ or $\pi(B) + \pi(B'_j) = 1$, since $\pi(B \cup B'_j) > 0$ for some j and $\pi(A \setminus A_i) < 1$ for some i. Hence, $k \geqslant 2$ and $l \geqslant 2$, and, for all $\pi \in \mathcal{P}$,

$$\pi(A) - \pi(B) \geqslant \pi(A_i) + \pi\big(B'_j\big), \quad \text{for all } i, j \quad \Rightarrow$$
$$(k + l - 1)\big(\pi(A) - \pi(B)\big) \geqslant 1$$

and the proof is completed with $\delta = 1/(k + l)$, for instance. □

Lemma 1. *The probability measures in $\mathcal{P}$ are uniformly strongly continuous.*

Proof. Let B be an event, and suppose that $\mu'(B) > 0$ for some $\mu' \in \mathcal{P}$. The inequality implies that $\neg(\emptyset \succsim B)$, therefore by Non-Atomicity there exists a partition of B, $\{B_1, \dots, B_n\}$, such that for all i, $B_i \succ\succ \emptyset$. Specifically, $\mu(B_1) > \delta > 0$ for all $\mu \in \mathcal{P}$ according to the previous claim, proving part (a) of uniform strong continuity.

Non-Atomicity also implies that $\neg(\emptyset \succsim B \setminus B_i)$. Therefore, there must be at least two events B_i in the partition of B, and for each one, $\mu(B_i) > \delta' > 0$ for some δ' and all $\mu \in \mathcal{P}$. Hence for all $\mu \in \mathcal{P}$, $\mu(B) > \mu(B \setminus B_1) > 0$. All probability measures in $\mathcal{P}$ are thus non-atomic.

According to the above arguments, there exists an event F_1 such that $0 < \mu(F_1) < 1$ for all $\mu \in \mathcal{P}$. As this implies $\neg(F_1^c \succsim S)$, it follows from Non-Atomicity that there exists a partition of F_1, $\{E_1, \dots, E_m\}$, such that $E_i \succ\succ \emptyset$ and $\neg(\emptyset \succsim F_1 \setminus E_i)$ for $i = 1, \dots, m$.

For a fixed i, the preference $E_i \succ\succ \emptyset$ entails that there exists a partition of S, $\{G_1, \dots, G_{r_i}\}$, that satisfies $E_i \succsim G_j$, $j = 1, \dots, r_i$. Taking the refinement of the partitions for each i, there exists a partition $\{G_1, \dots, G_r\}$ such that $E_i \succsim G_j$ for all $i = 1, \dots, m$ and $j = 1, \dots, r$. It follows that for each i, j, $\mu(G_j) \leqslant \mu(E_i)$ for every $\mu \in \mathcal{P}$. As $\emptyset \succsim \emptyset$, the partition $\{E_1, \dots, E_m\}$ must consist of at least two atoms. Hence, for each j,

$$\mu(G_j) \leqslant \frac{1}{m}\sum_{i=1}^{m} \mu(E_i) \leqslant \frac{1}{2}\mu(F_1) < \frac{1}{2}, \quad \text{for all } \mu \in \mathcal{P}.$$

Let $F_2 = G_j$ for G_j such that $\mu(G_j) > 0$ (there must exist such j since the G_j's partition S). Again $\neg(\emptyset \succsim F_2)$, and by Non-Atomicity there exists a partition of F_2, $\{E'_1, \dots, E'_l\}$, such that $E'_i \succ\succ \emptyset$ and $\neg(\emptyset \succsim F_2 \setminus E'_i)$ for $i = 1, \dots, l$.

