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Reward schemes [☆]

David Lagziel [a,*], Ehud Lehrer [b,c]

[a] Ben-Gurion University of the Negev, Beer-Sheva 8410501, Israel
[b] Tel Aviv University, Tel Aviv 6997801, Israel
[c] INSEAD, Bd. de Constance, 77305 Fontainebleau Cedex, France

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A B S T R A C T

An investor has some funds invested through portfolio managers. By the end of the year, she reallocates the funds among these managers according to the managers' performance. While the investor tries to maximize her subjective utility (that depends on the total expected earnings), each portfolio manager tries to maximize the overall amount of funds bestowed in his hands to manage. A *reward scheme* is a rule that determines how funds should be allocated among the managers based on their performance. A reward scheme is *optimal* if it induces the (self-interested) managers to act in accordance with the interests of the investor. We show that an optimal reward scheme exists under quite general conditions.

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1. Introduction

Consider an investor who has some funds already invested through several portfolio managers.[1] Due to several restrictions, the investor can only reallocate her funds among the managers according to their performance. While the goal of the investor is to maximize her subjective utility that depends on the total earnings, each portfolio manager tries to maximize his payoff, that depends on the overall amount of funds bestowed in his hands to manage.[2] The rule by which the investor reallocates her funds determines the environment in which the portfolio managers operate: it determines their incentives

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* Corresponding author.
E-mail addresses: davidlag@bgu.ac.il (D. Lagziel), lehrer@post.tau.ac.il (E. Lehrer).

[1] We sometimes refer to the investor as decision maker (DM), and to managers as agents or players.

[2] Portfolio managers usually get asset-under-managements fees, making fund flows a first-order incentive. The relation between performance, fees, and fund flows was examined in many studies, such as Ippolito (1989), Chevalier and Ellison (1997), and Sirri and Tufano (1998).

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and ultimately their modus operandi. This rule is referred to as a *reward scheme*. Reward schemes are supposed to guarantee that financial managers produce the optimal possible investments for their investors.

The core of the problem we address lies in the discrepancy between the motivations of the economic entities involved.[3] While the managers wish to maximize the total expected funds they manage, the DM wishes to maximize her expected utility based on the returns of her investment. Typically, these two motivations do not agree. By and large, the competition between managers along with the standard asset-under-management fees pushes them to take riskier actions. To make things even worse, the DM is restricted to the use of fund flows and cannot fully monitor the precise actions of the managers. She can typically observe only the quarterly, or annual, earnings reports. As a result, the managers may abuse this situation to increase their own expected payoffs at the expense of the DM's welfare.

Our objective in this paper is twofold: to establish a formal model for the analysis of this problem, and to introduce constructive methods that will incentivize the portfolio managers to act in accordance with the DM's goals.

In our model and contrarily to the individual contracts previously studied,[4] the share of the funds a specific manager gets to manage (and therefore, his payoff) depends not only on his own past performance, but on other managers' performance as well. The reward scheme introduced by the DM will actually induce a competition between the managers, or an *investment game*, as we call it. A reward scheme is said to be *optimal* if in every equilibrium (of the investment game) all managers act according to the best interests of the DM.

We divide our results into two categories: positive and negative ones. Our first positive results show that for every market, i.e., for every set of possible actions of the portfolio managers, an expected-utility maximizer DM can find an optimal reward scheme (Theorems 1, 2, and 5). This means that by properly designing the reward scheme and solely reallocating funds among managers, the DM can have the managers act in any equilibrium so as to maximize the DM's expected utility. The proof we provide is constructive and holds for a general number of managers and actions. More specifically, we present optimal reward schemes that are linear in the sum of differences of the managers' utility-based returns. In addition, we also prove (Theorem 3) that linearity is a necessary condition for optimality.

On the negative side, we tackle the problem of a *universal* reward scheme that could cater to any set of possible actions. In other words, we ask whether the DM can design, in advance, a reward scheme that remains optimal even when the market changes frequently and the set of actions changes considerably. It turns out that here things are less optimistic. In Theorem 4 and Corollary 1, we show that the only scheme that enables an optimal result is a constant scheme, independent of the managers' returns, such that no fund transfers are made. In some respects, we show that the contracts making the managers 'shareholders' of the fund without any fund flows, is the only scheme that works in every scenario. Yet, these contracts can also produce suboptimal equilibria. Another negative result, given in Proposition 1, shows that an investment game might not even have an equilibrium.

1.1. Contribution and related literature

This paper differs from the related literature in several respects. In most previous works,[5] the DM faces managers of various types, various abilities, and costly efforts. While the DM cannot distinguish between managers of different types, their types do affect the DM's utility. For instance, different managers vary in their productivity rates, thus affecting their investors' profit. A leading question in the literature is whether the DM can design a contract that efficiently rewards skilled managers, and screens out unskilled ones. In our setting, in contrast, all managers are potentially of the same type. They are all experts, all exposed to the same data and, most importantly, all have the same set of possible actions. These assumptions distinguish our work from previous studies. Our focus is neither on the screening process, nor on effectively exerting effort from managers. We study the ways by which an investor, who can *only reallocate funds among managers*, can still produce optimal results in an homogeneous moral-hazard set-up.

Our results apply to cases where the investor (either an individual, a syndicate, or a firm) is limited to the use of fund flows to motivate financial managers. For example, if the investor is a fund family,[6] then our reward scheme notion produces an allocation of investment resources among the different funds. Another example relates to mandatory pension plans. In such cases (commonly exercised in Chile and Israel, among others), employees must contribute to a pension plan and cannot manage their funds directly. Thus, they are limited to the transfer of funds between different firms.[7]

Taking a broad perspective of the subject, our work lies between the economic literature and the financial one. On one hand, we are using basic game theory and mechanism design approach to tackle the problem of an optimal incentive scheme in a general principle–agent problem. On the other hand, we apply our results mainly to the problem of delegated portfolio

[3] A clear evidence for this tension could be found in the words of the 13th Federal Reserve chairman, Alan Greenspan: "I made a mistake in presuming that the self-interests of organizations, specifically banks and others, were such as that they were best capable of protecting their own shareholders and their equity in the firms", New York Times, "Greenspan 'shocked' that free markets are flawed", October 23, 2008.

[4] See, among many others, Barry and Starks (1984), Admati and Pfleiderer (1997), Dasgupta and Prat (2006), Dasgupta and Prat (2008), Foster and Young (2010), Chassang (2013), Carroll (2015).

[5] See, e.g., Sharpe (1981), Bhattacharya and Pfleiderer (1985), Stoughton (1993), Berk and Green (2004), Dybvig et al. (2010), and Malamud and Petrov (2014).

[6] A group of mutual funds marketed under a single fund company.

[7] A broader discussion on this example is given in Subsection 3.1.

management in financial markets. Both fields are well-studied, and therefore we will not be able to relate specifically to all previous works (see Stracca (2006) for a comprehensive survey of the field).

The model we use is quite robust with respect to the managers' available actions. We restrict ourselves to neither specific distributions, nor correlations or dependence between the available portfolios.[8] This assumption is consistent with recent work of Chassang (2013) and Carroll (2015). Both studies also considered a similar general environment, but were mainly concerned with the exertion of effort by a single manager. Though Carroll (2015) uses a worst-case criterion, his main conclusion is similar to ours suggesting that the class of linear contracts is optimal and robust, compared to other classes. Nevertheless, one should not confuse this linearity and our proposed linear reward scheme. In Carroll (2015) the conclusion is that a fixed share of the return is optimal, whereas we suggest a contract where the linearity is taken with comparison to other agents.

Chassang (2013) and Carroll (2015) also limit their analysis to risk-neutral investor and managers. Though this limitation is similar to the restriction we impose in Section 4, we later generalize our results to any expected utility maximizer investor in Section 5. Another significant difference between the set-up of Chassang (2013) and ours is the use of dynamics. The result of Chassang (2013) is asymptotic, allowing a certain amount of learning, whereas we deal with a single period problem.

The need to use multiple managers also differs our work from previous studies. In the financial literature, Sharpe (1981) was one of the first to deal with the issue of using multiple asset managers, and the need to provide good incentives (or otherwise coordinate between different managers). This work was later followed by Barry and Starks (1984) and more recently by Van Binsbergen et al. (2008). In general, these papers focus on the impact of decentralized investment management when using multiple managers. They require much more structure over the available assets in the market, and for each manager, separately. In contrast, our use of more than one manager originates from the fund-flows limitation. Specifically, we use multiple managers to align their incentives, as the investor cannot control the managers' contract, but can only transfer funds.

The linearity of our proposed solution along with the use of multiple managers leads to a certain amount of funds still invested through under-preforming managers. Though this may seem counter-intuitive at first, there are theoretical and empirical studies suggesting such behavior is rational. Recent work by Cornell et al. (2016) shows higher excess return to investors who chose funds with poor recent performance, than the excess return to investors who chose funds with superior recent performance. This notion is consistent with the theoretical work of Berk and Green (2004). Berk and Green constructed a perfect-competition rational model where marginal trading costs increase as a function of the managed funds. Specifically, the price impact of large funds becomes significant compared to small funds, thus decreasing returns. Therefore, our investor's commitment, via the optimal reward scheme, to maintain some funds with poor recent-performance managers is not unprecedented.

1.2. Outline of the paper

The paper is organized as follows. Section 2 presents a simple 2-manager reward-scheme problem that illustrates the drawbacks of results-based incentives in competitive non-deterministic markets. In Section 3 we present the model along with the main assumptions. Section 4 includes the main results given a risk-neutral investor. In Subsections 4.1 and 4.2 we present the optimal schemes for bounded and unbounded assets, respectively. In Subsection 4.3 we prove the robustness of the linear schemes, and in Subsection 4.4 we show that, unless independent of the outcomes, every reward scheme might fail as the market evolves. The positive results (existence of optimal schemes) are extended to a general expected-utility maximizer investor in Section 5. Concluding remarks and additional comments are given in Section 6.

2. A motivating example: a 2-manager reward-scheme problem

An investor has some funds invested through two portfolio managers: Manager 1 and Manager 2. The goal of the DM is to maximize the expected returns of her investments. By the end of the period, she collects and uses the profits for her own needs, and reallocates the funds according to some predetermined rule that depends on the managers' yearly earnings.

Both managers operate under a fixed contract that gives them a share of the profits and a share of the managed funds.[9] Hence, the DM's reallocation rule (i.e., reward scheme) is her only ability to incentivize the managers. However, as she is not aware of the possible bonds in the market, she chooses to allocate the entire available amount to the manager who presents the highest earnings by the end of the year (i.e., a winner-takes-all reward scheme). In case both managers present the same earnings, the funds are equally divided between the two managers.

Suppose that the managers can invest either in Bond X_1, which yields 5% per year with probability (w.p.) 1, or in Bond X_2, which yields 5.1% per year w.p. 0.6 and 0% per year w.p. 0.4%. The goal of the managers is to maximize their expected earnings and to maximize the overall amount of funds they manage, according to the utility functions given below.

[8] Namely, normal distributions and Brownian motion as in Bhattacharya and Pfleiderer (1985), Holmstrom and Milgrom (1987), Stoughton (1993), Admati and Pfleiderer (1997), and Berk and Green (2004).

[9] This contract bares some resemblance to the commonly-used 2/20 contracts, where active portfolio managers get 2% of the funds and 20% of the excess profits relative to some exogenous benchmark.

