Set-Valued Capacities: Multi-Agenda Decision Making*

Ehud Lehrer[†] and Roee Teper[‡]

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Abstract

We study the problem in which a set of agents are required to produce across several different projects (or more generally, agendas) and we consider environments in which resources are constrained and investing (say, time or effort) in one agenda reduces the ability to invest in other agendas. To this end we introduce a class of capacities we refer to as set-valued: the value of each coalition is a subset of a vector space. For a particular coalition, each vector in its value is associated with a different distribution of the resources invested across the different agendas. In this context the Choquet and the concave integrals are defined, characterized, and shown to be identical if and only if the underlying set-valued capacity is supermodular. We apply the tools developed and introduce a new decision theory.

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†School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel and INSEAD, Bd. de Constance, 77305 Fontainebleau Cedex, France. e-mail: lehrer@post.tau.ac.il. This research was supported in part by ISF through Grant #963/15.

‡Department of Economics, University of Pittsburgh, 230 South Bouquet St., Pittsburgh, 15260, USA. email: rteper@gmail.com.

1 Introduction

Typically a *capacity* measures the 'worth' of any sub-group of individuals. This value can be interpreted as the productivity of the sub-group under consideration when investing time and effort in a particular enterprise, its social or political power, etc.[1] Often times though there are multiple enterprises that could be invested in while the resources, say time, are constrained. In such scenarios there are many different time allocations to each of the different projects, each allocation resulting in a different outcome. It seems that if one wishes to model such tradeoffs without pre-committing to a specific time allocation, and thereby to a specific level of production in each project, one needs to consider a more general notion of a capacity than the classical ones.[2]

In this paper we introduce and study the concept of *set-valued capacities*: each sub-group of individuals is associated with a *set of real valued vectors*. The set of vectors associated with a subgroup is *all* possible production possibilities across the different enterprises. Note that this approach does not take a stand on the aggregation of (or preferences over) payoffs across the different agendas[3]. We seek to study the primitive, as opposed to the reduced form, and wish to consider a model that is robust to the aggregation process across the different agendas. As in the classical theory, we address the issue of integration.

Suppose that each worker i out of a set of workers N can invest only X_i time units. If we wish to measure the productivity of the grand group conditional on these individual time constraints, we need to extend the set-valued capacity (henceforth, SVC), $\mathbf{v}$, from coalitions to a given time profile $(X_1, ..., X_n)$. We generalize the definition of the concave integral (Lehrer [15] and Lehrer and Teper [17]), which is a natural aggregator when the underlying objective is maximizing productivity. We also introduce the

[1] Another common use of capacities is in the context of uncertainty. The capacity is defined over a set of states and "measures" the likelihood of each subset.

[2] A different problem is introduced in Gonzalez and Grabisch [8]. They study a classical one project problem with coalitional-time constraints.

[3] These, for example, could be the preferences of the different individuals (or a third party, say, a manager, the market, etc.). This point is further discussed in Section 5.

counterpart of the Choquet integral[4] in the current setting[5]. We study both integrals and characterize them. Lastly, in order to understand exactly when it is that the two aggregators yield different outcomes, a necessary and sufficient condition is provided under which the two notions coincide, generalizing the classical result by Lovàsz [18].

We apply the idea of integration with respect to SVCs and introduce a decision theory. The primitive in this theory is a preference relation over individuals' time investment profiles ($\mathbb{R}^n_+$). Such preferences are induced by the respective profiles with respect to an SVC. These preferences are subjective and are determined by the SVC and the integration mechanism. These preferences are clearly incomplete, since we do not take a stand on how the different agendas are aggregated (this is related to the point discussed above). At this stage we consider a general utility function (or, preferences) over the different production possibilities across the different agendas, and resort to such utility to complete our preferences over time investment profiles.

The paper is organized as follows. Section 2 introduces the notion of set-valued capacities. Section 3 defines the concave integral w.r.t. SVCs and studies its characterization. Section 4 addresses similar questions regarding the Choquet integral. It also provides a necessary and sufficient condition for the two integration schemes to coincide. Section 5 introduces a decision theory founded on integration with respect to SVCs. Final comments are discussed in Section 6. All proofs appear in the Apendix.

[4]The incorporation of Choquet's theory [3] to decision sciences is due to Schmeidler [22]. He proposed an alternative to the classical subjective expected utility theory due to findings by Ellsberg [5] that such theory is not consistent with experimental findings (later Schmeidler's theory was extended to the domain of risk due to similar paradoxes that were raised by Allais [1]).