As in the previous step, it follows that $l \geqslant 2$ and there exists a partition $\{G'_1, \dots, G'_k\}$ with $\mu(G'_j) \leqslant \mu(E'_i)$ for all indices i, j and all probabilities $\mu \in \mathcal{P}$. Thus, for all j,

$$\mu\big(G'_j\big) \leqslant \frac{1}{l}\sum_{i=1}^{l} \mu\big(E'_i\big) \leqslant \frac{1}{2}\mu(F_2) < \frac{1}{4}, \quad \text{for all } \mu \in \mathcal{P}.$$

In the same manner, for all $n \in \mathbb{N}$ there exists a partition $\{G_1, \dots, G_r\}$ such that for all j, $\mu(G_j) < \frac{1}{2^n}$ for all $\mu \in \mathcal{P}$. It follows that the probabilities in $\mathcal{P}$ are uniformly strongly continuous. □

The proof of the direction (i) ⇒ (ii) is completed.

4.2.2. Proof of the direction (ii) ⇒ (i)

Suppose that for every $A, B \in \Sigma$, $A \succsim B$ if and only if $\pi(A) \succsim \pi(B)$ for every probability measure π in a $\succsim$-maximal set $\mathcal{P}$, and that $\mathcal{P}$ is uniformly strongly continuous, and all probabilities in the set are non-atomic. Assumptions P1 through P5 are shown to hold.

P1. Reflexivity and P2. Positivity. For every $A \in \Sigma$ and every $\pi \in \mathcal{P}$, $\pi(A) \geqslant \pi(A)$ and $\pi(A) \geqslant 0$, hence $A \succsim A$ and $A \succsim \emptyset$.

P3. Non-Triviality. The $\succsim$-maximal set $\mathcal{P}$ is nonempty, thus $\pi(B) > \pi(A)$ for some $A, B \in \Sigma$ and $\pi \in \mathcal{P}$, implying $\neg(A \succsim B)$.

P4. Generalized Finite Cancellation. Let $(A_i)_{i=1}^n$ and $(B_i)_{i=1}^n$ be two collections of events in Σ, such that $A_i \succsim B_i$ for all i, and $\sum_{i=1}^n (\mathbf{1}_{A_i}(s) - \mathbf{1}_{B_i}(s)) \leqslant k(\mathbf{1}_A(s) - \mathbf{1}_B(s))$ for all $s \in S$, for some $k \in \mathbb{N}$ and events $A, B \in \Sigma$. Then for every π in $\mathcal{P}$, $k\mathbb{E}_\pi(\mathbf{1}_A - \mathbf{1}_B) \geqslant \sum_{i=1}^n \mathbb{E}_\pi(\mathbf{1}_{A_i} - \mathbf{1}_{B_i}) \geqslant 0$. It follows that $\pi(A) \geqslant \pi(B)$ for every $\pi \in \mathcal{P}$, hence $A \succsim B$.

Claim 8. *If $\mu(F) > 0$ for an event F and some probability measure $\mu \in \mathcal{P}$, then $F \succ\succ \emptyset$.*

Proof. Suppose that F is an event with $\mu(F) > 0$ for some probability $\mu \in \mathcal{P}$. Using part (a) of uniform strong continuity, $\mu(F) > 0$ for *every* $\mu \in \mathcal{P}$. As the set $\mathcal{P}$ is weak* compact, the infimum of $\mu(F)$ over it is attained (see Lemma I.5.10 in Dunford and Schwartz [6]), yielding that $\inf_{\mu \in \mathcal{P}} \mu(F) = \mu'(F)$ for some μ'. Hence there exists $\delta > 0$ such that $\mu(B) > \delta$ for every probability $\mu \in \mathcal{P}$. By part (b) of uniform strong continuity there exists a partition $\{G_1, \dots, G_r\}$ of S such that $\mu(F) - \mu(G_k) > \mu(G_j)$ for every k, j and every $\mu \in \mathcal{P}$, yielding $F \setminus G_k \succsim G_j$ for every k, j. By definition, $F \succ\succ \emptyset$. □

P5. Non-Atomicity. Suppose that $\neg(A \succsim B)$. By the representation assumption, $\mu'(B) > \mu'(A)$ for some $\mu' \in \mathcal{P}$. Note that necessarily $\mu'(A^c) > 0$. It is required to show that there exists a partition $\{A'_1, \dots, A'_k\}$ of A^c such that for all i, $A'_i \succ\succ \emptyset$ and $\neg(A \cup A'_i \succsim B)$.

