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Competitive economy as a ranking device over networks ☆

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ABSTRACT

We propose a novel approach to generating a ranking of items in a network (e.g., of web pages connected by links or of articles connected by citations). We transform the network into an exchange economy, and use the resulting competitive equilibrium prices of the network nodes as their ranking. The widely used Google's PageRank comes as a special case when the nodes are represented by Cobb–Douglas utility maximizers. We further use the economic metaphor to combine between the Citation Count and PageRank by imposing a redistributive taxing scheme.

1. Introduction

In a world with abundant information, ranking systems are of utmost importance. Well known examples include Google's PageRank, which helps Internet users to identify web pages that are more likely to interest them, and the citation count,[4]

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4 The citation count, which simply counts the number of citations received by a paper, is used to measure the quality of an individual paper. A close related measurement is the famous Impact Factor, which is used to measure the quality of an academic journal. The Impact Factor shares the same spirit of the citation count as it essentially counts the average number of citations received by papers published on it. In this paper, we focus on the problem of ranking individual papers.

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which helps academic researchers to assess the quality of academic articles. In both cases, as well as in many other contexts, the ranking of items is based solely on the information embodied in the network such as links between web sites or citations between scientific articles.

Whereas the citation count merely counts the number of citations from other articles and does not discriminate by the source of the citation, more sophisticated ranking systems attempt to give more weight to votes from items which are of a higher rank (as ascribed by the system itself). The approach taken by PageRank, for example, is based on the idea of translating the link structure to a Markov process as follows: each web page is viewed as a state, and after the random walk hits a state it moves on randomly to one of the states that the current state gives them a link. The ranking of a web page is defined as the long-run proportion of time that the process spends in a given state. Since this value depends not only on the number of incoming links but also on the proportion of time the system spends on the states that send these links, the induced ranking indeed grants more weight to links from higher-ranked pages.

We propose a different approach to ranking that consists of constructing an economy based on the network of links and deriving the ranks from the competitive equilibrium prices.[5] Our approach employs a neoclassical pure-exchange economy. Each web page is represented by one consumer, who is initially endowed with one unit of a specific good. The market price of his good becomes his budget, which in turn serves to buy other goods. The consumer derives utility from consuming the specific goods provided by exactly those consumers that he or she sends a link to. For example, if page i has links only to pages j and k, then consumer i has utility $u(x^i_j, x^i_k)$ where x^i_j, x^i_k are the quantities that i consumes from goods j and k.

The main idea of this paper is to use the competitive prices of this economy as a ranking system. That is, the ranking of a web page is defined as the price of the corresponding good. In this pure-exchange economy, higher-ranked pages correspond to more expensive goods. Moreover, since the budget of a consumer equals the price of the good he initially owns, the initial owners of highly demanded goods are rich. These owners demand larger quantities of the goods they like, thus pushing their prices higher. Hence, those web pages that are pointed to by highly ranked web pages, are highly ranked themselves.

The specific ranking obtained depends on the modeler's choice of utility functions. We assume throughout that all consumers, although consuming different goods, have the same type of utility function. We first consider the Cobb–Douglas utility function which is perhaps the most widely used in economic modeling. We show that when all consumers have Cobb–Douglas preferences, the resulting vector of competitive equilibrium prices coincides with the PageRank ranking system.

We next show that the citation count cannot be derived from our economy. The reason is that, in this type of economy, the value of a paper is identified with the budget of the corresponding consumer, which in turn is his reviewing power. However, in citation count, the reviewing power of a paper is a fixed constant. This impossibility extends also to the 'Normalized' Citation Count, in which the value of a citation from an article is inversely proportional to the number of citations the article makes.

A ranking system based solely on the network information[6] actually considers each item (e.g., web page, article) both as a reviewer, whose judgment (link, references) determines the rank of others, and as a refereed, whose assessed quality depends on the links it obtained from others. The following economic metaphor can sharpen this distinction: the value of an agent as refereed is the market price of its good. Its power as a reviewer equals its budget. In the formulation we employed so far, these two powers are, by definition, the same, since the budget comes from selling the agent's specific good.

The citation count, that does not make any connection between the value of a paper and its refereeing power seems to lack the desirable property of allocating greater weight to citations originating from more important papers. However, there are many cases in which also PageRank, or in fact any ranking system derived from an exchange economy, fails to produce meaningful results. Consider, for example, a sequence of papers published sequentially, each at a distinct time. As citations can only refer to earlier work, all these papers will have zero value. This happens precisely because the value of an item as a reviewer is equated with its value as a refereed entity. The latest article has no incoming links, implying that its price and budget are 0. In turn, it has 0 demand for the articles it has links to. Thus, the penultimate article has 0 value as well, and so forth. In similar cases the citation count – where the reviewing power of all items are the same and independent of their power as refereed items – may perform better than the PageRank.

One can combine the advantages of these two ranking systems by developing the economic metaphor a bit further. We add to the exchange economy a taxation scheme, thus allowing to disentangle the value of an item as a reviewer and its quality as determined by others. Each consumer pays a proportion, say α, of his income as a tax. The tax revenue is then equally redistributed between all the consumers. When $\alpha = 0$, a consumer's budget equals the price of his specific good, leading us back to the original model. With a 100% tax (i.e., $\alpha = 1$), the budgets of all consumers are equal. As a result, their reviewing power is equal regardless of the prices of their goods. The competitive equilibrium prices in this case could serve

[5] More precisely, under some conditions (shown in the paper) the economy has a competitive equilibrium, while under weaker conditions (also shown) the economy is only guarantied to have a quasi-equilibrium (Debreu, 1962). For brevity we refer in both cases to the vector of prices as "competitive prices".

[6] This is as opposed to information obtained from other sources, such as the quality of the journal in which an article was published, etc.

as a ranking system that bears the spirit of the citation count.[7] By choosing a tax rate between the two extremes, one can control the extent to which the reviewing power of an item depends on its quality (as assessed by the ranking system). We show examples in which the ranking system derived from an intermediate tax seems to do better than both PageRank and the citation count.

Related literature. In the 1960s, Garfield introduced the first citation index for papers published in academic journals: the Science Citation Index (SCI). This index was followed by the Social Sciences Citation Index (SSCI) and later by the Arts and Humanities Citation Index (AHCI). In 1972 Garfield established a ranking of scientific journals, known as Impact Factor. Liebowitz and Palmer (1984) analyzed the impact factors of economic journals by using an iteration (impact adjusted) method, without making the mechanism explicit.