[5]Beyond Choquet integral and the concave integral there are other integral schemes that are not discussed here (see Grabisch [10] for a recent comprehensive monograph on the topic). Examples are the partitional integral (also known as Riemann or Pan integral; see Wang et al. [26]) and Shilkret [23]. The first two are types of decomposition integrals (see Even and Lehrer [6]) that can be expressed in terms of decompositions of the time X between different subgroups of N.

2 Set-Valued Capacities

2.1 The definition

Let $N = \{1, ..., n\}$ be a finite set of players and $k \in \mathbb{N}$. A *coalition* S is any subset of N, $S \subseteq N$. According to the classical definition, a *capacity* (also known as a cooperative game) v associates a real valued number, $v(S)$, to each coalition $S \subseteq N$. A *set-valued capacity* is a generalization of this notion and associates to each coalition a *subset of vectors in* $\mathbb{R}^k_+$.

Definition 1. *A* set-valued capacity (SVC) *over a collection* N *of players, is a function* $\mathbf{v} : 2^N \to 2^{\mathbb{R}^k_+}$ *defined over all subsets of* N*, and satisfies:*

1. $\mathbf{v}(\emptyset) = \{0\}$*;*

2. Closedness*. For every* $S \subseteq N$, $\mathbf{v}(S)$ *is a closed set;*

3. Comprehensiveness*. If* $x \in \mathbf{v}(S)$ *and* $y \in \mathbb{R}^k_+$ *is such that*[6] $y \leq x$*, then* $y \in \mathbf{v}(S)$*.*[7]

4. Monotonicity*. If* $S \subseteq T \subseteq N$*, then* $\mathbf{v}(S) \subseteq \mathbf{v}(T)$*.*

We will explicitly refer to a classical capacity as such, as opposed to the more general set-valued capacity that will be referred to as *SVC*. In the remainder of this section we discuss possible properties of an SVC and its relation to capacities. The role of the definitions and examples we introduce is to motivate the notion of an SVC, and to familiarize the reader with this notion and related definitions pertinent to the analysis in subsequent sections.

2.2 Set-valued and classical capacities

The values that an SVC takes are subsets of $\mathbb{R}^k_+$. A capacity, on the other hand, is defined, like an SVC, on subsets of N, but takes numbers as values. However, there is a natural connection between the two notions. A capacity v is related to SVC $\mathbf{v}$

[6] $y \leq x$ means that $y_j \leq x_j$ for every $j = 1, ..., k$.

[7] More generally, if φ is a function with domain $\mathbb{R}^N_+$ and range $2^{\mathbb{R}^k_+}$, then we say that φ is *closed* if $\varphi(X)$ is closed for every $X \in \mathbb{R}^n_+$. Furthermore, φ is *comprehensive* if for every $X \in \mathbb{R}^N_+$, whenever $x \in \varphi(X)$ and $y \in \mathbb{R}^k_+$ such that $y \leq x$, then $y \in \varphi(X)$.

that takes values in $\mathbb{R}_+$, that is, $k = 1$. In this case the values that $\mathbf{v}$ takes are closed intervals whose left side is 0. The capacity associated with such $\mathbf{v}$ is defined as $v(S) = \max \mathbf{v}(S)$ for every $S \subseteq N$. And vice versa: if v is a capacity then the SVC associated with it is the one defined as $\mathbf{v}(S) = [0, v(S)]$ for every $S \subseteq N$.

2.3 A multi-capacity based SVC

Consider a family of $k \in \mathbb{N}$ capacities $v_1, ..., v_k$, and define a SVC $\mathbf{v}$ by

$$\mathbf{v}(S) := \left\{ (\alpha_1 v_1(S), ..., \alpha_k v_k(S)) \in \mathbb{R}^k_+;\ \alpha_j \geq 0\ \forall j, \sum_{j=1}^{k} \alpha_j \leq 1 \right\}. \tag{1}$$

Note, if all v_j's are capacities (in particular monotonic), then $\mathbf{v}$ satisfies all the properties in Definition 1 and is indeed an SVC.[8]

This definition could be interpreted as follows. Consider $v_j(S)$ to be the amount of project j coalition S can complete in one unit of time. Therefore, if the coalition invests $\alpha_j(S)$ in project j, and if $\sum_{j=1}^k \alpha_j(S) \leq 1$, then in one time unit, a vector $(\alpha_1 v_1(S), ..., \alpha_k v_k(S))$ can be produced by coalition S. An SVC of this kind is referred to as *multi-capacity based*.