In this example we prove that, although investing in Bond X_2 is substantially worse than investing in Bond X_1 (in terms of both expected return and risk), the unique equilibrium in the induced game is when both managers invest in X_2.

Formally, let $A = \{X_1, X_2\}$ be the set of (pure) actions available to the managers. The distributions of X_1 and X_2 are

$$X_1 = 1.05 \text{ per year w.p. } 1, \quad X_2 = \begin{cases} 1.051, & \text{per year w.p. } \frac{3}{5}, \\ 1.0, & \text{per year w.p. } \frac{2}{5}. \end{cases}$$

These distributions are common knowledge between the managers. The managers can also invest in a combination of X_1 and X_2. That is, Manager i may decide to invest, say, a portion $\alpha_i \in [0,1]$ of the money he manages in X_1 and $1-\alpha_i$ in X_2. To such a strategy we refer as a diversified strategy.

The utility functions of both managers depend on a parameter $\lambda \in [0,1]$. Let $\sigma_i = \alpha_i X_1 + (1-\alpha_i)X_2$ be the strategy of Manager i. The utility U_1 of Manager 1 is defined as follows[10]:

$$\begin{aligned} U_1(\sigma_1,\sigma_2) &= \lambda \mathbf{E}(\sigma_1) + (1-\lambda)\mathbf{E}\left(\mathbf{1}_{\{\sigma_1>\sigma_2\}} + \frac{\mathbf{1}_{\{\sigma_1=\sigma_2\}}}{2}\right) = \\ &= \lambda \mathbf{E}(\alpha_1 X_1 + (1-\alpha_1)X_2) + \\ &+ (1-\lambda)\left(\mathbf{Pr}\left((\alpha_1-\alpha_2)[X_1-X_2]>0\right) + \frac{\mathbf{Pr}\left((\alpha_1-\alpha_2)[X_1-X_2]=0\right)}{2}\right). \end{aligned} \tag{1}$$

In words, the utility function of Manager 1 is a weighted average (λ vs. $1-\lambda$) of its earnings (e.g., $\mathbf{E}(\alpha_1 X_1 + (1-\alpha_1)X_2)$) and the probability that the additional funds are allocated to Manager 1. Manager 2's utility function is defined in a similar fashion.

The following lemma shows that while the managers maximize their expected utilities (i.e., their share of the managed funds and expected returns), the result is unfavorable for the DM.

Lemma 1. *For every* $0 \le \lambda < \frac{1}{1.194} \approx 0.83$*, the unique equilibrium is when both managers choose to invest only in* Bond X_2.

The proof is given in the Appendix.

While the expected earnings per year of Bond X_2 is 3.006%, that of Bond X_1 is 5%. It turns out that the chosen reward scheme, the winner-takes-all mechanism, is adversarial to the interests of the DM: the unique equilibrium is (X_2, X_2), which is the worst possible result from the DM's perspective. Moreover, even from the managers' perspective the equilibrium (X_2, X_2) is Pareto-dominated by any other profile (σ_1, σ_2), such that $\sigma_1 = \sigma_2$.

This example illustrates the problem in case there are two managers, two pure actions, a risk-neutral investor, and a specific winner-takes-all reward scheme. The model presented in the following section concerns a more general case. However, in order to maintain simplicity we assume that managers care only about the volume of their allocations (related to the second summand of the RHS of Eq. (1)) and not about the actual performance (the LHS of Eq. (1)) of the fund they manage. That is, $\lambda = 0$.

3. The model

There are k portfolio managers (also referred to as *players*) in the market. Let $A = \{X_1, \ldots, X_n\}$ be a set of random variables (assets) with a finite expectation. This is the set of possible investment options available to every player. The yield of the i-th investment[11] is represented by the random variable X_i. The elements composing A will be referred to later as assets or *pure strategies*.

A *diversified strategy* σ_i of player i is a mixture of random variables in A. Formally,[12] $\sigma_i = \sum_{j=1}^n \sigma_i^j X_j$, where $\sigma_i^j \ge 0$ and $\sum_{j=1}^n \sigma_i^j = 1$. The set of diversified strategies is denoted by Q. For instance, a player taking a pure action X_j invests all his managed funds in the j-th asset. However, in case he chooses to use the diversified strategy $\sigma_i = \sum_{j=1}^n \sigma_i^j X_j$, he invests a share σ_i^j of his managed funds in the j-th asset.[13]

Initially, the investor, or the decision maker (DM), has some funds invested through the k players. For simplicity and without loss of generality, we assume that the funds are equally divided between the players and each player gets a normalized initial amount of 1. Since the DM is an expected utility maximizer with a general utility function $U : \mathbb{R} \to \mathbb{R}$ and an initial amount of k (so that each of the k portfolio managers gets 1), her main goal is to maximize $\mathbf{E}[U(kq)]$, where $q \in Q$

[10] In what follows $\mathbf{E}$ stands for the expectation.

[11] Any investment in a financial asset, such as a bond, a stock, or an option, as well as any other sort of investment, such as in real estate or in a commodity.

[12] We sometimes denote a diversified strategy σ_i as a distribution $(\sigma_i^1, \ldots, \sigma_i^n)$ over the set of pure actions A. Nevertheless, the formal definition states that σ_i is the new random variable $\sum_{j=1}^n \sigma_i^j X_j$, which is a convex combination of pure strategies given the weights $(\sigma_i^1, \ldots, \sigma_i^n)$.

[13] In game theory, a diversified strategy is commonly perceived as a pure strategy.

is some diversified action. Nevertheless, the DM cannot invest directly in the available assets and can only reallocate her funds among the players based on their performance.[14]

For this purpose we introduce the notion of a reward scheme. Let r_i be the measurement of player i's performance. That is, r_i denotes the realization of player i's diversified strategy σ_i. Note that the DM is not familiar with the assets included in A. She does not know their distributions, nor their expected payoffs, and can only observe the performances $(r_1, \ldots, r_k)$ of the players at the end of a single time period.

Definition 1. A *reward scheme* is a function $f : \mathbb{R}^k \to [0,1]^k$ such that for every $r \in \mathbb{R}^k$,

$$\sum_{i=1}^{k} f_i(r) = 1. \tag{2}$$

In words, given a vector $(r_1, \ldots, r_k)$ of the players' performances, a proportion $f_i(r_1, \ldots, r_k)$ of the available funds is to be allocated to player i. Since the condition given by Eq. (2) is a significant part of our model, we provide it with a broad discussion and relevant concrete examples in Section 3.1.

The DM publicly commits to a reward scheme f. This, in turn, defines a k-player game, called an *investment game* and denoted G_f, as follows. Player i chooses a strategy (i.e., a diversified portfolio) $\sigma_i \in Q$. Player i's payoff depends not only on its own strategy, but on all other players' strategies as well. When $\sigma = (\sigma_1, \ldots, \sigma_k) \in Q^k$ is the profile of strategies used by the players, the expected payoff of player i is

$$\mathbf{E}\left[f_i(\sigma)\right].$$

In words, the payoff of player i is the expected proportion of the funds he is going to manage. This game is symmetric in all respects: all the players are homogeneous in their utility function and have the same set of strategies.

Definition 2. A profile of strategies $\sigma \in Q^k$ is a *dominant-strategy equilibrium* in the investment game G_f if

$$\mathbf{E}\left[f_i\left(\sigma_i, \sigma'_{-i}\right)\right] \geq \mathbf{E}\left[f_i\left(\sigma'\right)\right],$$

for every Player i and for every strategy profile $\sigma' \in Q^k$.

In the situation under consideration, the DM is actually a mechanism designer. She announces a reward scheme and thereby defines an investment game. The portfolio managers are the players in this game. They wish to maximize their expected payoffs. The goal of the DM, on the other hand, is to design a game G_f in a way that in any dominant-strategy equilibrium $\sigma = (\sigma_1, ..., \sigma_k)$, the profile σ maximizes the expected return of the DM given the utility function U.

Definition 3. A reward scheme f is *optimal*, if

(i) a dominant-strategy equilibrium exists in G_f; and
(ii) in every dominant-strategy equilibrium $\sigma = (\sigma_1, ..., \sigma_k)$,

$$\mathbf{E}\left[U\left(\sum_{i=1}^{k}\sigma_i\right)\right] = \max_{q \in Q} \mathbf{E}\left[U(kq)\right]. \tag{3}$$

In words, f is *optimal* if in every dominant-strategy equilibrium σ in G_f, the combined portfolio $\sum_{i=1}^{k}\sigma_i$ produces the maximal expected utility for the DM.[15] A profile σ that sustains Eq. (3) is called *optimal*.

The existence condition in Definition 3 is not trivial. In Subsection 4.5 we show that one can produce an investment game with a continuous reward scheme f, such that there exists no Nash equilibrium.

3.1. Interpreting the reward scheme

Though the concept of incentive schemes in the context of delegated portfolio management is well-studied, the idea of a reward scheme that sustains the conditions given in Definition 1, is relatively novel. Most previous works considered individual contracts based on monetary transfers (see, e.g., Barry and Starks, 1984; Admati and Pfleiderer, 1997; Dasgupta and Prat, 2006, 2008; Chassang, 2013, among many others). These contracts do not focus on fund flows and are not limited by the constraint of Eq. (2). Nevertheless, we think that our reward-scheme model captures an important property of the

[14] Concrete examples and applications relating to this assumption are given in Subsection 3.1.
[15] The factor k in the RHS of Eq. (3) follows from the initial amount of 1 given to each player.

portfolio-management market, and since these conditions go to the very core of our model, we wish to explain them thoroughly.

The conditions of Definition 1, and specifically Eq. (2), relate to the standard contract exercised, in practice, in the portfolio-management market where portfolio managers and investment firms normally get asset-under-management fees. These fees generate indifference towards the risk preferences of the investor, since the managers are mainly concerned with the expected returns and the amount of managed funds. Our reward-scheme approach solves this problem by tying the managed funds (and, therefore, the managers' payoffs) to the preferences of the DM. Second, the commonly-used standard fees limit the ability of the DM to implement complicated individual contracts. Our approach is designed to use the one ability an investor always possesses – reallocating funds among managers.

Consider, for example, a parent that wishes to save money for her kids' college tuition. She decides on a year-by-year plan specifying a monetary goal to be achieved: if her investments under preform, relative to this goal, she increases the fund by putting more money, while if the investments over preform, she uses the extra profits for her own needs. Since she does not manage the funds, she needs to devise a plan such that the portfolio managers are optimizing on a yearly basis according to her personal preferences. As most managers get asset-under-management fees, the ability of such a retail client to negotiate their incentives is limited. An optimal reward scheme (designed separately for each year) keeps the managers in accordance with the best interests of the DM, by taking advantage of her elementary ability to redistribute the funds.

A similar situation arises when saving for a pension. In some cases and due to regulatory restrictions, a worker cannot invest her pension funds directly. In addition, throughout the years the worker's relation towards risk changes. Using an optimal reward scheme periodically and based on the worker's ability to maneuver funds, she can guarantee that managers are optimizing according to her updated preferences. This is, in fact, a major part of the Chilean pension reform. The reform obligates every worker to save in a pension fund, while enabling her to switch (almost) freely between the different funds and plans, based on her subjective preferences.[16]

Though the two previous examples only considered small retail clients, one could also give a different interpretation to our reward-scheme model. Consider a scenario where the DM is a fund family or a syndicate of investors. Such an institution needs to allocate its resources among different managers, where each manager works under a fee-based contract. The reward scheme, in this case, is a simple performance-based allocation rule.