The Markov-chain approach behind PageRank has been originally proposed by Wei (1952) and Kendall (1955). The approach was applied to create PageRank by Brin and Page who describe the algorithm in detail in Brin and Page (1998).

Another major direction that was applied in the literature has been the axiomatic approach. By postulating a number of desired requirements that the ranking system must satisfy, one attempts to identify a specific ranking. This approach was adopted by Palacios-Huerta and Volij (2004) and Altman and Tennenholtz (2005) who axiomatized PageRank. Demange (2011) also takes an axiomatic approach and derives an index based on treating members in the network both as referees and being refereed at the same time.

Posner (2000) criticizes many ranking methods and states that 'citation analysis is not an inherently economic methodology' due to its lack of theoretical or empirical grounding. Amir (2002) studies properties of various indices. The influence model of Demange (2011) describes a dynamics whereby the ranking of journals affect the intensities of citations.

Structure of the paper. Section 2 introduces the model of an exchange economy induced by a network and derives the ranking scheme. In Section 3 we show that if an exchange economy is governed by Cobb–Douglas utility maximizers, then the resulting ranking coincides with PageRank. Section 4 studies the citation count and shows that it cannot be derived from an exchange economy. Section 5 introduces taxation into the economy and deals with the resulting distinction between quality and refereeing power. Final remarks appear in Section 6.

2. Exchange economy as a ranking system

2.1. The graph of citations

Our initial data is a directed graph G with the set of nodes $V = \{1, \dots, n\}$ and the set of directed edges $E \subseteq V \times V$. Each node (vertex) represents a web page (or an article) and each directed edge $(i, j) \in E$ represents a link (or citation) from web page i to web page j. We refer to $(i, j) \in E$ as 'i has a link to j', 'i cites j', 'j is cited by i' and alike. It is also useful to define the *coincidence matrix* $\Pi = (\pi_{ij})_{ij}$ in which $\pi_{ij} = 1$ if $(i, j) \in E$ and $\pi_{ij} = 0$, otherwise. Denote by $I(i) = \{j : (j, i) \in E\}$ the set of nodes sending links to i (incoming links to i) and by $O(i) = \{j : (i, j) \in E\}$ the set of the nodes that i has a link to (outgoing links from i). Denote the cardinality of $O(i)$ by $c(i)$.

In case the graph G has a node i that has no outgoing links ($O(i) = \emptyset$), we modify the graph by adding to this node a self link (i, i). To save on notation we abuse notation and refer to the modified graph also as G. Thus, in the modified graph every node has at least one outgoing link (i.e., for every $i \in V$, $O(i) \neq \emptyset$).

A ranking system is a function from such directed graphs to vectors of valuations, one coordinate for each node.

2.2. Connectedness and irreducibility

We say that there is a path from i to j if there is a sequence of edges, $(i, i_1), (i_1, i_2), \dots, (i_{k-1}, i_k), (i_k, j)$ in E. The graph G is *connected* if for every $i, j \in G$ there is either a path from i to j or from j to i.

Assumption 1. The graph G is connected.

In Section 6 we explain how our results are extended to the case of non-connected graphs. Note that in a connected graph there can be at most one node with no outgoing links (sink). Thus, we have added, at most, one self link.

For any node i define $V(i)$ to be the set of all nodes j such that there is a path from i to j. The set $V(i)$ is non-empty for every $i \in V$. Note that if there is a path from i to j, then $V(j) \subseteq V(i)$. The set $V^* = \cap_{i \in V} V(i)$ is the *recurrent component* of G. A node in V^* is called *recurrent* and in $V \setminus V^*$ *transient*. Note that for every node $i \in V$ and any recurrent node $j \in V^*$ there is a path from i to j. Since G is connected and $V(i) \neq \emptyset$, we have:

[7] What is actually obtained is a normalized citation count, in which any article obtains an allotment of one unit to be equally shared between all the articles it cites. The normalized citation count of an article is then the sum of all the (normalized) citations it obtains. A similar idea applied to clusters of papers, based on their field, was proposed by Moed et al. (1995) and applied by the University of Leiden.

Lemma 1. *The graph G has recurrent nodes (i.e., the irreducible component V^* is nonempty).*

All the proofs are deferred to Appendix A.

We say that the graph is *irreducible* if all the nodes are recurrent (i.e., $V = V^*$). In words, the graph is irreducible if for every two nodes there are paths connecting them is both directions.

2.3. The "governing" utility function

The second element needed in order to define the exchange economy is a set of utility functions $\{u^k\}_{k=1,\dots,n}$ where $u^k : \mathbb{R}^k_+ \to \mathbb{R}$ will be the utility function of a consumer who interested in consuming k out of the n goods. For convenience, we use the normalization $u^k(\vec{0}) = 0$ for every k ($\vec{0}$ being the zero vector). Throughout, we impose the following restrictions on the utility functions:

Assumption 2. For every k, the function u^k is

- Continuous.
- Non-degenerate: There is at least one bundle x such that $u^k(x) > 0$.
- Quasi-concave: For any x, the set $\{y;\ u^k(y) \geq u^k(x)\}$ is convex.
- Satisfies an Inada condition: For every $x \neq 0$ in the boundary[8] of $\mathbb{R}^k_+$ and every $M > 0$ there is $\varepsilon > 0$ such that[9] $u^k(x - M\varepsilon \cdot x^+ + \varepsilon \cdot x^0) > u^k(x)$ and $x - M\varepsilon \cdot x^+ + \varepsilon \cdot x^0 \geq 0$.
- Conditionally increasing: For every $x \in \mathbb{R}^k_+$ and j, the function[10] $f(d) = u^k(x^{-j}, d)$ is either constant or strictly increasing.

The first three conditions are rather standard. The Inada condition states that for any $x \neq 0$ in the boundary of $\mathbb{R}^k_+$, the "MRS" between the goods with zero quantity and those with positive quantities tends to infinity. The purpose of this condition is to ensure that a consumer purchases a positive amount of every good on his domain. Conditionally increasing means that, keeping the quantities of all goods but j unchanged, utility is either constant in the quantity of good j or strictly increasing in this quantity.

2.4. The induced exchange economy

We now construct the u-exchange-economy induced by the graph G. There are n consumers, one for each node. There are also n distinct goods. Consumer i has an initial endowment of one unit of good i. The utility of consumer i with set of outgoing links $O(i)$ is $u^{c(i)}((x^j_i)_{j\in O(i)})$. Thus, his utility depends on the quantities that he consumes from the $c(i)$ goods that correspond to the vertices he has links to; he is not interested in consuming other goods.