The base capacities (v_j's) capture the stand-alone productivity (or, value) for each project. The SVC ($\mathbf{v}$) defined through these capacities reflects the natural time constraints and tradeoffs between investing in the different projects. Clearly, not every SVC is multi-capacity based, but the interpretation remains: $\mathbf{v}(S)$ captures the productivity of coalition S and reflects the time constraints faced by the coalition when contemplating how much time (or, more generally, resources) to invest in each project.

The following example illustrates the structure of a multi-capacity based SVC.

Example 1. *Let* $N = \{1, 2, 3\}, k = 2$, *and define the following SVC:* $\mathbf{v}(N) = \{(w_1, w_2);\ w_1 + w_2 \leq 3\}, \mathbf{v}(1, 2) = \{(w_1, w_2);\ \ 2w_1 + w_2 \leq 3\}, \mathbf{v}(2, 3) = \{(w_1, w_2);\ w_1 + 2w_2 \leq 3\}, \mathbf{v}(1, 3) = \{(w_1, w_2);\ w_1 + w_2 \leq 2\}$ *and* $\mathbf{v}(i) = \{(w_1, w_2);\ w_1 + w_2 \leq 1\}, i = 1, 2, 3.$

This SVC is multi-capacity based. Indeed, define two capacities v_1, v_2 *in the following manner.* $v_1(N) = 3, v_1(1, 2) = 1.5, v_1(2, 3) = 3, v_1(1, 3) = 2$, *and* $v_1(i) = 1$ *for*

[8] If $v_j(\emptyset) = 0$ then $\mathbf{v}(\emptyset) = \{0\}$. Closedness is obtained by the weak inequalities of Eq. 1. Comprehensiveness is obtain by the requirement that the sum $\sum_{j=1}^k \alpha_j$ may be smaller, and not necessarily equal, to 1. Lastly, monotonicity is obtain by the monotonicity of all the v_j's.

every player $i \in N$. Similarly, $v_2(N) = 3, v_2(1,2) = 3, v_2(2,3) = 1.5, v_2(1,3) = 2$, and $v_2(i) = 1$ for every player $i \in N$. It is easy to verify that $\mathbf{v}$ is a multi-capacity based (on v_1, v_2) SVG.

2.4 Additive SVCs

An SVC $\mathbf{v}$ is *additive* if for every two disjoint coalitions, $S, T \subseteq N$,

$$\mathbf{v}(S) + \mathbf{v}(T) = \mathbf{v}(S \cup T). \tag{2}$$

It is clear that $\mathbf{v}$ is additive if and only if for every $S \subseteq N$, $\mathbf{v}(S) = \sum_{i \in S} \mathbf{v}(i)$. In terms of productivity, each worker's productivity is independent of the group she is working with: her contribution to a group is always the set of vectors she can produce alone.

Gould integral (see [9]) is defined in a general setup where $\mathbf{v}$ is additive and taking values that are subsets of a Banach space. In our finite case, when $\mathbf{v}$ is additive, the Gould integral is defined as

$$\mathbb{E}_{\mathbf{v}}(X) = \sum_{i=1}^{n} X_i \mathbf{v}(i) \tag{3}$$

for every $X \in \mathbb{R}^n_+$. The notation $\mathbb{E}$ refers to the expectation and is chosen to distinguish it from an integral. Indeed, Eq. (3) resembles that of the expectation and we will later refer to it as the expectation w.r.t. $\mathbf{v}$.

2.5 Convex-valued functions

We say that an SVC v is *convex-valued* if $\mathbf{v}(S)$ is convex for every $S \subseteq N$. More generally, if φ is a function with domain $\mathbb{R}^N_+$ and range $2^{\mathbb{R}^k_+}$, then we say that φ is convex-valued if $\varphi(X)$ is a convex set for every $X \in \mathbb{R}^N_+$. Whenever an SVC $\mathbf{v}$ (or, more generally, a set-valued function φ) is convex-valued and closed we say that it is *convex-closed*.

2.6 The induced capacity

The following definition will be helpful in analyzing the core of an SVC in subsequent sections. Let λ be a non-negative vector in $\mathbb{R}^k$ (i.e., $\lambda \in \mathbb{R}^k_+$) and $\mathbf{v}$ be an SVC over N.

Definition 2. *The* λ-induced-capacity of **v** *is the capacity* v_λ *defined by*[9]

$$v_\lambda(S) = \max_{y \in \mathbf{v}(S)} y \cdot \lambda \tag{4}$$

for every $S \subseteq N$.

One possible interpretation for λ could be that it is a weight function across the different agendas,[10] and that payoff vectors are evaluated according to a separable utility function with weights being λ.

Remark 1. *Note that for every* $S \subseteq N$, $\mathbf{v}(S) \subseteq \{y \in \mathbb{R}^k_+;\ y \cdot \lambda \leq v_\lambda(S)\}$.