4. Optimal schemes under risk neutrality

We begin our analysis with a risk-neutral DM, as in Chassang (2013), Carroll (2015), and the motivating example presented in Section 2. That is, throughout the current section (Section 4) the considered DM is an expected-return maximizer with a linear utility function U. Given a linear utility, Definition 3 becomes a straightforward maximization problem, where an optimal scheme f is a scheme that generates the maximal expected return in every dominant-strategy equilibrium of the investment game G_f.

The results in this section are divided into positive and negative ones. We start with the positive results proving that for every set of actions A, either bounded (Subsection 4.1) or unbounded (Subsection 4.2), the DM can construct an optimal scheme. Our proofs are constructive and the optimal scheme that we propose is linear in terms of the differences between the players' earnings. The next positive result extends the first ones by showing that linearity is a necessary condition for optimality (Subsection 4.3).

The negative results are given in Subsections 4.4 and 4.5. The first negative result shows that the only scheme that remains optimal, independently of A, is constant. In other words, for every non-trivial (i.e., non-constant) reward scheme f, there exists a set A such that f is not optimal. The next negative result proves that the existence condition of Definition 3 is necessary, since one can construct an investment game with a continuous reward scheme f, but without a Nash equilibrium.

The combination of the positive and negative results is significant. On the positive side, in any non-deterministic market the DM can design rules to ensure that her interests are kept by the players. On the negative side however, when the market is dynamic – meaning that the market is constantly changing in terms of possible actions and assets' yields are constantly growing – and when the DM cannot keep track of these changes, then no single reward scheme can produce optimal results (in any possible market). That is, any non-trivial reward scheme can lead to suboptimal results. More formally, the only reward scheme that induces also equilibria in which players act according to the DM's preferences, is a reward scheme which, paradoxically, is independent of the players' actions.

4.1. Bounded portfolios

We start with the case of bounded assets. Assume that there exists an $M \in \mathbb{R}$ such that $\mathbf{Pr}(|X_i| < M) = 1$ for every asset $X_i \in A$. When such an M exists, we say that A is *uniformly bounded*.

[16] For a detailed description see the book by the Superintendencia de Administradoras de Fondos de Pensiones (2003) and the OECD report, "Chile: review of the private pensions system", October 2011.

Define the *Linear Reward Scheme* f as,

$$f_i(r) = \frac{1}{k} + \begin{cases} \frac{\sum_{j\neq i}(r_i - r_j)}{2k(k-1)M}, & \text{if } \forall i,\ |r_i| \leq M, \\ 0, & \text{if } \exists i \text{ s.t. } M < |r_i|. \end{cases}$$

One can verify that f is well-defined, since for every $r \in \mathbb{R}^k$, the equality $\sum_i f_i(r) = 1$ holds and $f(r) \in [0,1]^k$.

The Linear Reward Scheme f can be rewritten as

$$f_i(r) = \frac{1}{k} + \frac{1}{2Mk}\left[r_i - \frac{1}{k-1}\sum_{j\neq i} r_j\right]\mathbf{1}_{\{\forall i,\ |r_i|\leq M\}}.$$

This presentation provides an important economic insight on the optimal reward scheme f: this reward scheme distributes to all players, before the results are considered, the same basic share $1/k$. When the results are considered, every player gains or loses relatively to the basic share, a portion that depends on the difference between his result and the average result of the other players.

In other words, the performance of every player is assessed relative to the other players' average performance, and not to some exogenous benchmark portfolio (which is the case in most previous studies). This is one crucial aspect of the Linear Reward Scheme. It generates a competition that, in return, generates a specific benchmark for each player, which is the average performance of his competitors.

In reality, the implementation of Linear Reward Scheme is trivial. The DM only needs to fix an estimated large M (e.g., the maximal result obtained in previous years), and to observe the actual performances of the players at the end of a single time period. This simplicity allows a complete layman in terms of investments to employ such an incentives-coordinating mechanism. Moreover, in case the portfolio managers are not risk neutral, the ability to incentivize them without an exogenous benchmark might be crucial (see discussion in Section 5.1).

Theorem 1. *For every uniformly-bounded set of assets A and given a risk-neutral* DM, *the* Linear Reward Scheme *is optimal.*

Being of a technical nature, all proofs henceforth are deferred to the Appendix.

4.1.1. The motivating example — revisited

Before extending the basic result of Theorem 1, we present the effect of the Linear Reward Scheme on the motivating example (Section 2). Recall that there are two players ($k = 2$), and two pure actions X_1 and X_2. Fix $M = 2$ and note that $|X_i| < M$ for every $i = 1, 2$. The utility function of Player 1, given the profile of actions (σ_1, σ_2) and the Linear Reward Scheme f, is

$$\begin{aligned} U_1(\sigma_1,\sigma_2) &= \lambda\sigma_1 + (1-\lambda) f_1(\sigma_1,\sigma_2) \\ &= \lambda\sigma_1 + (1-\lambda)\left[\frac{1}{2} + \frac{\sigma_1 - \sigma_2}{8}\right] \\ &= \frac{(1-\lambda)(4-\sigma_2) + (1+7\lambda)\sigma_1}{8}. \end{aligned}$$

This utility increases with σ_1, rendering X_1 a dominant strategy, and (X_1, X_1) the unique dominant-strategy equilibrium.

4.2. Unbounded portfolios

In this subsection, we extend Theorem 1 to unbounded assets. For that purpose, a few additional notation are needed. Let $O \subseteq A$ be the set of optimal assets with respect to an expected return-maximizer DM:

$$O = \{X_j \in A;\ \mathbf{E}\left[X_j\right] \geq \mathbf{E}[X_\ell] \text{ for every } X_\ell \in A\}. \tag{4}$$

Namely, there exists an $\epsilon > 0$ such that for every $X_j \in O$ and $X_\ell \notin O$,

$$\mathbf{E}[X_j] > \mathbf{E}[X_\ell] + \epsilon. \tag{5}$$

Let $Q_1 = \{q \in Q :\ q^i = 0, \forall X_i \notin O\}$ be the set of diversified strategies in Q where all the sub-optimal pure strategies are taken with probability 0, and let $Q_2 = \{q \in Q :\ q^i = 0, \forall X_i \in O\}$ be the set of diversified actions in Q where all the optimal pure actions in O are assigned probability 0. The following lemma enables us to define a reward scheme f adapted to the infinite set-up.

Lemma 2. *Recall ϵ from Ineq. (5). There exists an $M > 0$ such that for every $(X, q, \alpha) \in Q_1 \times Q_2 \times [0, 1]$ and every $m \geq M$,*

$$\mathbf{E}\left[|X - q|\mathbf{1}_{\{|(1-\alpha)X+\alpha q|>m\}}\right] < \frac{\epsilon}{2}.$$

We use the M given in Lemma 2 to define a new reward scheme f, similar to the Linear Reward Scheme used in Theorem 1. First, define the real-valued function $\phi : \mathbb{R} \to \mathbb{R}$ as

$$\phi(x) = \begin{cases} -M, & \text{if } x < -M, \\ x, & \text{if } -M \le x \le M, \\ M, & \text{if } x > M. \end{cases} \tag{6}$$

For every player i and every vector $r = (r_1, \dots, r_k) \in \mathbb{R}^k$, define the *Truncated Reward Scheme* f such that

$$f_i(r) = \frac{1}{k} + \frac{1}{2Mk}\left[\phi(r_i) - \frac{1}{k-1}\sum_{j \neq i}\phi(r_j)\right].$$

One can verify that f is well-defined, since

$$0 \le \frac{1}{k} + \frac{1}{2Mk}\left[\phi(r_i) - \frac{1}{k-1}\sum_{j \neq i}\phi(r_j)\right] \le \frac{2}{k} \Leftrightarrow \left|\phi(r_i) - \frac{1}{k-1}\sum_{j \neq i}\phi(r_j)\right| \le 2M,$$

and the last inequality holds for every $r_i, r_j \in \mathbb{R}$.

The next theorem extends Theorem 1 to the infinite risk-neutral case. It states that an optimal reward scheme exists also when the set A is not uniformly bounded, implying that the linear utility function is unbounded as well.

Theorem 2. *For every set of assets A and given a risk-neutral DM, the* Truncated Reward Scheme *f is optimal.*

Note that Theorem 1 and Theorem 2 considered managers that are concerned only with the share they obtain from the managed funds. Using the terminology of the example in Section 2, we actually assumed that $\lambda = 0$ for every manager. It is important to emphasize, though, that our results could be extended beyond this restriction. One can take $\lambda > 0$ and show that the Truncated Reward Scheme remains optimal. The intuition is clear: once λ is positive, every manager has an additional incentive to choose an optimal asset rather than a sub-optimal one. Therefore, the reward schemes, that were previously considered optimal, remain so even when managers are concerned with their actual performance and not only with their share. In addition, we extend these results to a general utility function U in Theorem 5 (Section 5).

4.3. Uniqueness

Theorems 1 and 2 provide optimal schemes for bounded and unbounded cases. These theorems suggest that a sufficient condition for an optimal reward scheme is linearity with respect to the returns. The next question we tackle is whether this is also a necessary condition. The following theorem proves that linearity is indeed crucial for optimality. Specifically, if one wants a reward scheme f to generate an investment game G_f with a dominant-strategy optimal equilibrium, then $f_i(r)$ must be linear in r_i.

Theorem 3. *Fix $M > 0$ and assume a risk-neutral DM. If for every uniformly-bounded set of assets A w.r.t. M, the investment game G_f has a dominant-strategy optimal equilibrium, then $f_i(r)$ is linear in r_i for every r_{-i}.*

The result of Theorem 3, as do previous results, holds for every large M. This reflects the understanding that, under risk-neutrality, giving the managers a share of the fund can generate an optimal result. However, when fixing an absurdly large M, the managers' incentives to invest optimally might be too weak. In such cases, almost every strategy becomes an equilibrium. This motivation leads us to the impossibility results given in the following subsection.

4.4. A universal reward scheme

The conditions leading to Theorems 1, 2, and 3 suggest that the set of assets A is either fixed or uniformly bounded. That is, the investor and managers work under the assumption that either the set A or its bound, do not change. A natural question that arises is whether there exists a reward scheme that is optimal for every set of actions A. Such a scheme is referred to as *universal* and defined as follows.

Definition 4. A reward scheme f is said to be *universal* if for every set of assets A, the induced investment game G_f has a Nash equilibrium $(\sigma_1, ..., \sigma_k)$ where $\mathbf{E}[\sigma_i] = \max_{X_j \in A} \mathbf{E}\left[X_j\right]$ for every $i = 1, \dots, k$.

In words, f is a universal reward scheme if for every finite set of actions, the investment game G_f has an equilibrium that sustains the optimality condition given in Eq. (3) w.r.t. a linearly-increasing utility function. When comparing a universal reward scheme and an optimal reward scheme, one should notice two differences. First, a reward scheme is optimal if there exists a dominant-strategy equilibrium and *every dominant-strategy* equilibrium is optimal. Whereas, a universal reward scheme requires a weaker existence condition of an *optimal Nash equilibrium*. Secondly, a reward scheme f is universal if for every set of actions A the induced investment game G_f has an optimal equilibrium. An optimal reward scheme, on the other hand, relates only to specific sets A (either uniformly bounded, or fixed).

The following theorem states that a non-constant universal reward scheme does not exist.