2.5. Competitive equilibrium and quasi-equilibrium

We intend to base our ranking of web pages on the equilibrium prices of the goods in the u-exchange-economy. We show below that if the graph is irreducible, then a competitive equilibrium exists and all prices are positive. However, if this is not the case, then there are graphs and utility functions for which a competitive equilibrium fails to exist. Roughly, the reason is that there are consumers whose budget is zero and some of the goods they consume have zero price. With positive marginal utility from consuming the zero-priced goods, they will demand large amounts without violating their budget constraint, therefore leading demand to exceed supply.[11] Yet, in such cases we can still apply Debreu's weaker notion of quasi-equilibrium (see Debreu, 1962), which does exist.

As in a competitive equilibrium, also in a quasi-equilibrium aggregate demand equals aggregate supply, and consumers must choose, among all the baskets of goods that they can afford, the one that maximizes their utility. However, the second requirement is less strict and applies only to consumers with positive budget; consumers whose initial basket is worth zero at the equilibrium prices are not required to maximize their utility and consume only leftovers. Formally,

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8 That is, x has zero coordinates.

9 We denote by x^+ and x^0 the indicator functions of the positive and the zero coordinates of x: $x^+_j = 1$ iff $x_j > 0$ and $x^0_j = 1$ iff $x_j = 0$.

10 (x^{-j}, d) is the vector that coincides with x on all the coordinates but j and equals to d on the j-th coordinate.

11 As an example consider a graph with nodes 1, 2, 3 and edges $(1, 2)$ and $(2, 3)$, and the induced economy with Cobb–Douglas utility functions. In this economy, node 1 should have price 0, since its good does not get any demand. Since the budget of node 1 is 0, the price of the good 2 is also 0. But then 1 will consume unbounded amounts of good 2. This will make a competitive equilibrium fail to exist.

Definition 1. Consider the u-exchange-economy induced by the graph G where $x_i = (x_i^j)_{j \in O(i)} \in \mathbb{R}^{O(i)}$ is the basket consumed by consumer i and p is a nonzero vector in $\mathbb{R}_+^n$. A tuple $(x; p)$ is:

A *competitive equilibrium* if

(1) All consumers maximize their utility: $y_i \in \mathbb{R}^{O(i)}$ and $u^{c(i)}(y_i) > u^{c(i)}(x_i)$ imply $\sum_{j;\ j \in O(i)} p_j y_i^j > p_i$; and
(2) All markets clear: For every good j, $\sum_{i;\ j \in O(i)} x_i^j \leq 1$.

A *quasi-equilibrium* if

(1′) Consumers with positive budgets maximize their utility: For every i with $p_i > 0$, $y_i \in \mathbb{R}^{O(i)}$ and $u^{c(i)}(y_i) > u^{c(i)}(x_i)$ imply $\sum_{j;\ j \in O(i)} p_j y_i^j > p_i$;
(2) All markets clear: For every good j, $\sum_{i;\ j \in O(i)} x_i^j \leq 1$.

Note that since (1) is more restrictive that (1′), every competitive equilibrium is a quasi-equilibrium. The following proposition states that under the conditions stated above, quasi-equilibrium exists. A sketch of the proof, which is rather standard but not identical to any other in the literature, is provided in Appendix A.

Proposition 1. *A quasi-equilibrium exists.*

The following proposition and corollary state that for irreducible graphs, i.e., when all nodes are recurrent, the two equilibrium notions coincide:

Proposition 2. *For every quasi-equilibrium $(x; p)$ in the u-exchange-economy, $p_i > 0$ iff i is recurrent.*

Corollary 1. *If the graph is irreducible then any quasi-equilibrium is a competitive equilibrium. In particular, a competitive equilibrium exists.*

Since in some interesting applications graphs are not irreducible, we state all the results in the reminder of the paper without this assumption and thus employ the weaker solution concept of quasi-equilibrium. Yet, whenever the conditions of Corollary 1 hold, then in each of the results one can replace "quasi-equilibrium" by "competitive equilibrium".

2.6. Economy-based ranking

Our ranking system is based on quasi-equilibrium prices (or competitive prices if a competitive equilibrium exists). We define the rank, or quality, of node i as the price of the corresponding good p_i in the (quasi- or competitive) equilibrium. Since a quasi-equilibrium always exists, the definition is non-empty. However, since a (quasi- or competitive) equilibrium is in general not unique, our approach might generate multiple rankings. In the sequel we look for utility functions and network structures that generate a unique equilibrium and thereby a uniquely defined ranking.

Different utility functions typically produce different rankings. Thus, the exchange economy could serve as a mechanism that generates different ranking systems. By simply 'plugging in' different utility functions the exchange economy produces different rankings. It is important to note though that whatever the utility function, the budget of consumer i coincides with the price of good i. Moreover, this price depends on the individual purchasing power of its consumers, which coincides with the prices of their own goods. The latter, on their part, depend on the purchasing power of their own consumers, and so forth. Hence, the price of a good depends on the entire network's structure. In particular, a good consumed by a rich consumer (i.e., one who initially owns an expensive good) is likely to be expensive itself.

2.7. Symmetry and anonymity

A desirable property of the ranking system is that any difference between items in their induced ranking stems only from the structure of links and is not imposed by the utility function. Thus, while not needed for the results above, it is natural to further assume that the utility functions are symmetric:

- Symmetry assumption: $u^k(x) = u^k(\pi(x))$ for any k-vector of goods x and any permutation π of the k goods.

Symmetry implies that a consumer's utility depends only on the quantities of the goods consumed and not on their identity; moreover, all the consumers who are interested in k goods have the same utility function. Symmetry, however, yields only partial anonymity. While it implies that consumers who are interested in the same number of goods k must have the same utility function u^k, it does not impose any relation between u^ℓ and u^m for $\ell \neq m$.

Nevertheless, in all the concrete examples we study (e.g., Cobb–Douglas, minimum, CES) the utility functions employed induce the same consumption patterns across all consumers, regardless of the number of goods they consume. We discuss additional possible restrictions on the utility function in the concluding comments (Section 6).

3. Cobb–Douglas utility yields Google's ranking

In this section we illustrate the main idea using the Cobb–Douglas utility function and show the connection with Google's PageRank.

3.1. PageRank in detail

Google's ranking, known as PageRank, attempts to capture not only the number of links a site receives from others, but also the significance of each link. The rank of a site depends on the values of the links it receives, where the value of a link is determined by the rank of the site that gave the link (divided by the number of links coming out from that site). The approach taken by Brin and Page (1998) to tackle the circular definition, is to transform the link data into a Markov chain; PageRank then takes the ranks from the resulting invariant distribution. We now describe their methodology in detail.