Uniform strong continuity of the set $\mathcal{P}$ implies that there exists a partition $\{G_1, \dots, G_r\}$ of S, such that for all j, $\mu(G_j) < \mu'(B) - \mu'(A)$ for all μ, thus specifically for μ'. The partition $\{G_1, \dots, G_r\}$ induces a partition $\{A'_1, \dots, A'_k\}$ of A^c such that $\mu'(A'_i) > 0$ and $\mu'(A \cup A'_i) < \mu'(B)$ for all i, thus by the representation, $\neg(A \cup A'_i \succsim B)$. According to the previous claim, $A'_i \succ\succ \emptyset$ for all i.

4.3. Proof of Proposition 3

Suppose first that $\succsim$ is represented as in (3), and in addition Monotone Continuity P6 holds. Denote the representing set of probabilities by $\mathcal{P}$. For countable additivity it suffices to prove that for any sequence of events that decreases to the empty set, the limit of the probabilities of the sets is zero.

Let E_n be a sequence of events such that $E_1 \supseteq E_2 \supseteq \cdots$ and $\bigcap_n E_n = \emptyset$. Take $\varepsilon > 0$. According to the representation there exists a partition of S, $\{G_1, \dots, G_r\}$, such that for every j and every $\mu \in \mathcal{P}$, $\mu(G_j) < \varepsilon$. Let G_j be such that $\mu'(G_j) > 0$ for some μ', thus by the representation $\mu(G_j) > 0$ for all $\mu \in \mathcal{P}$, which implies $G_j \succ\succ \emptyset$ by Claim 8. Monotone Continuity them renders $G_j \succsim E_n$ for some n, yielding $\mu(E_n) < \varepsilon$ for every $\mu \in \mathcal{P}$. As $E_n \supseteq E_{n+1} \supseteq E_{n+2} \supseteq \cdots$, the same inequality holds for all probabilities from this n on and $\lim_{n\to\infty} \mu(E_n) = 0$.

In the other direction suppose that each probability in the set is sigma-additive, and consider a decreasing sequence of events, $E_1 \supseteq E_2 \supseteq \cdots$, with limit event $\bigcap_n E_n = \emptyset$. Let F be an event such that $F \succ\succ \emptyset$. According to Claim 7, $\mu(F) > \delta$ for some $\delta > 0$ and all $\mu \in \mathcal{P}$. By weak* compactness of the set $\mathcal{P}$, $\sup_{\mu\in\mathcal{P}} \mu(E_n)$ is attained for every n. Denote by μ_n a probability in $\mathcal{P}$ for which $\mu_n(E_n) = \sup_{\mu\in\mathcal{P}} \mu(E_n)$. It suffices to prove that there exists n for which $\mu_n(E_n) \leqslant \delta$ (as the sequence of events E_n is decreasing the same will hold for all larger n's as well).

Suppose on the contrary that $\mu_n(E_n) > \delta$ for every n. As the set $\mathcal{P}$ is weak* compact, it is also sequentially weak* compact (see Theorem 1 in [15]). Thus $(\mu_n)_n$ has a convergent subsequence with limit in $\mathcal{P}$. That is to say, there is a subsequence, denote it $(\mu_k)_k$, such that μ_k event-wise converges to a probability $\mu \in \mathcal{P}$. For every k, $\mu_k(E_k) > \delta$, therefore, by the structure of the sequence of events, $\mu_{k+\ell}(E_k) > \delta$ for all $\ell = 1, 2, \dots$. It follows that all measures in the sequence (except maybe for a finite number) assign a probability larger than δ to each event E_k, therefore $\mu(E_k) \geqslant \delta > 0$ for every k. But μ is in $\mathcal{P}$, therefore by assumption σ-additive, satisfying $\lim_{k\to\infty} \mu(E_k) = 0$. Contradiction. It is concluded that $\mu(F) > \delta \geqslant \mu_n(E_n) \geqslant \mu(E_n)$ for all $\mu \in \mathcal{P}$, hence $F \succsim E_n$, from some n on. □

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