Theorem 4. *In the case of two managers, there is no non-constant universal reward scheme.*

A generalization of Theorem 4 to any number of managers $k \geq 3$ is not trivial though. For example, take any non-constant reward scheme $f : \mathbb{R}^k \to \mathbb{R}^k$ such that $f_i(r) = 1/k$ for every $r \in \mathbb{R}^k$ that has at least two identical coordinates. In this case, for every action X_j, the profile of strategies $(X_j, \ldots, X_j)$ is an equilibrium, because a unilateral deviation would still leave at least two identical coordinates of at least two other managers, in which case the share would be determined as $1/k$. In other words, any deviation of a single manager will not bear any influence on the payoffs. On the other hand, requiring that all equilibria satisfy Eq. (3) will not hold when f is constant, because in this case all profiles are equilibria. Therefore, when the number of managers is three or more, we must introduce a stronger requirement, as given in the following corollary.

Corollary 1. *If for every set of assets A, every optimal profile of strategies in G_f constitutes a Nash equilibrium, then every profile of strategies in a Nash equilibrium.*

4.5. An investment game without an equilibrium

An optimal reward scheme requires the existence of an equilibrium. Such a requirement is not an issue when discussing finite, strategic-form games, played with vN-M utility maximizers and mixed actions. However, when considering investments games, this restriction becomes significant. The reason for the lack of equilibrium might be two-fold: discontinuity or lack of linearity of the utility functions. The lack of equilibrium in the example of Section 2 (for a small range of λ's) is due to discontinuity: the winner takes all, but in case of a draw, the players divide the funds evenly.

The following example shows that even when the payoff functions in investment games are continuous, equilibrium might not exist. To show that the issue of equilibrium existence is rather subtle, we introduce just a slight modification of the Truncated Reward Scheme discussed in Theorem 2 and show that there is no equilibrium in the induced investment game.

Consider the reward scheme

$$f_i(r) = \frac{1}{k} + \frac{\sum_{j=1}^{k} \phi(r_i - r_j)}{2k(k-1)M}, \tag{7}$$

where ϕ is given by Eq. (6). The reward scheme f is well-defined and might induce an optimal result for a specific uniformly-bounded set of assets A. However, this is not true in general. The investment game G_f induced by f defined in Eq. (7) might not have an equilibrium.

Proposition 1. *There is a set of assets A such that the game G_f, induced by f defined in* Eq. (7)*, has no Nash equilibrium.*

The set of assets A used in the proof of Proposition 1 contains only two variables $A = \{X, Y\}$ where $X \equiv 0$. Asset Y, however, has a certain oscillating property, not enabling an equilibrium to exist when f is restricted to only two players.

5. Optimal schemes under general utility functions

In this section we generalize previous results to the case of a general utility function U. Since we impose no restrictions over the set of available assets A, we do need to draw two assumptions over the utility function U.

The first assumption is that the utility function is bounded by some constant M. Specifically, a utility function U is *bounded* if there exists an $M > 0$ such that $|U(x)| \leq M$ for every $x \in \mathbb{R}$. Though this assumption might seem restrictive from a theoretical perspective, it is quite basic in practice (especially when the assets are bounded).

The second assumption relates to the number of optimal diversified actions w.r.t. U. Formally, a utility function U is *uniquely maximized* if there exists a diversified action $q^* \in Q$ such that

$$E[U(kq^*)] > E[U(kq)], \quad \forall q \in Q \setminus \{q^*\}.$$

In words, we say that U is *uniquely maximized* in case the maximal expected utility in the RHS of Eq. (3) is obtained by a unique diversified action. This assumption is later omitted in Corollary 2.

In the following theorem, we define the *General Reward Scheme*, and along with these two assumptions, we prove optimality.

Theorem 5. *If U is bounded and uniquely maximized, then the following* General Reward Scheme *f is optimal:*

$$f_i(r) = \frac{1}{k} + \frac{1}{2Mk}\left[U(kr_i) - \frac{1}{k-1}\sum_{j\neq i} U(kr_j)\right].$$

The economic interpretation of the General Reward Scheme is important. The DM uses her own utility function in order to produce an incentive scheme that is linear with respect to U. The performance of each player is first assessed via the DM's utility function, and later compared to the utility-based performance of the other players.

The insertion of the k factor into the utility function of the General Reward Scheme is important. The evaluation of manager i's performance using $U(kr_i)$ instead of $U(r_i)$ ensures that manager i is compensated as if he was managing all the investor's wealth, though he invests only a fraction $1/k$ of the total sum. Such scaling is important in a non risk-neutral environment since the investor's preferences may depend on the volume of the investment made. Moreover, in our model the managers' returns are independent of their investments volume. That is, investments are constant-return-to-scale. Some theoretical and empirical studies (such as Berk and Green, 2004 and Pollet and Wilson, 2008) showed that returns might be decreasing with respect to the size of the investment. Though it goes beyond our model, the need to use multiple managers could be supported by such arguments.

The requirement that every equilibrium is optimal in Definition 3 is quite restrictive, especially in the context of delegated portfolio management. Another possibility is to require only the existence of an optimal equilibrium. The following corollary shows that, under the weaker existence assumption, the uniquely-maximized condition in Theorem 5 is redundant.

Corollary 2. *If U is bounded, then the investment game induced by the* General Reward Scheme *has a dominant-strategy optimal equilibrium.*

The proof follows directly from the proof of Theorem 5. Nevertheless, we should point out that, given the General Reward Scheme, *all equilibria are dominant-strategy equilibria in which every player chooses an optimal portfolio for the investor*. Thus, the cases that differ Theorem 5 and Corollary 2 are cases where different players choose different optimal portfolios such that the combination of these portfolios is sub-optimal.

5.1. Risk-averse players and the self-induced benchmark

Aside from the ability to bridge between the DM's preferences and the portfolio managers' payoffs using fund flows, our General Reward Scheme has another important and novel property we wish to discuss: a *self-induced benchmark*.

Admati and Pfleiderer (1997) presents the problem of using exogenous benchmarks in various scenarios. The main problem that arises is that the benchmark is not calibrated to the effort, and sometimes the risk, that the players incur. The same problem is met in Holmstrom and Milgrom (1987) and Sappington (1991). In fact, in most frameworks it is well-established that a linear scheme is only a second-best solution, as it leads to an underinvestment in effort and information, when needed.

The scheme we propose relates to these issues by generating a competition between the players. The General Reward Scheme defines a competition in which every player is measured in comparison to the other players, who are also evaluated in the same way. Therefore, the effort and risk invested by one player are taken with respect to those invested by the other players. Thus, the strategy of every player is correlated with its benchmark, via the equilibrium of the investment game.

Let us consider a concrete set-up where a risk-neutral DM faces two players, both with a constant absolute risk aversion (CARA) utility function. Specifically, assume that player $i = 1, 2$ tries to maximize $\mathbf{E}[p_i] - \frac{\gamma_i}{2}\text{Var}[p_i]$, where p_i is player i's payoff and $\gamma_i > 0$. Assume that each player gets an initial normalized amount of 1, and let Z be some exogenous benchmark, say the S&P 500 index.

The DM needs to choose between two single-period incentives plans. The first plan suggests that each player gets a fixed share $C \in (0, 1)$ of the portfolio's final value, where profits are assessed relative to the benchmark Z. Hence, the payoff equals $p_i = C(1 + \sigma_i - Z)$ and player i approaches the optimization problem[17]

$$\max_{\sigma_i \in A} \mathbf{E}[\sigma_i] - \frac{C\gamma_i}{2}\text{Var}[\sigma_i - Z].$$

The second plan is the General Reward Scheme where player i gets a fixed share C of the basic fund according to f after the realized returns are observed. Namely, $p_i = C\left[1 + \frac{1}{2M}(\sigma_i - \sigma_{-i})\right]$ and the optimization problem becomes

$$\max_{\sigma_i \in A} \mathbf{E}[\sigma_i] - \frac{C\gamma_i}{4M}\text{Var}[\sigma_i - \sigma_{-i}].$$

[17] The case where every player receives a share C of the portfolio's value is also considered by taking $Z \equiv 1$.

To clarify, all payments are made at the end of the period, and the DM collects the overall funds (including the profits) after payments to the players are made.

Lemma 3. *Let $X_1 \neq Z$ be the portfolio that maximizes expected returns for the investor. The profile (X_1, X_1) is an equilibrium of the investment game induced by the* General Reward Scheme. *However, if $\mathbf{E}[Z] > \mathbf{E}[X_1] - \frac{C\gamma_i}{2}\mathrm{Var}[X_1 - Z]$, then the players will not invest in X_1 given the first incentives plan.*

Proof. The proof of the first part is straightforward since a deviation of one player can only increase his variance without increasing the expected value. That is, assuming that one player chooses to invest in X_1, the other player will minimize his risk by choosing X_1, a choice that also maximizes the portfolio's expected return.

The proof of the second part follows from a comparison of an investment in Z relative to X_1. Taking the difference between a portfolio consisting completely of Z and another consisting of X_1 yields

$$\mathbf{E}[Z] - \frac{C\gamma_i}{2}\mathrm{Var}[Z - Z] - \left(\mathbf{E}[X_1] - \frac{C\gamma_i}{2}\mathrm{Var}[X_1 - Z]\right) > 0,$$

where the inequality follows from the given assumption, thus proving the lemma. □

The condition suggesting that players prefer Z than X_1 is clearly not unique. One could also think, e.g., of a case where X_1 and Z are positively correlated and a risk-free asset is available with a rate of $r_f > \mathbf{E}[X_1] - \frac{C\gamma_i}{2}V(X_1)$. Again, in such a case the players will not invest in X_1. In addition, a similar result could be easily extended to cases where the DM has a bounded utility function U. In fact, Corollary 2 also holds when players have a CARA utility function.

Lemma 3 relates to cases where active portfolio managers tend to stay too close to their benchmarks, investing more passively than they are supposed to. The against-the-average benchmarking used in the General Reward Scheme tries to solve this issue by regulating the benchmark in equilibrium. This approach might also solve the second-best problem presented in Admati and Pfleiderer (1997), where is it proven that the best use of information is achieved when the benchmark is optimal. Moreover, we conjecture that Lemma 3 could be further extended to any risk-averse set-up along with the appropriate condition. The models of costly efforts and general risk-aversion are left to future research.

6. Concluding remarks

The current paper explores the portfolio delegation problem with multiple homogeneous managers, where the investor is limited to fund flows. The first natural extension is to eliminate homogeneity among managers. One can consider a similar scenario, where managers have different capabilities and different assets in play, such that the managers is required to use the induced fund-flows competition to screen out low-level managers, while efficiently incentivizing high-level ones. Moreover, our fund-flows induced-competition model could prove useful in future research of the costly-effort problem, given a general risk-neutral set-up similar to ours, that was raised by Chassang (2013).

In addition, our static set-up assures that the overall amount of managed funds is fixed. That is, the managers are rewarded only through the reallocation of a fixed amount, rather than by additional funds gained through profitable investment. However, one can naturally extend our model to a dynamic one, where the overall managed funds (of all the managers combined) increase or decrease with respect to past investments. This extension suggests that the DM redistributes available funds, including her yearly earnings, such that every manager needs to balance between getting a bigger share and increasing the amount of managed funds. To a certain extent, such a dynamic model resembles the model in Huck et al. (2012), where workers consider both the common prosperity and their own personal good.

Another possible extension to our model is the influence of additional investors on the managers. In general, it is clear that the reward schemes of additional investors can distort incentives generated by a single DM assuming all investors are pooled into a single portfolio. We leave this problem along with its implications on regulatory decisions for future research.