Let the set of states of the Markov chain be V. The probability of transition from state i to state j, denoted m_{ij}, is defined by:

$$m_{ij} = \frac{\pi_{ij}}{c(i)}. \tag{1}$$

In words, the total probability to exit state i is equally divided between all the vertices j to which i refers. Note that m_{ij} is well defined. Denote by M the transition matrix $(m_{ij})_{(i,j)\in V\times V}$.

PageRank is based on an invariant distribution of M. Suppose that p is a distribution over V, that is, $p \in \Delta(V)$. We say that p is an *invariant distribution* of M, if

$$pM = p. \tag{2}$$

The interpretation of p, which makes it a good candidate for ranking, is that the probability p_i assigned to i by the distribution p is the frequency in which state i is visited by the system, when run for a long period of time.

A technical comment is due here. Typically, since M might not be ergodic, there could be a few invariant distributions. In practice, M is slightly perturbed in order to make all its transition probabilities strictly positive. Brin and Page (1998) make use of the "damping factor" technique to make the matrix ergodic, which guarantees uniqueness of its invariant distribution. Later on (in Section 5.2) we propose an alternative way of guaranteeing uniqueness other than perturbing the matrix.

Example 1. Consider the following graph with 3 nodes:

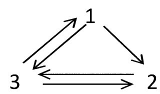

Fig. 1. Example 1.

The coincidence and transition matrices are:

$$\Pi = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad M = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

There is only one vector whose coordinates sum up to 1 that satisfies Eq. (2), which is $p = (2/9, 3/9, 4/9)$. Indeed,

$$(2/9, 3/9, 4/9)\begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix} = (2/9, 3/9, 4/9).$$

3.2. The Cobb–Douglas economy

Let u be the symmetric Cobb–Douglas utility function, and consider the u-exchange-economy defined above. That is, i's utility is the product of the quantities he consumes from all the goods in $O(i)$:

$$u_i^{c(i)}(x) = \Pi_{j\in O(i)} x_j, \tag{3}$$

where $x \in \mathbb{R}_+^{O(i)}$, and x_j is the quantity of good j in basket x. We denote this economy by $E(\Pi, \text{CD})$.

The following proposition states that there is a unique normalized vector of quasi-equilibrium prices and this vector coincides with PageRank.

Proposition 3. *There is a unique quasi-equilibrium price system of in* $E(\Pi, \mathrm{CD})$*, denoted* p*. Moreover,* $pM = p$ *and it coincides with PageRank or the Invariant Method.*

Remark 1. Eaves (1985) showed that the computation of an equilibrium for a Cobb–Douglas market can be reduced to solving a linear equation system. Although it was not explicitly claimed in Eaves (1985), Eaves's result implies that an equilibrium of a Cobb–Douglas market actually corresponds to a principal eigenvector of a stochastic matrix. In Proposition 3, we show the reverse direction of Eaves' reduction. That is, a principal eigenvector of a stochastic matrix corresponds to an equilibrium of a special Cobb–Douglas economy.

4. The citation count method

A common ranking method of a scientific paper is the citation count (CC), which simply counts the number of citations it received from other papers. This method does not take into consideration the ranking of the citing paper nor its venue. One can also define a modified version of this method, in which a citation from an article that makes n citations is counted as $1/n$ rather than 1. Thus, each article has the same reviewing power independently of the number of citation it makes. We call this the Normalized Citation Count, henceforth NCC.[12]

Proposition 4. *There is no utility function* u *such that the quasi-equilibrium price system of the* u*-exchange-economy coincides with CC (NCC) for every irreducible citations graph.*

It is clear, but still worth noting, that since there is no exchange economy whose quasi-equilibrium prices coincide with CC or with NCC on the set of all irreducible graphs, this statement holds true on the larger set of all connected graphs.

The reason why the citation count cannot be captured by an exchange economy is that it makes a complete separation between an item's rank and its refereeing power. That is, the value of a link from item i is independent of i's rank. In our exchange economy demand from a 'richer' consumer, ceteris paribus, has a higher impact on a good's price. One may view this result as an argument against the use of the citation count (whether normalized or not).

Remark 2. The CC and NCC can be obtained as an equilibrium of another exchange economy, called the Fisher economy (Nisan et al., 2008). In this economy each node of the graph is represented by two consumers: one who has one unit of a specific good (like in the economy above) and derives utility only from "gold" and a second one, with Cobb–Douglas utilities, who has an initial endowment of "gold" (for the CC method, the initial endowment is k units if the item has k references, while for the NCC method, the initial endowment is 1 unit). By a similar argument to that in the proof of Proposition 3, it is straightforward to see that the equilibrium price of goods is corresponding to the CC or NCC. The key difference between a ranking based on the Fisher economy and a ranking based on u-exchange-economy is that in a Fisher economy the referencing power of a consumer is fixed, while in a u-exchange-economy the referencing power of a consumer is based on its authority rank, i.e., the price of his good.

5. Quality and refereeing power

Any ranking systems induced by an exchange economy equates, by definition, the refereeing power of a paper with its quality (its rank). The reason is that the price of good i, which is identified as i's rank, becomes the budget of consumer i. This budget is then split between the papers that i cites, so it is exactly i's refereeing power. By contrast, the citation count disregards completely the rank of an article when it determines its refereeing power. In the CC, a citation from any article has the same importance. In the NCC, the weight of a citation from an article is also independent of its rank, but decreases with the number of citations an article makes. In this section we introduce a new element – taxation – that allows to control the extent to which the price and the budget of i are tied to each other.

5.1. Ranking induced by an exchange economy with taxes

We add the following taxation scheme: every consumer is required to pay a fraction α of his income as a tax, and receives back $1/n$ of the total tax revenue as a lump sum transfer.

The purpose of the tax is to allow a separation between the rank of an item as evaluated by others (i.e., the *price* of its specific good) and its reviewing power (the *budget* of its representing consumer). Without the tax, the two coincide. With a 100% tax rate, the reviewing power of all articles coincide, regardless of their ranking, because all have the same budget

[12] The NCC can be helpful in comparing articles coming from different fields, as these often differ in the average length of their citations' lists.

($1/n$ of the tax revenue). In this case the induced ranking system is the Normalized Citation Count, namely, every article has the same budget, and this budget is equally split between the articles it cites. Hence, the money each article (consumer) gets amounts to the sum of all its normalized citations. By fixing a tax rate in-between the two extremes, one could attain control over the extent of the entanglement or disentanglement between the reviewing power and the ranking of an item.

With tax α, agent i's budget comes from two sources. The first is from the redistribution of the tax revenue, which amounts to $\frac{\alpha}{n}$. The second part is the net income received from selling the agent's endowment: $(1-\alpha)p_i$. Thus, the total budget of consumer i is

$$q_i = (1-\alpha)p_i + \alpha/n. \tag{4}$$

Multiplying the vector of budgets by the matrix M yields the agents' expenditures on the different goods in equilibrium. Since the supply of each good is unitary, this equals the vector of equilibrium prices. Thus, in the Cobb–Douglas case with tax rate α, the resulting competitive equilibrium price system p satisfies (compare with Eq. (2))

$$\left(\alpha \cdot \left(\frac{1}{n}, \ldots, \frac{1}{n}\right) + (1-\alpha)p\right) M = p. \tag{5}$$

In order to receive a clearer picture of the effect of taxation, consider the following example that deals with articles of two generations: two old papers, denoted $2a, 2b$, and two young ones, denoted $1a, 1b$. The two articles in each generation cite each other, and one of the younger generation makes an intergenerational citation and cites an article from the old generation.

Example 2. Four items, $V = \{1a, 1b, 2a, 2b\}$, have citations given by

$$E = \{(1a, 1b), (1b, 1a), (2a, 2b), (2b, 2a), (2a, 1a)\}.$$

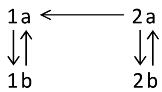

Fig. 2. Two generations.

The rankings provided by the various systems are shown in the following table:

Ranking system	Economy	$p_{1a}, p_{1b}, p_{2a}, p_{2b}$
PageRank	Any exchange economy without tax ($\alpha = 0$) (except for perfect substitutes)	$\frac{1}{2}, \frac{1}{2}, 0, 0$
	CD with tax $\alpha = 0.5$	$\frac{11}{28}, \frac{9}{28}, \frac{5}{28}, \frac{3}{28}$
NCC	CD with tax $\alpha = 1$	$\frac{3}{8}, \frac{2}{8}, \frac{2}{8}, \frac{1}{8}$
CC	NONE	$\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}$

Indeed, the articles of the old generation ($1a, 1b$) are more important, since they receive a citation from the younger generation ($2a, 2b$) and not vice versa. However, giving the articles of the younger generation a rank of 0 makes PageRank too extreme in differentiating between the two generations. Moreover, since the younger generation has 0 rank, its refereeing power is also 0, and thus article $1a$ has the same rank as $1b$ ($1/2$), even though the former enjoys an additional citation.

The CC does grant positive ranking to the articles of the younger generation, and it also differentiates between the two articles of the old generation. However, it does not differentiate between $2a$ and $2b$, even though – while each of them cites the other – $2b$ enjoys only part of the citations of $2a$ and $2a$ enjoys all the citations of $2b$.

The NCC does take this last argument into consideration, and gives $2a$ a rank higher than that of $2b$. But like the CC, article $1b$, which receives a unique link from the most important article $1a$, has the same rank as $2a$, which has a unique link from the least important article $2b$.

The economy with an intermediate tax rate does succeed to grant appropriate ranking to all the articles: the old generation has a higher ranking than that of the young one, and the articles are properly ranked within each generation as well.

While in the above example the tax rate affects only the cardinal ranking, this is not the general case. The following example shows that different tax rates may induce different ordinal rankings:

Example 3. Consider the following variation of Example 1, with Cobb–Douglas preferences and tax α:

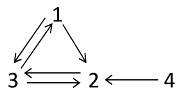

Fig. 3. Node No. 4 is added to Fig. 1.

With no tax ($\alpha = 0$), we have $p_4 = 0$, and thus the prices of the other node remain the same as before: $p_1 = \frac{2}{9}$, $p_2 = \frac{3}{9}$, $p_3 = \frac{4}{9}$. In particular, the price of node 3 is strictly higher than that of node 2. With tax $\alpha = 1$ we obtain the NCC, and the price of node 2 (which receives $1 + \frac{1}{2} + \frac{1}{2}$ normalized citations) is strictly higher than that of node 3 (which receives only $1 + \frac{1}{2}$ normalized citations). Thus, by changing the tax rate, the ordinal ranking of nodes 2 and 3 is reversed.

5.2. An alternative approach: adding a government node

Consider the budget vector in the case of Cobb–Douglas utilities (recall Eq. (4)). It can be calculated using the eigenvector idea. Consider the original n states and the transition probability matrix M. Now we add another state that represents, say, the government. We introduce a new Markov matrix that depends on the tax rate α, denoted $T(\alpha)$, applied now to $n+1$ states: n original ones plus the newly added state.

The probability T_{ij} of moving from i to j, where $i, j \le n$, is $(1-\alpha)m_{ij}$, while $T_{ij} = \alpha$ when $i < j = n+1$. In other words, the probability of moving to $n+1$ is always α and the remaining probability is divided proportionally to the probabilities in M. Finally, the probability of moving from $n+1$ to j ($j \le n$) is always $1/n$. Formally, let

$$T(\alpha) = \begin{pmatrix} & & & & \alpha \\ & & & & . \\ & (1-\alpha)M & & & . \\ & & & & . \\ & & & & \alpha \\ 1/n & . & . & . \;\; 1/n & 0 \end{pmatrix} \tag{6}$$

The idea of adding a state has two advantages. First, for every Markov matrix M, the matrix $T(\alpha)$ is ergodic and therefore there is no need to perturb the probability in M in order to obtain ergodicity. Thus, uniqueness of an invariant distribution of $T(\alpha)$ is automatically guaranteed. The second advantage is that the algorithm that finds the invariant distribution in the same way as in the PageRank, could find the equilibrium prices with tax, if applied to $T(\alpha)$, rather than to M.

The following proposition summarizes the relation between the price vector and that of the budgets. It also states the existence and uniqueness of a competitive equilibrium. While the mathematical derivations have close relations to existing results in the computer science literature (see, e.g., Langville et al., 2003), the focus here is on the economic interpretation in terms of our model.