Appendix

Lemma 1. *For every $0 \le \lambda < \frac{1}{1.194} \approx 0.83$, the unique equilibrium is when both managers choose to invest only in* Bond X_2.

Proof. Fix $\lambda \in \left[0, \frac{1}{1.194}\right)$. Assume that Manager 1 employs $\sigma_1 = \alpha_1 X_1 + (1-\alpha_1)X_2$ and Manager 2 employs $\sigma_2 = \alpha_2 X_1 + (1-\alpha_2)X_2$, where $0 \le \alpha_1, \alpha_2 \le 1$. The first term $\lambda\mathbf{E}(\sigma_1)$ of Manager 1's utility equals $\lambda(1.0306 + 0.0194\alpha_1)$, which is linearly increasing in α_1 when $\lambda > 0$. The second term equals

$$(1-\lambda)\left(\mathbf{Pr}(\sigma_1 > \sigma_2) + \frac{1}{2}\mathbf{Pr}(\sigma_1 = \sigma_2)\right) = (1-\lambda)\cdot\begin{cases} 3/5, & \text{if } \alpha_1 < \alpha_2, \\ 1/2, & \text{if } \alpha_1 = \alpha_2, \\ 2/5, & \text{if } \alpha_1 > \alpha_2. \end{cases}$$

Table 1
The two pure-action game.

	X_1	X_2
X_1	$0.5+0.55\lambda,\ 0.5+0.55\lambda$	$0.4+0.65\lambda,\ 0.6+0.4306\lambda$
X_2	$0.6+0.4306\lambda,\ 0.4+0.65\lambda$	$0.5+0.5306\lambda,\ 0.5+0.5306\lambda$

When $\lambda=0$, the strategy $\sigma_1=X_2$ (i.e., $\alpha_1=0$) is a dominant strategy and the result holds. We may thus assume that $\lambda\in\left(0,\frac{1}{1.194}\right)$. By the linearity in α_i, a profile of strategies in which $\alpha_i<\alpha_{-i}$ cannot be an equilibrium, since Manager i has a profitable deviation to $\frac{\alpha_i+\alpha_{-i}}{2}$. This deviation increases the first term of Manager i's utility without affecting the second term.

In addition, if $\alpha_1=\alpha_2>0$, then any manager can make an infinitesimal deviation to $\alpha_i-\epsilon\in(0,\alpha_i)$ and gain $\frac{3(1-\lambda)}{5}$ instead of $\frac{1-\lambda}{2}$ with an infinitesimal loss in $\lambda\mathbf{E}(\sigma_i)$. Therefore, we only need to consider the profile where $\alpha_1=\alpha_2=0$, that is, (X_2,X_2). A direct computation shows that no profitable deviation exists and the result holds. We point out that the linearity of $\lambda\mathbf{E}(\sigma_i)$ in α_i implies that we only need to verify that deviating to X_1 is not profitable.

It is important to note that the utility functions in this example are not linear. If we would let the managers invest their entire allocation either in X_1 or in X_2, without allowing diversified investments, we would get a game where each manager has only two possible actions at its disposal. In such a case the payoff matrix would be Table 1.

Table 1 presents a 2-player game with the relevant expected values. It is easy to see that X_2 is dominating X_1 for every $\lambda<\frac{1}{1.194}$. Note, however, that this is not sufficient for showing that X_2 is a dominant strategy. The reason is that the utility function of Manager 1 (see Eq. (1)), as well as that of Manager 2, are not linear. That is, $U_1(\alpha_1X_1+(1-\alpha_1)X_2,\sigma_2)$ is typically not equal to $\alpha_1U_1(X_1,\sigma_2)+(1-\alpha_1)U_1(X_2,\sigma_2)$. This implies that although X_2 is dominant over X_1, the action X_2 is not necessarily dominant over all other mixtures between the two. The lack of linearity (let alone the lack of continuity) might also impede the existence of an equilibrium. In our example, one can verify that an equilibrium does not exist[18] when $\frac{1}{1.194}<\lambda<1$. □

Theorem 1. *For every uniformly-bounded set of assets A and given a risk-neutral* DM, *the* Linear Reward Scheme *is optimal.*

Proof. For the sake of simplicity, and without loss of generality, we assume that X_1 is the unique expected-return maximizing asset. That is, $\mathbf{E}[X_1]>\mathbf{E}[X_i]$ for every $2\le i\le n$. This implies that a reward scheme f is optimal if and only if $(X_1,\dots,X_1)$ is a unique equilibrium in G_f.

We prove that for every Player i, for every profile of diversified actions $(\sigma_1,\sigma_2,\dots,\sigma_k)\in Q^k$ of players $1,\dots,k$ respectively, and for every strategy $\sigma_i\neq X_1$ of Player i, the inequality

$$\mathbf{E}\left[f_i\left(\sigma_1,\dots,\sigma_{i-1},X_1,\sigma_{i+1},\dots,\sigma_k\right)\right]>\mathbf{E}\left[f_i\left(\sigma_1,\dots,\sigma_k\right)\right],$$

holds.

Without loss of generality, assume that $i=1$. Therefore

$$\begin{aligned}\mathbf{E}\left[f_1\left(\sigma_1,\sigma_2,\dots,\sigma_k\right)\right]&=\mathbf{E}\left[\frac{\sum_{j=1}^k(\sigma_1-\sigma_j)}{2k(k-1)M}+\frac{1}{k}\right]\\&=\mathbf{E}\left[\frac{(k-1)\sigma_1-\sum_{j=2}^k\sigma_j}{2k(k-1)M}+\frac{1}{k}\right]\\&<\mathbf{E}\left[\frac{(k-1)X_1-\sum_{j=2}^k\sigma_j}{2k(k-1)M}+\frac{1}{k}\right]\\&=\mathbf{E}\left[\frac{\sum_{j=2}^k\left(X_1-\sigma_j\right)}{2k(k-1)M}+\frac{1}{k}\right]\\&=\mathbf{E}\left[f_1\left(X_1,\sigma_2,\dots,\sigma_k\right)\right],\end{aligned}$$

where the first and the last equalities follow from the definition of f, and the inequality follows from the fact that $\sigma_1\neq X_1$ and $\mathbf{E}[\sigma_1]=\mathbf{E}\left[\sum_{i=1}^n\sigma_1^jX_j\right]<\mathbf{E}[X_1]$. □

[18] A simple computation shows that (X_2,X_2) is no longer an equilibrium, because a deviation to X_1 is profitable. All other profiles are not equilibria by the same reasoning given in the proof of Lemma 1.

Lemma 2. *Recall ϵ from Ineq. (5). There exists an $M > 0$ such that for every $(X, q, \alpha) \in Q_1 \times Q_2 \times [0, 1]$ and every $m \geq M$,*

$$\mathbf{E}\left[|X - q|\mathbf{1}_{\{|(1-\alpha)X+\alpha q|>m\}}\right] < \frac{\epsilon}{2}.$$

Proof. By the optimality of the actions in Q_1 and the sub-optimality of the actions in Q_2, we know that $\mathbf{E}[X - q] > \epsilon$ for every $X \in Q_1$ and every $q \in Q_2$. In addition, for every $\alpha \in [0, 1]$, one can choose a sufficiently large $M_{X,q,\alpha} > 0$ such that for every $m \geq M_{X,q,\alpha}$,

$$\mathbf{E}\left[|X - q|\mathbf{1}_{\{|(1-\alpha)X+\alpha q|>m\}}\right] < \frac{\epsilon}{2}. \tag{8}$$

This follows from the fact that for every $(X, q, \alpha) \in Q_1 \times Q_2 \times [0, 1]$, the set of random variables $\{|X - q|) \times \mathbf{1}_{\{|(1-\alpha)X+\alpha q|>m\}}\}_{m\in\mathbb{N}}$ is a sequence of real-valued measurable functions that are weakly dominated by an integrable function $|X - q|$. That is,

$$|X - q|\mathbf{1}_{\{|(1-\alpha)X+\alpha q|>m\}} \leq |X - q|$$

for every $m \in \mathbb{N}$. The sequence converges pointwise to 0 as $m \to \infty$. Hence, by the dominated convergence theorem,

$$\mathbf{E}\left[|X - q|\mathbf{1}_{\{|(1-\alpha)X+\alpha q-Y|>m\}}\right] \to 0 \text{ as } m \to \infty.$$

Since Ineq. (8) is strict, there exists an open set $B_{X,q,\alpha} \subseteq Q^2 \times \mathbb{R}$ containing (X, q, α), such that this inequality holds for every $(X', q', \alpha') \in B_{X,q,\alpha}$ and every $m \geq M_{X,q,\alpha}$.

The collection of open sets $\{B_{X,q,\alpha}\}_{(X,q,\alpha)\in Q_1\times Q_2\times[0,1]}$ is an open cover of the compact set $Q_1 \times Q_2 \times [0, 1]$, hence a finite subcover B exists. Fix a positive number $M = \max_{B_{X,q,\alpha}\in B} M_{X,q,\alpha}$ and note that (8) holds for every $(X, q, \alpha) \in Q_1 \times Q_2 \times [0, 1]$ and every $m \geq M$. □

Theorem 2. *For every set of assets A and given a risk-neutral DM, the* Truncated Reward Scheme *f is optimal.*

Proof. Fix a strategy $\sigma_1 \in Q \setminus Q_1$ and $\sigma_2, \ldots, s_k \in Q$. There exist $X \in Q_1$ and $q \in Q_2$ such that $\sigma_1 = (1 - \alpha)X + \alpha q$, where $\alpha > 0$. Without loss of generality, we relate only to manager 1 and prove that $\mathbf{E}[f_1(X, \sigma_2, \ldots, s_k)] > \mathbf{E}[f_1(\sigma_1, \sigma_2, \ldots, s_k)]$. In words, for every profile of strategies $\sigma = (\sigma_1, \ldots, \sigma_k)$, manager 1 can increase his expected payoff by passing to a diversified action that includes only optimal actions.

By the linearity of the sum in f_1, it suffices to prove that

$$\mathbf{E}[\phi(X)] > \mathbf{E}[\phi((1 - \alpha)X + \alpha q)], \tag{9}$$

and every diversified action $\sigma_1 \notin Q_1$, that includes a sub-optimal action q, is dominated by some diversified action $X \in Q_1$.

Assume to the contrary that (9) does not hold, i.e.,

$$\mathbf{E}[\phi(X)] \leq \mathbf{E}[\phi((1 - \alpha)X + \alpha q)]. \tag{10}$$

Consider the real-valued function $\psi(x) = x - \phi(x)$, and note that $\phi(x) = x - \psi(x)$. Then, Ineq. (10) is recast as

$$\mathbf{E}[X - \psi(X)] \leq \mathbf{E}[(1 - \alpha)X + \alpha q - \psi((1 - \alpha)X + \alpha q)]$$

or, equivalently,

$$\mathbf{E}[X] - \mathbf{E}[(1 - \alpha)X + \alpha q] \leq \mathbf{E}[\psi(X)] - \mathbf{E}[\psi((1 - \alpha)X + \alpha q)]. \tag{11}$$

Since X is a convex combination of optimal actions and q is a convex combination of sub-optimal actions, it follows from (4) that

$$\begin{aligned}\mathbf{E}[X] - \mathbf{E}[(1 - \alpha)X + \alpha q] &= \mathbf{E}[X - (1 - \alpha)X - \alpha q] \\ &= \alpha\mathbf{E}[X - q] > \alpha\epsilon.\end{aligned}$$

Combining the last inequality with Ineq. (11) we obtain

$$\begin{aligned}\mathbf{E}[\psi(X) - \psi(X - \alpha(X - q))] &= \mathbf{E}[\psi(X)] - \mathbf{E}[\psi((1 - \alpha)X + \alpha q)] \\ &\geq \mathbf{E}[X] - \mathbf{E}[(1 - \alpha)X + \alpha q] > \epsilon.\end{aligned}$$

Denote $\gamma = \psi(X) - \psi(X - \alpha(X - q))$. We contradict the last inequality by showing that $\mathbf{E}[\gamma] < \alpha\epsilon$.

Consider the intervals $I_1 = (-\infty, -M)$, $I_2 = [-M, M]$, and $I_3 = (M, \infty)$. One can write ψ explicitly as

$$\psi(x) = \begin{cases} x + M, & \text{if } x \in I_1, \\ 0, & \text{if } x \in I_2, \\ x - M, & \text{if } x \in I_3. \end{cases}$$

Note that $\psi(x) < 0$ iff $x \in I_1$, and $\psi(x) > 0$ iff $x \in I_3$.

Overall, there are 9 cases we need to consider where $X \in I_i$ and $X - \alpha(X-q) \in I_j$, for every $i, j = 1, 2, 3$ (denote these events by A_{ij}):

Event A_{11}. If $i = j = 1$, then

$$\gamma = X + M - [X - \alpha(X-q) + M] = \alpha(X-q).$$

Event A_{33}. If $i = j = 3$, then

$$\gamma = X - M - [X - \alpha(X-q) - M] = \alpha(X-q).$$

Event A_{22}. If $i = j = 2$, then $\gamma = 0 - 0 = 0$.
Event A_{12}. If $i = 1$ and $j = 2$, then $\gamma = X + M < 0$. The inequality $\gamma < 0$ also holds in events A_{23} and A_{13}.
Event A_{32}. If $i = 3$ and $j = 2$, then $X - \alpha(X-q) < M$, or equivalently, $X - M < \alpha(X-q)$. This means that $\gamma = \psi(X) = X - M < \alpha(X-q)$.
Event A_{31}. If $i = 3$ and $j = 1$, then

$$\begin{aligned}\gamma &= X - M - (X - \alpha(X-q) + M) \\ &= \alpha(X-q) - 2M < \alpha(X-q).\end{aligned}$$

Event A_{21}. If $i = 2$ and $j = 1$, then $-M < X < M$, which implies that $-X - M < 0$. Thus,

$$\begin{aligned}\gamma &= 0 - (X - \alpha(X-q) + M) \\ &= -X - M + \alpha(X-q) < \alpha(X-q).\end{aligned}$$

This covers all nine possible cases. To conclude, we showed that in A_{i1} when $i = 1, 2$, and in A_{3j} when $j = 1, 2, 3$, the inequality $\gamma < \alpha(X-q)$ holds, and in all other events $\gamma < 0$. Note that $\bigcup_{j=1}^{3} A_{3j} = \{X > M\} \subseteq \{|X| > M\}$ and

$$\begin{aligned}A_{11} \cup A_{21} &= \{X \le M,\ X - \alpha(X-q) < -M\} \\ &= \{X \le M,\ (1-\alpha)X + \alpha q < -M\} \\ &\subseteq \{|(1-\alpha)X + \alpha q| > M\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{E}[\gamma] &= \sum_{i,j=1}^{3} \mathbf{E}\left(\gamma \mathbf{1}_{A_{ij}}\right) \\ &< \sum_{i=1}^{2} \mathbf{E}\left([\alpha(X-q)]\mathbf{1}_{A_{i1}}\right) + \sum_{j=1}^{3} \mathbf{E}\left([\alpha(X-q)]\mathbf{1}_{A_{3j}}\right) \\ &= \alpha \mathbf{E}[(X-q)\mathbf{1}_{\{X>M\}}] + \alpha \mathbf{E}\left[(X-q)\mathbf{1}_{\{X \le M,\ X-\alpha(X-q)<-M\}}\right] \\ &\le \alpha \mathbf{E}[|X-q|\mathbf{1}_{\{X>M\}}] + \alpha \mathbf{E}\left[|X-q|\mathbf{1}_{\{X \le M,\ X-\alpha(X-q)<-M\}}\right] && (12) \\ &\le \alpha \mathbf{E}[|X-q|\mathbf{1}_{\{|X|>M\}}] + \alpha \mathbf{E}\left[|X-q|\mathbf{1}_{\{|(1-\alpha)X+\alpha q|>M\}}\right] && (13) \\ &< \alpha \frac{\epsilon}{2} + \alpha \frac{\epsilon}{2} = \alpha \epsilon. && (14)\end{aligned}$$

Here, Ineq. (12) follows from the absolute values, Ineq. (13) follows from increasing the subset over which the expected values are taken, and Ineq. (14) follows from Ineq. (8). A contradiction.

We proved that for every manager, every optimal strategy (pure or diversified) $X \in Q_1$ dominates every sub-optimal strategy $\sigma_1 = (1-\alpha)X + \alpha q \in Q \setminus Q_1$. Hence, by eliminating sub-optimal strategies, the managers will play only optimal strategies.

Let Δ_{Q_1} be the probability simplex over the respective pure actions in Q_1. That is, every diversified optimal action of a manager i could be represented by a probability vector in Δ_{Q_1}. By the fixed point theorem on a convex compact set Δ_{Q_1}, we know that an equilibrium exists, and the result follows. □

Theorem 3. *Fix $M > 0$ and assume a risk-neutral DM. If for every uniformly-bounded set of assets A w.r.t. M, the investment game G_f has a dominant-strategy optimal equilibrium, then $f_i(r)$ is linear in r_i for every r_{-i}.*

Proof. Without loss of generality, fix $i=1$ and take $r_{-i}=(r_2,\dots,r_k)\in(-M,M)^{k-1}$ such that $r_i\ge r_{i+1}$ for every $2\le i\le k-1$. Define $g(t)=f_1(t,r_{-i})$. We need to prove that $g(t)$ is linear. Fix $r_1>r_2$ and consider $-M<y<x<M$ and $\lambda\in(0,1)$ such that $r_1=\lambda x+(1-\lambda)y$. We start by proving that $g(\lambda x+(1-\lambda)y)=\lambda g(x)+(1-\lambda)g(y)$.

Define the constant random variables $X_j=r_j$ for every $1\le j\le k$. Take $\epsilon>0$ such that $\lambda+\epsilon<1$ and define Z_+ such that

$$Z_+=\begin{cases}x, & \text{w.p. } \lambda+\epsilon,\\ y, & \text{w.p. } 1-\lambda-\epsilon.\end{cases}$$

Clearly, $\mathbf{E}[Z_+]>\mathbf{E}[X_j]$ for every $1\le j\le k$. Let $A_+=\{X_1,X_2,\dots,X_k,Z_+\}$ be a set of strictly-bounded actions. Since an optimal dominant-strategy equilibrium exists, it follows that Z_+ is a dominant strategy of manager 1. Thus,

$$\begin{aligned}g(\lambda x+(1-\lambda)y)&=f_1(\lambda x+(1-\lambda)y,r_{-i})\\&=\mathbf{E}[f_1(X_1,X_2,\dots,X_k)]\\&\le\mathbf{E}[f_1(Z_+,X_2,\dots,X_k)]\\&=(\lambda+\epsilon)f_1(x,r_{-i})+(1-\lambda-\epsilon)f_1(y,r_{-i})\\&=(\lambda+\epsilon)g(x)+(1-\lambda-\epsilon)g(y).\end{aligned}$$

Taking the limit when ϵ tends to 0, gives the inequality $g(\lambda x+(1-\lambda)y)\le\lambda g(x)+(1-\lambda)g(y)$.

Now, take $\epsilon>0$ such that $\lambda-\epsilon>0$ and define Z_- such that

$$Z_-=\begin{cases}x, & \text{w.p. } \lambda-\epsilon,\\ y, & \text{w.p. } 1-\lambda+\epsilon.\end{cases}$$

Let $A_-=\{X_1,X_2,\dots,X_k,Z_-\}$ be a set of strictly-bounded actions, and similarly to the previous reasoning, we get that X_1 is a dominant strategy, as $\mathbf{E}[X_1]=r_1>\mathbf{E}[Z_-]$. Therefore,

$$\begin{aligned}g(\lambda x+(1-\lambda)y)&=f_1(\lambda x+(1-\lambda)y,r_{-i})\\&=\mathbf{E}[f_1(X_1,X_2,\dots,X_k)]\\&\ge\mathbf{E}[f_1(Z_-,X_2,\dots,X_k)]\\&=(\lambda-\epsilon)f_1(x,r_{-i})+(1-\lambda+\epsilon)f_1(y,r_{-i})\\&=(\lambda-\epsilon)g(x)+(1-\lambda+\epsilon)g(y).\end{aligned}$$

Taking the limit where $\epsilon\to0$, it follows that $g(\lambda x+(1-\lambda)y)\ge\lambda g(x)+(1-\lambda)g(y)$. To conclude, we proved that $g(\lambda x+(1-\lambda)y)=\lambda g(x)+(1-\lambda)g(y)$ for the specific case where $\lambda x+(1-\lambda)y=r_1\ge r_2$. Nevertheless, this result holds for every $y<r_1<x$, and a straightforward examination shows that

$$\frac{g(y)-g(r_1)}{y-r_1}=\frac{g(x)-g(r_1)}{x-r_1}=\frac{g(y)-g(x)}{y-x},$$

which implies linearity, as required. □

Theorem 4. *In the case of two managers, there is no non-constant universal reward scheme.*

Proof. Let $x<y<z$. We first prove that $f_1(y,y)\ge f_1(x,y)$. Assume to the contrary that $f_1(y,y)<f_1(x,y)$. Let $A=\{X_1,X_2\}$ be a set of two actions X_1 and X_2 with a joint probability distribution

$X_1\backslash X_2$	x	y
x	0	0
y	1	0

Note that $\mathbf{E}[X_1]>\mathbf{E}[X_2]$. However,

$$\mathbf{E}[f_1(X_2,X_1)]=f_1(x,y)>f_1(y,y)=\mathbf{E}[f_1(X_1,X_1)],$$

implying that (X_1,X_1) is not an equilibrium in G_f, since manager 1 can benefit from deviating to X_2. Thus,

$$f_1(y,y)\ge f_1(x,y). \tag{15}$$

For similar reasons,

$$f_2(y,y) \geq f_2(y,x). \tag{16}$$

Next we prove that $f_1(x,x) \geq f_1(y,x)$. Let p be a number in $(0,1)$ and let $A = \{X_1, X_2\}$ be a set of two actions X_1 and X_2 with a joint probability distribution,

$X_1 \backslash X_2$	x	y
x	0	p
z	$1-p$	0

A direct computation shows that

$$\mathbf{E}[f_1(X_1,X_1)] - \mathbf{E}[f_1(X_2,X_1)] = p(f_1(x,x) - f_1(y,x)) + (1-p)(f_1(z,z) - f_1(x,z)).$$

Recall that $f_1(z,z) - f_1(x,z)$ is bounded. If $f_1(x,x) < f_1(y,x)$, then there is p smaller than, but sufficiently close to 1, such that for every z, $\mathbf{E}[f_1(X_1,X_1)] - \mathbf{E}[f_1(X_2,X_1)] < 0$. In other words,

$$\mathbf{E}[f_1(X_1,X_1)] < \mathbf{E}[f_1(X_2,X_1)]. \tag{17}$$

Now one can choose z to be sufficiently large, so that $\mathbf{E}[X_1] > \mathbf{E}[X_2]$. Inequality (17) implies that (X_1, X_1) is not an equilibrium in G_f, since manager 1 can benefit from deviating to X_2. Hence,

$$f_1(x,x) \geq f_1(y,x). \tag{18}$$

A similar argument shows that

$$f_2(x,x) \geq f_2(x,y). \tag{19}$$

We now sum up inequalities (15), (16), (18), and (19) to obtain, $f_1(y,y) + f_2(y,y) + f_1(x,x) + f_2(x,x) \geq f_1(x,y) + f_2(y,x) + f_1(y,x) + f_2(x,y)$. Due to Eq. (2), equality holds. Thus, (15), (16), (18), and (19) are actually equalities. Therefore,

$$f_1(x,x) = f_1(x,y) = f_1(y,x) = f_1(y,y),$$

and the proof is complete. □

Corollary 1. *If for every set of assets A, every optimal profile of strategies in G_f constitutes a Nash equilibrium, then every profile of strategies in a Nash equilibrium.*

Proof. Let f be a reward scheme sustaining the conditions of the corollary. Clearly, Theorem 4 implies that the result holds for the case of $k = 2$. Fix $k \geq 3$. We prove the theorem by showing that for every manager i and for every vector of outcomes $r \in \mathbb{R}^k$, the ith coordinate $f_i(r)$ of the reward scheme is non-decreasing and non-increasing in r_i.