Proposition 5. *Suppose that the tax rate is $0 < \alpha < 1$. Let G be a graph, M be the matrix defined in Eq. (1) and T the one defined in Eq. (6). Then,*

(i) *There is a unique invariant distribution of $T(\alpha)$, denoted $q = (q_1, \ldots, q_{n+1})$, which corresponds to the budget of each agent in equilibrium;*
(ii) $p = \frac{(1+\alpha)(q_1,\ldots,q_n) - \alpha(\frac{1}{n},\ldots,\frac{1}{n})}{1-\alpha}$ *is an equilibrium price vector of the exchange economy ruled by Cobb–Douglas utilities and tax rate of α; and*
(iii) *There is a unique competitive equilibrium of the exchange economy ruled by Cobb–Douglas utilities and tax rate of α, which is a solution of Eq. (5). Moreover, if $p = (p_1, \ldots, p_n)$ is the normalized equilibrium price vector, then $q = (q_1, \ldots, q_{n+1})$ is an invariant distribution of $T(\alpha)$, where $(q_1, \ldots, q_n) = \frac{(1-\alpha)p + \alpha(\frac{1}{n},\ldots,\frac{1}{n})}{1+\alpha}$ and $q_{n+1} = \frac{\alpha}{1+\alpha}$.*

Remark 3. The Web graph is almost certainly reducible, which makes its PageRank vector not unique. Brin and Page (1998) solved the problem by introducing a "damping factor". Langville et al. (2003) showed that the technique of damping factor is mathematically equivalent to introducing a new state, called "teleportation state", which is formally equivalent to the government node introduced above. While the purpose of these techniques is to perturb slightly the original Markov matrix in order to make it irreducible, our main purpose of our taxation approach is to obtain ranking systems that are "convex combinations" of the NCC and PageRank.

6. Concluding comments

6.1. Non-connected graphs

Throughout the paper we have made the assumption that the graph of links is connected. When this is not the case, the ranking we obtain is not unique. Each connected component of the graph is in fact an isolated economy, an "island". Within each island the relative prices are indeed determined by the equilibrium; however, there are no restrictions on the intra-island relative prices.

More formally, assume that the graph G is the union of m connected components $G_1, \dots, G_m$, and let $p(i)$ be a vector of equilibrium prices of the u-exchange economy that corresponds to the graph G_i alone. Then, for every non-negative m-vector $\alpha \neq 0$, $p = (\alpha_1 p(1), \dots, \alpha_m p(m))$ is an equilibrium price system of the u-exchange economy that corresponds to the graph G.

This should be of no surprise: if two academic fields, for example, do not cite each other (neither directly or indirectly), then the information embodied in the graph of citations has nothing to say about the relative importance of the two fields. Furthermore, similarly in the theory of exchange rates, if there are no trade relation between the different countries, then any exchange-rate vector is an equilibrium of the world economy.

6.2. Anonymous preference relations over an arbitrary number of goods

In the general formulation of the exchange economy we proposed the restriction that the utility function for a given number of goods k, u^k, is a symmetric utility function. The symmetry of the utility functions guaranties that all goods are treated in the same fashion, so that an item's name does not affect its ranking. We did not suggest, however, any restrictions on the relationship between the different u^k's, i.e., the utility functions from consuming different number of goods may be very different. As a result, changes in the graph may lead to undesired outcomes. For example, the share of i's budget spent on j may increase if we add an outgoing link from i to another node because adding the link changes i's utility function.

The concrete examples on which we have focused, however, employed CES utility functions. These induce similar consumption patterns across all consumers, regardless of the number of goods they consume. Defining a restriction on $u = (u^1, u^2, \dots, u^n)$ that would generate a natural relationship between different u^k's is not straightforward. One condition that would naturally link between different u^k's is separability:

Let $\succeq$ be a reflexive complete order over $\mathbb{R}^n_+$. We say that $\succeq$ is *separable*[13] (Debreu, 1960) if for every $k = 1, \dots, n-1$, $x^{(k)}, y^{(k)} \in \mathbb{R}^k_+$, $x^{(n-k)}, y^{(n-k)} \in \mathbb{R}^k_+$ the following holds

$$(x^{(k)}, x^{(n-k)}) \succeq (y^{(k)}, x^{(n-k)}) \text{ if and only if } (x^{(k)}, y^{(n-k)}) \succeq (y^{(k)}, y^{(n-k)}).$$

Most commonly used utility functions, such as minimum, Cobb–Douglas and CES induce separable preference orders, and if symmetric, induce symmetric separable preference order.

6.3. Economies and rankings

The question remains open as to what ranking schemes are economy-based. In other words, what are the conditions that characterize the ranking schemes generated by an exchange economy. This question can be refined further by restricting the generating economies to those governed by a smaller set of utility functions, such as separable or symmetric CES utility functions.

Another important question concerns the relationship between the economic properties of the utility function, such as the CES parameter β (which captures the extent to which goods are substitutes or complements of each other) and the ranking scheme they generate.

The answers to these questions lie beyond the scope of the current paper and are left for a future project.

6.4. Employing other economic concepts

We dealt so far with ranking of individual nodes. A direction for further application of these ideas is to ranking cluster of nodes. This gives rise to using ideas from the theory of (international) trade between countries (that stand for clusters).

Appendix A

Proof of Proposition 1. Let Δ be the set of n-dimensional non-negative vectors whose coordinates sum up to 1 and let $p = (p_1, \dots, p_n) \in \Delta$. For every i, denote by $C_i(p)$ the set of all bundles that maximize i's utility subject to his budget, p_i, with the restriction that quantities may not exceed n (which might happen when the price of a good is zero). Since

[13] We found no paper that characterizes utility functions representing separable preference orders. We did find however, an extensive study of preference orders representable by separably additive utility functions. The latter, in particular, are separable.

each utility function is quasi-concave, each $C_i(p)$ is a convex set. Denote $C(p) = \sum C_i(p)$ the sum of all the $C_i(p)$'s. $C(p)$ is convex and compact. Set $E(p) = C(p) - 1_V$, where $1_V = (1, \dots, 1)$ is the indicator of N. Note that in our economy 1_V is the aggregate supply. Thus, $E(p)$ is the set of aggregate excess demand. $E(p)$ is a translation of $C(p)$ and as such is convex and compact. Due to the continuity of all utility functions, the correspondence $p \hookrightarrow E(p)$ is upper semi-continuous. Furthermore, since every i does not spend more than his budget, for every $p \in \Delta$ and $e \in E(p)$, $\langle p, e\rangle \le 0$.

Denote by E the set of n-dimensional vectors whose coordinates do not exceed n^2. For every $e \in E$, let $P(e)$ be the set of all the vectors $p \in \Delta$ that maximize $\langle p, e\rangle$. That is, $P(e) = \text{argmax}\langle \cdot, e\rangle$. $P(e)$ is convex. Moreover, the correspondence $e \hookrightarrow P(e)$ is upper semi-continuous. Thus, the correspondence $(p, e) \hookrightarrow P(e) \times E(p)$ is semi-continuous and convex-values from $\Delta \times D$ to itself. By Kakutani theorem this correspondence has a fixed point, say (p^*, e^*), where $e^* = \sum_i x_i^* - 1_V$ and $x_i^* \in C_i(p^*)$ for every $i \in V$. We show that $((x_i^*)_i; p^*)$ (with a slight change) is quasi-equilibrium.