Assume to the contrary that there exists a manager i, a vector of outcomes $r \in \mathbb{R}^k$, and $w_i \in \mathbb{R}$, such that $f_i(w_i, r_{-i}) > f_i(r_i, r_{-i})$ where $w_i < r_i$. Define the random variable X such that $\mathbf{Pr}(X = x) > 0$ if $x = r_j$ when $1 \leq j \leq k$. Assume that $\mathbf{Pr}(X = r_i) > \mathbf{Pr}(X = r_j)$ for every $j \neq i$. In addition, define a set of i.i.d. random variables $X_j \sim X$ where $1 \leq j \leq k$. Define the vector-valued random variable (W, X_{-i}) by

$$\mathbf{Pr}((W, X_{-i}) = x) = \mathbf{Pr}((X_i, X_{-i}) = x), \ \ \forall x \neq r,$$

and

$$\mathbf{Pr}((W, X_{-i}) = (w_i, r_{-i})) = \mathbf{Pr}((X_i, X_{-i}) = r).$$

Clearly, (W, X_{-i}) and W are well defined. A direct computation shows that $\mathbf{E}[W] < \mathbf{E}[X]$. However, the vector (X_i, X_{-i}) is not an equilibrium, as manager i can deviate to W and increase his payoff, since $f_i(w_i, r_{-i}) > f_i(r_i, r_{-i})$. Hence, $f_i(\cdot, r_{-i})$ is non-decreasing for every i and every r_{-i}.

Now assume to the contrary that $f_i(r_i, r_{-i})$ is strictly increasing in r_i. That is, there exists a manager i, a vector of outcomes $r \in \mathbb{R}^k$, and $y_i \in \mathbb{R}$, such that $f_i(y_i, r_{-i}) > f_i(r_i, r_{-i})$ where $y_i > r_i$.

Let $\bar{z}, \underline{z} \in \mathbb{R}$ be two real numbers such that $\bar{z} > r_j > \underline{z}$ for every $1 \leq j \leq k$ and let p be a number in $(0,1)$. Define the random variable Y such that, w.p. p, it follows that $\mathbf{Pr}(Y = r_j) > 0$ for every $1 \leq j \leq k$. Assume that $\mathbf{Pr}(Y = r_i) > \mathbf{Pr}(Y = r_j)$ for every $j \neq i$. In addition, w.p. $1 - p$, the random variable Y equals $\bar{z}$. Define a set of i.i.d. random variables $Y_j \sim Y$ where $1 \leq j \leq k$. Define the vector-valued random variable (Z, Y_{-i}) by

$$\mathbf{Pr}((Z, Y_{-i}) = y) = \mathbf{Pr}((Y_i, Y_{-i}) = y) \ \ \forall y \neq r, y_j \neq \bar{z} \, \forall j,$$

$$\mathbf{Pr}((Z, Y_{-i}) = (y_i, r_{-i})) = \mathbf{Pr}((Y_i, Y_{-i}) = r),$$

and if there exists a coordinate j of $y \in \mathbb{R}^k$ such that $y_j = \bar{z}$, then

$$\mathbf{Pr}((Z, Y_{-i}) = (\underline{z}, y_{-i})) = \mathbf{Pr}((Y_i, Y_{-i}) = y).$$

Clearly, (Z, Y_{-i}) and Z are well defined. Note that

$$\begin{aligned}
\mathbf{E}[f_i(Z, Y_{-i})] &= \mathbf{E}\left[f_i(Y_i, Y_{-i})\mathbf{1}_{\{Y_{-i}\neq r, Y_j \neq \bar{z}\ \forall j\}}\right] + f_i(y_i, r_{-i})\mathbf{Pr}((Y_i, Y_{-i}) = r) \\
&\quad + \sum_{\substack{y \in \mathbb{R}^k: \\ \exists j, y_j = \bar{z}}} f_i(\underline{z}, y_{-i})\mathbf{Pr}((Y_i, Y_{-i}) = y) \\
&> \mathbf{E}\left[f_i(Y_i, Y_{-i})\mathbf{1}_{\{Y_j \neq \bar{z}\ \forall j\}}\right] + \sum_{\substack{y \in \mathbb{R}^k: \\ \exists j, y_j = \bar{z}}} f_i(\underline{z}, y_{-i})\mathbf{Pr}((Y_i, Y_{-i}) = y) &(20) \\
&= \mathbf{E}\left[f_i(Y_i, Y_{-i})\right] + \sum_{\substack{y \in \mathbb{R}^k: \\ \exists j, y_j = \bar{z}}} (f_i(\underline{z}, y_{-i}) - f_i(y))\mathbf{Pr}((Y_i, Y_{-i}) = y), &(21)
\end{aligned}$$

where Ineq. (20) follows from the assumption that $f_i(y_i, r_{-i}) > f_i(r_i, r_{-i})$. The sum in Eq. (21) is bounded, therefore we can choose a p sufficiently close to 1 (but still smaller than 1), such that $\mathbf{E}[f_i(Z, Y_{-i})] > \mathbf{E}[f_i(Y_i, Y_{-i})]$ for every $\bar{z}$. Taking a sufficiently large $\bar{z}$ and a sufficiently low $\underline{z}$ guarantees that $\mathbf{E}[Y] > \mathbf{E}[Z]$.

In conclusion, the vector (Y_i, Y_{-i}) is not an equilibrium, as manager i can deviate to Z and increase his payoff. A contradiction. Hence, $f_i(\cdot, r_{-i})$ is non-increasing for every i and every r_{-i}. The combination of the two results proves that f_i is independent of the ith coordinate. This implies that the expected payoff of every manager i is independent of the manager's actions and that every profile of actions is an equilibrium. □

Proposition 1. *There is a set of assets A such that the game G_f, induced by f defined in* Eq. (7)*, has no Nash equilibrium.*

Proof. The proof is divided into three parts. First, we define two pure actions X and Y. Next, we formulate and prove a claim regarding the properties of the random variable Y. Finally, we use the second part (specifically, Claim 1) to show that the investment game G_f has no equilibrium.

First part: fixing a set of actions A.

Assume that there are only two players ($k = 2$), and only two actions X and Y, both optimal. Fix a large $M > 0$. The reward scheme f, defined in Eq. (7), equals

$$f_i(r_1, r_2) = \frac{1}{2} + \frac{\phi(r_i - r_{-i})}{4M}.$$

Let $X \equiv 0$ be a constant random variable. For every $n \in \mathbb{N}$, denote $Y_n^{\pm} = \pm 2^n + \frac{5n-4}{10}(-1)^n$, and define the random variable Y such that

$$\mathbf{Pr}\left(Y = Y_n^+\right) = \mathbf{Pr}\left(Y = Y_n^-\right) = \frac{3}{2 \cdot 4^n}.$$

Note that Y is well defined because

$$\sum_{n=1}^{\infty}\left[\mathbf{Pr}(Y = Y_n^{\pm})\right] = 2\sum_{n=1}^{\infty}\frac{3}{2 \cdot 4^n} = 3 \cdot \frac{1}{4} \cdot \frac{1}{1 - \frac{1}{4}} = 1.$$

Since $\sum_{n\in\mathbb{N}} \frac{3Y_n^+}{2\cdot 4^n}$ and $\sum_{n\in\mathbb{N}} \frac{3Y_n^-}{2\cdot 4^n}$ converge absolutely,

$$\begin{aligned}
\mathbf{E}[Y] &= \sum_{n\in\mathbb{N}}\left[\frac{3(Y_n^+ + Y_n^-)}{2 \cdot 4^n}\right] \\
&= \frac{3}{2}\sum_{n=1}^{\infty}\left[\frac{2 \cdot \frac{5n-4}{10}(-1)^n}{4^n}\right] \\
&= \frac{3}{2}\left[\sum_{n=0}^{\infty} n\left(-\frac{1}{4}\right)^n\right] + \frac{3}{10}\left[\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n\right] \\
&= \frac{3}{2}\left[\frac{-\frac{1}{4}}{(1+\frac{1}{4})^2}\right] + \frac{3}{10}\left[\frac{1}{1+\frac{1}{4}}\right] = 0,
\end{aligned}$$

where in the last line we used the Taylor expansions $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ and $\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} nx^n$ taking $x = -\frac{1}{4}$. Therefore, $\mathbf{E}[X] = \mathbf{E}[Y] = 0$.

Second part: properties of the random variable Y.

Claim 1. *For every $M > 0$, there exists $n_+, n_- > M$ such that,*

$$\mathbf{Pr}(Y > n_\pm) = \mathbf{Pr}(Y < -n_\pm), \tag{22}$$

and

$$\pm\mathbf{E}[Y\mathbf{1}_{\{|Y| \le n_\pm\}}] > 0. \tag{23}$$

Proof. Fix $M > 0$. For every $n \in \mathbb{N}$, denote $\gamma_n = (-1)^n \cdot \frac{5n-4}{10}$ and recall that $Y_n^\pm = \pm 2^n + \gamma_n$. As n grows unbounded, γ_n, which is linear in n, becomes relatively small compared to $\pm 2^n$. Hence, we can choose $N > M$ such that $\max\{\pm\gamma_n\} + 2 < \min\{2^n \pm \gamma_{n+1}\}$ for every $n \ge N$. There are two cases we consider:

- If $n \ge N$ is even, then $\gamma_{n+1} < 0 < \gamma_n$, and we can choose $k_n \in \mathbb{N}$ such that
$$k_n \in (\gamma_n, 2^n + \gamma_{n+1}) = (\max\{\pm\gamma_n\}, \min\{2^n \pm \gamma_{n+1}\}).$$
Note that $k_n \in (-\gamma_n, 2^n - \gamma_{n+1})$.
- If $n \ge N$ is odd, then $\gamma_n < 0 < \gamma_{n+1}$, and we can choose $k_n \in \mathbb{N}$ such that
$$k_n \in (-\gamma_n, 2^n - \gamma_{n+1}) = (\max\{\pm\gamma_n\}, \min\{2^n \pm \gamma_{n+1}\}).$$
Note that $k_n \in (\gamma_n, 2^n + \gamma_{n+1})$.

In any case $\gamma_n < k_n < 2^n + \gamma_{n+1}$, which implies that

$$Y_n^+ = 2^n + \gamma_n < k_n + 2^n < 2^{n+1} + \gamma_{n+1} = Y_{n+1}^+,$$

and $-\gamma_n < k_n < 2^n - \gamma_{n+1}$, which implies that

$$Y_{n+1}^- = -2^{n+1} + \gamma_{n+1} < -k_n - 2^n < -2^n + \gamma_n = Y_n^-.$$

To conclude, we proved that for every $M > 0$, one can choose $N > M$, such that for every $n \ge N$, there exists a natural number $k_n \in \mathbb{N}$ such that $2^n + k_n \in (Y_n^+, Y_{n+1}^+)$, and $-2^n - k_n \in (Y_{n+1}^-, Y_n^-)$.

To prove Eq. (22) holds, take $n_+ = 2^n + k_n$ such that n is even (the previous conclusions hold for every $n \ge N$). Clearly,

$$\{Y > n_+\} = \{Y = Y_k^+ : k \ge n+1\}, \text{ and } \{Y < -n_+\} = \{Y = Y_k^- : k \ge n+1\}. \tag{24}$$

Since $\{Y = Y_n^+\}_{n\in\mathbb{N}}$ and $\{Y = Y_n^-\}_{n\in\mathbb{N}}$ are symmetric in terms of probability, in the sense that $\mathbf{Pr}(Y = Y_n^+) = \mathbf{Pr}(Y = Y_n^-) = \frac{3}{2\cdot 4^n}$ for every $n \in \mathbb{N}$, Eq. (22) holds. As this holds for every $n \ge N$, we can fix $n_- = 2^{n+1} + k_{n+1}$ and get the same result. Note that $n+1$ is odd.

Now, we prove Ineq. (23) using the previously defined n_+ and n_-. By the same reasoning presented in the LHS of line (24),

$$\begin{aligned}
\mathbf{E}[Y\mathbf{1}_{\{|Y| \le n_+\}}] &= \mathbf{E}[Y] - \mathbf{E}[Y\mathbf{1}_{\{|Y| > n_+\}}] \\
&= 0 - \sum_{k=n+1}^{\infty} \left[(Y_k^+ + Y_k^-)\mathbf{Pr}(\omega_k)\right] \\
&= -\sum_{k=n+1}^{\infty} \left[(2^n + \gamma_k - 2^n + \gamma^k)\frac{3}{2\cdot 4^k}\right] \\
&= -3\sum_{k=n+1}^{\infty} \frac{\gamma_k}{4^k}.
\end{aligned}$$

Inserting $\gamma_k = (-1)^k \cdot \frac{5k-4}{10}$ yields

$$\mathbf{E}[Y\mathbf{1}_{\{|Y| \le n_+\}}] = \frac{3}{10}\sum_{k=n+1}^{\infty} \frac{(-1)^{k+1}(5k-4)}{4^k}.$$

The last sum is a Leibniz Series, and it is bounded between the first term (which is positive since n is even) and 0. Thus,

$$0 < \sum_{k=n+1}^{\infty} \frac{(-1)^{k+1}(5k-4)}{4^k} < \frac{(-1)^{n+1+1}(5(n+1)-4)}{4^{n+1}} = \frac{5n+1}{4^{n+1}},$$

and $\mathbf{E}[Y\mathbf{1}_{\{|Y|\leq n_+\}}] > 0$. Since $n_- = 2^{n+1} + k_{n+1}$ and $n+1$ is odd, the same computation shows that

$$\mathbf{E}[Y\mathbf{1}_{\{|Y|\leq n_-\}}] = \frac{3}{10}\sum_{k=n+2}^{\infty} \frac{(-1)^{k+1}(5k-4)}{4^k} < 0,$$

which proves Ineq. (23), and concludes the proof of Claim 1. □

Third part: the investment game G_f has no equilibrium.

To simplify the computations, we take the induced investment game G_f and subtract 0.5 for each player's utility function to get a symmetric, 2-player, zero-sum game G. In this part we prove that the auxiliary game G has no equilibrium, implying that G_f has no equilibrium as well.

Assume, to the contrary, that G has an equilibrium $\sigma = (\sigma_1, \sigma_2)$. Since G is a symmetric zero-sum game, every player can guarantee a payoff of at least 0 by playing the same action as the other player. Therefore the payoffs of both players given σ are 0. For every $i = 1, 2$, assume that

$$\sigma_i = (1-\alpha_i)X + \alpha_i Y = \alpha_i Y$$

where $\alpha_i \in [0,1]$. Hence, $\mathbf{E}[\phi(\sigma_1 - \sigma_2)] = \mathbf{E}[\phi([\alpha_1 - \alpha_2]Y)] = 0$. Since σ is an equilibrium, it follows that $\mathbf{E}[\phi([\alpha - \alpha_2]Y)] \leq 0$ for every $0 \leq \alpha \leq 1$. We will show that there exists $\alpha \in [0,1]$ such that $\mathbf{E}[\phi([\alpha - \alpha_2]Y)] > 0$.

There are two cases we need to consider: $\alpha_2 < 1$ and $\alpha_2 = 1$. We begin with the first case. Assume that $\alpha_2 < 1$ and consider $\epsilon = \alpha - \alpha_2 > 0$ such that $\alpha \in (\alpha_2, 1)$.

$$\begin{aligned}\mathbf{E}[\phi([\alpha-\alpha_2]Y)] &= \mathbf{E}[\phi(\epsilon Y)] \\ &= \mathbf{E}[\epsilon Y\mathbf{1}_{\{|\epsilon Y|\leq M\}}] + M\left[\mathbf{Pr}(\epsilon Y > M) - \mathbf{Pr}(\epsilon Y < -M)\right] \\ &= \epsilon\mathbf{E}[Y\mathbf{1}_{\{|Y|\leq \frac{M}{\epsilon}\}}] + M\left[\mathbf{Pr}\left(Y > \frac{M}{\epsilon}\right) - \mathbf{Pr}(Y < -\frac{M}{\epsilon})\right].\end{aligned}$$

Claim 1 holds for every $M > 0$, hence there exist unbounded sequences of $\{n_\pm\} \subset \mathbb{R}$ such that Eq. (22) and Ineq. (23) hold. Thus, we can choose a small enough $\epsilon > 0$ such that $\alpha < 1$ and $\frac{M}{\epsilon} = n_+$. Inserting $\frac{M}{\epsilon} = n_+$ into the previous equation yields

$$\begin{aligned}\mathbf{E}[\phi([\alpha-\alpha_2]Y)] &= \epsilon\mathbf{E}[Y\mathbf{1}_{\{|Y|\leq n_+\}}] + M\left[\mathbf{Pr}(Y > n_+) - \mathbf{Pr}(Y < -n_+)\right] \\ &= \epsilon\mathbf{E}[Y\mathbf{1}_{\{|Y|\leq n_+\}}] + 0 > 0,\end{aligned}$$

where the last line follows from Eq. (22) and Ineq. (23). Therefore, there exists an $\alpha \in (\alpha_2, 1)$ such that $\mathbf{E}[\phi([\alpha - \alpha_2]Y)] > 0$. Contradiction.

Now assume that $\alpha_2 = 1$, and consider $\alpha = \alpha_2 - \epsilon$ such that $\alpha \in (0, \alpha_2)$. Then

$$\begin{aligned}\mathbf{E}[\phi([\alpha-\alpha_2]Y)] &= \mathbf{E}[\phi(-\epsilon Y)] \\ &= \mathbf{E}[-\epsilon Y\mathbf{1}_{\{|-\epsilon Y|\leq M\}}] + M\left[\mathbf{Pr}(-\epsilon Y > M) - \mathbf{Pr}(-\epsilon Y < -M)\right] \\ &= -\epsilon\mathbf{E}[Y\mathbf{1}_{\{|Y|\leq \frac{M}{\epsilon}\}}] + M\left[\mathbf{Pr}\left(Y < -\frac{M}{\epsilon}\right) - \mathbf{Pr}(Y > \frac{M}{\epsilon})\right].\end{aligned}$$

Similarly to the previous conclusion, we use Claim 1 to choose a small enough $\epsilon > 0$ such that $\alpha > 0$ and $\frac{M}{\epsilon} = n_-$. Therefore,

$$\begin{aligned}\mathbf{E}[\phi([\alpha-\alpha_2]Y)] &= -\epsilon\mathbf{E}[Y\mathbf{1}_{\{|Y|\leq n_-\}}] + M\left[\mathbf{Pr}(Y < -n_-) - \mathbf{Pr}(Y > n_-)\right] \\ &= -\epsilon\mathbf{E}[Y\mathbf{1}_{\{|Y|\leq n_-\}}] + 0 > 0,\end{aligned}$$

and there exists an $\alpha \in (0, \alpha_2)$ such that $\mathbf{E}[\phi([\alpha - \alpha_2]Y)] > 0$. Contradiction. This completes the proof of Proposition 1. □

Theorem 5. *If U is bounded and uniquely maximized, then the following* General Reward Scheme f *is optimal:*

$$f_i(r) = \frac{1}{k} + \frac{1}{2Mk}\left[U(kr_i) - \frac{1}{k-1}\sum_{j\neq i} U(kr_j)\right].$$

Proof. Let $q^* \in Q$ be the unique diversified action such that $E[U(kq^*)] > E[U(kq)]$ for every $q \in Q \setminus \{q^*\}$. We prove that for every manager i, for every profile of diversified actions $(\sigma_1, \sigma_2, \ldots, \sigma_k) \in Q^k$ of managers $1, \ldots, k$ respectively, and for every strategy $\sigma_i \neq q^*$ of manager i, the inequality

$$\mathbf{E}\left[f_i\left(\sigma_1, \ldots, \sigma_{i-1}, q^*, \sigma_{i+1}, \ldots, \sigma_k\right)\right] > \mathbf{E}\left[f_i\left(\sigma_1, \ldots, \sigma_k\right)\right],$$

holds.

Assume w.l.o.g. that $i = 1$ and

$$\begin{aligned}
\mathbf{E}\left[f_1\left(q^*, \sigma_2, \ldots, \sigma_k\right)\right] &= \mathbf{E}\left[\frac{1}{k} + \frac{1}{2Mk}\left(U(kq^*) - \frac{1}{k-1}\sum_{j=2}^{k} U(k\sigma_j)\right)\right] \\
&> \frac{1}{k} + \frac{1}{2Mk}\left[\mathbf{E}\left[U(k\sigma_1)\right] - \frac{1}{k-1}\sum_{j=2}^{k}\mathbf{E}\left[U(k\sigma_j)\right]\right] \\
&= \mathbf{E}\left[f_1\left(\sigma_1, \sigma_2, \ldots, \sigma_k\right)\right].
\end{aligned}$$

Note that the inequality is independent of σ_{-i}, thus implying that q^* is a strictly dominant strategy for every manager i. Thus, $(q^*, \ldots, q^*) \in Q^k$ is a dominant-strategy equilibrium. Since all the managers use the same diversified action q^* in the unique equilibrium σ, Eq. (3) is satisfied and the result follows. □

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