If $p_k^* > 0$, since $\langle p^*, e^*\rangle \le 0$ and p^* maximizes $\langle p^*, e^*\rangle$, therefore $e_k^* = 0$. Thus, if $e_k^* > 0$, then $p_k^* = 0$. Steps II–III in the proof of Proposition 2 show that if $p_i^* > 0$ then for every $k \in O(i)$, $p_k^* > 0$. Therefore, $p_k^* = 0$ implies that $p_i^* = 0$ for every $i \in I(k)$. Thus, if there is an excess demand for good k, it must come from consumers (in $I(k)$) whose budget is zero. We modify $(x_i^*)_i$ as follow. Define $y_i^* = x_i^*$ if $p_i^* > 0$ and $y_i^* = 0$ if $p_i^* = 0$. Now, if $p_k^* = 0$, then $\sum_{i \in I(k)} y_i^k = 0$, namely the aggregate demand for k is 0. It implies that there is no excess demand also for goods whose prices are 0. Therefore, $((y_i^*)_i; p^*)$ is quasi-equilibrium. □

Proof of Lemma 1. Suppose that there is a sink, say i. Since the graph is connected, for every node j, $i \in V(j)$. Thus, $V^* = \{i\}$. Suppose now that there is no sink and assume, to the contrary of Lemma 1, that $V^* = \emptyset$. Let V' be a minimal set of nodes in the sense that $\cap_{i \in V'} V(i) = \emptyset$, while for every $i_0 \in V'$, $\cap_{j \in V' \setminus \{i_0\}} V(j) \neq \emptyset$. Fix $i_0 \in V'$. Since the graph is connected, i_0 is connected to every other node in V'. Case I: There are only paths from i_0 to all other nodes in V' (and no path in the inverse direction). In this case, $V(j) \subseteq V(i_0)$ for every $j \in V' \setminus \{i_0\}$. Thus, $\cap_{j \in V'} V(j) = \cap_{j \in V' \setminus \{i_0\}} V(j) \neq \emptyset$. This is a contradiction. Case II: There is a path from at least one node in V', say j_0, to i_0. In this case, $V(i_0) \subseteq V(j_0)$. By the choice of V', $\cap_{j \in V' \setminus \{j_0\}} V(j) \neq \emptyset$, but then $\cap_{j \in V'} V(j) = V(j_0) \cap (\cap_{j \in V' \setminus \{j_0\}} V(j)) \supseteq V(i_0) \cap (\cap_{j \in V' \setminus \{j_0\}} V(j)) = \cap_{j \in V' \setminus \{j_0\}} V(j)$, implying that $\cap_{j \in V'} V(j) \neq \emptyset$, which is a contradiction. □

Proof of Proposition 2. Let $((x_1, x_2, \dots, x_n); (p_1, \dots, p_n))$ be a quasi-equilibrium.

Step I: For steps I–III, fix i such that $p_i > 0$. We show first that if $p_j = 0$ and $j \in O(i)$, then $u^{c(i)}(x_i^{-j}, d)$ is constant as a function of d. Otherwise, by assumption, $u^{c(i)}(x_i^{-j}, d)$ is strictly increasing in d. However, this would contradict i's optimization, because $u^{c(i)}(x_i^{-j}, x_i^j + \varepsilon) > u^{c(i)}(x_i^{-j}, x_i^j)$ while the market value of both $(x_i^{-j}, x_{ij} + \varepsilon)$ and (x_i^{-j}, x_{ij}) is the same. Thus i could increase the consumption of j by a small $\varepsilon > 0$ and thereby his utility, without spending more money.

Step II: We show here that $u^{c(i)}(x_i) > 0$. It is sufficient to show that in any neighborhood of $\vec{0}$, there a bundle from which i derives positive utility. The reason is that in this case i could purchase a bundle that is available subject to his positive budget, p_i, that yields a positive utility, contradicting $u^{c(i)}(x_i) = 0$.

Fix an $\varepsilon > 0$. Since $u^{c(i)}$ is not degenerate, there is a bundle $x \in \mathbb{R}_+^{c(i)}$ such that $u^{c(i)}(x) > 0$. If $u^{c(i)}(x^{-j}, \cdot)$ is constant we replace x^j with 0. Since $u^{c(i)}$ is conditionally increasing, $u^{c(i)}(x^{-j}, 0)$ is also positive. Otherwise, $u^{c(i)}(x^{-j}, \cdot)$ is not constant. In this case we replace x^j with $\varepsilon > 0$. Since $u^{c(i)}$ is conditionally increasing, $u^{c(i)}(x^{-j}, \varepsilon) > 0$. We proceed inductively over all $j \in O(i)$. Eventually we obtain a bundle whose utility is positive. This bundle consists of 0 quantities of the goods that do not affect the utility of i and of ε quantities of those goods that do affect his utility. We therefore obtain a bundle in the ε-neighborhood of $\vec{0}$ whose utility is positive, as desired.

Step III: From the two previous steps we conclude that there is at least one $\ell \in O(i)$ such that $p_\ell > 0$. We show that for every $j \in O(i)$, $p_j > 0$. Otherwise, there is some $j \in O(i)$ with $p_j = 0$. Suppose there is a bundle x_i such that $u^{c(i)}(x_i) > 0$. As shown in Step I, $u^{c(i)}(x_i^{-j}, \cdot)$ should be a constant. Thus, $u^{c(i)}(x_i^{-j}, x_i^j) = u^{c(i)}(x_i^{-j}, 0)$, which contradicts the Inada condition. Therefore, $\forall j \in O(i)$, $p_j > 0$.

Step IV: For any $k \notin V^*$, $p_k = 0$. If to the contrary, there is $k \notin V^*$ with $p_k > 0$. By the definition of V^*, there is a path from k to any $i \in V^*$. Fix a path of this kind. By the previous step all the prices of all the nodes along this path are positive. In particular, there are nodes, say $k' \notin V^*$ and $i \in V^*$ on this path, such that $k' \in I(i)$. Since, by assumption, the utility function of k satisfies Inada condition, his optimal bundle must include a positive quantity form i. In other words, k spends money on good i. Thus, there is money going from $V \setminus V^*$ to V^*. But no money goes backward – from V^* to $V \setminus V^*$, simply because there is no path from V^* out. Recall that in quasi-equilibrium each individual has a balanced budget: his spending equals his budget. Therefore it applies also to groups of consumers. Here, however, the spending of $V \setminus V^*$ is greater than its budget, contradicting the fact that p is a quasi-equilibrium price system.

Step V: For every recurrent i, $p_i > 0$. Indeed, by the previous step there must be at least one $i \in V^*$ such that $p_i > 0$, otherwise all the prices would be zero. We use Step III inductively to conclude that for every $j \in V(i)$, $p_j > 0$. But, by definition $V^* = V(i)$, which completes the proof. □

Proof of Proposition 3. When p is normalized, it becomes a distribution over V. The budget of consumer i, who is the original owner of one unit of good i, is p_i. As the utility function is Cobb–Douglas, when his budget is positive, he divides it equally between the goods in $c(i)$, i.e., he purchases quantity $\frac{p_i \pi_{ij}}{c(i) p_j}$ of commodity j (note that for $j \notin O(i)$, $\pi_{ij} = 0$). His expenditure on good j is then $\frac{p_i \pi_{ij}}{c(i) p_j} p_j = \frac{p_i \pi_{ij}}{c(i)} = p_i m_{ij}$.

The total budget of consumer j is the sum of all the expenditures, made by all other agents, on good j. That is, for every j with $p_j > 0$,

$$p_j = \sum_i p_i m_{ij}. \tag{7}$$

However, Eq. (7) holds also when $p_j = 0$. Indeed, suppose that $p_j = 0$. Since Cobb–Douglas is not degenerate and conditionally increasing, we may use Proposition 2, which implies that j is not recurrent. If $\pi_{ij} = 1$, then i is also not recurrent and, again by Proposition 2, $p_i = 0$. Therefore, $p_i m_{ij} = 0$ for every i.

Finally, notice that Eq. (7) precisely means that p satisfies Eq. (2), and therefore coincides with PageRank.

Assume now that $p = (p_1, \ldots, p_n)$ and $p' = (p'_1, \ldots, p'_n)$ are two competitive equilibrium price systems. Without loss of generality we may assume that p is coordinate-wise greater than p' and that for one j, $0 < p_j = p'_j$. Due to the connectedness of the graph, we can assume that there is ℓ such that $\ell \in I(j)$ and $p_\ell > p'_\ell$.

We now reduce all p_i, $i \in V^* \setminus \{j\}$ to the level p'_i without touching p_k, $k \notin V^* \setminus \{j\}$. Two effects can be perceived from the perspective of each consumer i ($i \neq j$). First, the budget has been reduced from p_i, to p'_i and second, all prices are not higher compared to what they were before the reduction.

Consider $i \in I(j)$. In the case of Cobb–Douglas utility functions, when the prices of the goods i is interested in have been reduced and when his budget has been cut down, his demand for good j cannot grow up. Moreover, since the budget of ℓ is strictly lower than what it was prior to the reduction, his demand for good j is strictly smaller than what is was prior to the change. Consequently, following the change (i.e., after prices turned into p'), total demand for good j becomes smaller than 1 – the amount supplied. In other words, there is an excess supply of good j, which contradicts the assumption that p' is a competitive equilibrium price system. We conclude that there is a unique quasi-equilibrium price system. □

Proof of Proposition 4. Consider the graph with 4 nodes whose edges are: $(1, 2), (1, 3), (2, 4), (3, 4), (4, 1)$. This is an irreducible graph. The citation count of 1 and 4 are 1 and 2, resp. However, in any exchange economy, p_1 should be at least p_4, because 4 spends all its money on 1.

As for NCC, the indices of both 2 and 3 are 1/2 and that of 4 is 2, while in any exchange economy, $p_j \leq \sum_{i \in I(j)} p_i$. This implies that p_4 should not exceed $p_2 + p_3$. □

Proof of Proposition 5. Denote $e = (\frac{1}{n}, \ldots, \frac{1}{n})$.

(i) In the Markov matrix $T(\alpha)$ every state has a positive probability (i.e., α) to move to state $n+1$. Moreover, state $n+1$ has a positive probability (i.e., $\frac{1}{n}$) to move to any other state. Thus, the matrix $T(\alpha)$ is ergodic and therefore has a unique invariant distribution, $q = (q_1, \ldots, q_{n+1})$:

$$qT(\alpha) = q. \tag{8}$$

The $n+1$ coordinate of q satisfies, $q_{n+1} = (1 - q_{n+1})\alpha$. Thus, $q_{n+1} = \frac{\alpha}{1+\alpha}$. It implies that $q_1 + \ldots + q_n = \frac{1}{1+\alpha}$, and therefore $q' = (1+\alpha)(q_1, \ldots, q_n)$ is a probability distribution. Eq. (8) now implies,

$$(1-\alpha)q'M + \alpha e = q'. \tag{9}$$

Note that for every $i = 1, \ldots, n$, $q_i \geq q_{n+1}/n = \frac{\alpha}{n(1+\alpha)}$, implying that $q'_i = (1+\alpha)q_i \geq \alpha/n$. Define, $p = \frac{q' - \alpha e}{1-\alpha}$. p is a probability distribution and $q' = (1-\alpha)p + \alpha e$. Thus, q'_i ($i = 1, \ldots, n$) satisfies Eq. (9), meaning that q' is the vector of budgets, one for each agent. We obtain that the first n coordinates of q, $(q_1, \ldots, q_n)$, correspond (recall, $q' = (1+\alpha)(q_1, \ldots, q_n)$) to the budgets of each agent in equilibrium, as desired.

(ii) From Eq. (9) we obtain, $(1-\alpha)((1-\alpha)p + \alpha e))M + \alpha e = (1-\alpha)p + \alpha e$, which is equivalent to Eq. (5). Thus, p is a quasi-equilibrium price system of the exchange economy governed by Cobb–Douglas utilities and tax rate of α. However, since for every agent i, the budget q'_i is positive, p is actually an equilibrium price system.

(iii) Let $p = (p_1, \ldots, p_n)$ be a normalized solution of Eq. (5). Define, $q = (q_1, \ldots, q_{n+1})$ as follows: $(q_1, \ldots, q_n) = \frac{(1-\alpha)p + \alpha e}{1+\alpha}$ and $q_{n+1} = \frac{\alpha}{1+\alpha}$. Similar calculation to the previous one shows that q satisfies Eq. (8). Since $T(\alpha)$ has a unique invariant distribution, there must be a unique solution to Eq. (5). This implies that there is a unique equilibrium price system of the exchange economy governed by Cobb–Douglas utilities and tax rate of α. □

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