3 The Concave Integral w.r.t. SVCs

Definition 3. *Let* **v** *be an SVC and*[11] $X \in \mathbb{R}^n_+$. *The* concave integral *of* X *w.r.t.* **v** *is*

$$\int^{cav} X d\mathbf{v} = \left\{ \sum_{j=1}^{\ell} \alpha_j y_j;\ \sum_{j=1}^{\ell} \alpha_j \mathbf{1}_{A_j} \leq X, y_j \in \mathbf{v}(A_j), \alpha_j \geq 0, A_j \subseteq N,\ j = 1, ..., \ell \right\}. \tag{5}$$

The concave integral has a natural interpretation in the context of production. Indeed, an SVC, **v**, reflects the different production possibilities of every coalition (given one unit of time). Then, if each individual $i \in N$ is time constrained by X_i, then $\int^{cav} X d\mathbf{v}$ reflects *all* production possibilities given the time constraints profile X. In particular, the concave integral takes into account all the tradeoffs between how much time different coalitions invest in the different projects, when the overall time constraint for individual i (regardless of which coalitions she partakes in) is X_i.

Remark 2. *(i) In Eq. (5) one can replace the inequality condition* $\sum_{j=1}^{\ell} \alpha_j \mathbf{1}_{A_j} \leq X$ *with the equality condition* $\sum_{j=1}^{\ell} \alpha_j \mathbf{1}_{A_j} = X$ *to obtain an equivalent definition of the concave integral. This is due to comprehensiveness and monotonicity of* **v**.

(ii) The concave integral for a capacity v*, as defined by Lehrer [15], is given by:*

$$\int^{cav} X dv = \max \left\{ \sum_{i=j}^{\ell} \alpha_j v(A_j);\ \sum_{i=j}^{\ell} \alpha_j \mathbf{1}_{A_j} \leq X, \alpha_j \geq 0, A_j \subseteq N,\ j = 1, ..., \ell \right\}. \tag{6}$$

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[9] $y \cdot \lambda$ denotes the inner product of y and λ.

[10] See also the related discussion in Section 5.

[11] From here on and without explicitly specifying it, X stands for a non-negative vector in $\mathbb{R}^n$.

One can see the differences and the similarities between Eqs. (5) and (6). While the integral in Eq. (5) takes sets as values, the one in Eq. (6) takes numbers as values. When $k = 1$, the set Eq. (5) is an interval whose maximal number equals Eq. (6).

Lemma 1. *For every $\mathbf{v}$ (convex-valued or not), $\int^{cav} \cdot \, d\mathbf{v}$ is convex-valued.*

The following proposition is useful since it ties the concave integral w.r.t. an SVC to standard integration w.r.t. the induced capacities.

Proposition 1. *A set-valued function φ, defined on $\mathbb{R}^n_+$, coincides with $\int^{cav} \cdot \, d\mathbf{v}$ if and only if φ satisfies,*

(i) *For every $X \in \mathbb{R}^n_+$, $\varphi(X)$ is convex-closed and satisfies comprehensiveness; and*

(ii) *For every $\lambda \in \mathbb{R}^k_+$,*

$$\max_{y \in \varphi(X)} y \cdot \lambda = \int^{cav} X dv_\lambda. \tag{7}$$

3.1 A characterization of the concave integral

This section provides a characterization of the concave integral w.r.t. SVCs. It follows the spirit of the characterization of the integral w.r.t. capacities that appears in Lehrer [15]

Definition 4. *Let φ be a set-valued function defined on $\mathbb{R}^n_+$ whose values are subsets of $\mathbb{R}^k_+$.*
(i) φ is homogeneous *if for every $\alpha \geq 0$ and $X \in \mathbb{R}^n_+$, $\varphi(\alpha X) = \alpha\varphi(X)$.*
(ii) φ is concave *if for every $\alpha \in (0, 1)$ and $X, Y \in \mathbb{R}^n_+$,*

$$\alpha\varphi(X) + (1 - \alpha)\varphi(Y) \subseteq \varphi(\alpha X + (1 - \alpha)Y).$$

Fix an SVC $\mathbf{v}$. Beyond the properties in Definition 4, we introduce two additional axioms on φ:

DOM. If $\mathbf{p}$ is an additive capacity and for every $S \subseteq N$, $\mathbf{v}(S) \subseteq \mathbf{p}(S)$, then $\varphi \subseteq \mathbb{E}_{\mathbf{p}}$, that is,

$$\varphi(X) \subseteq \mathbb{E}_{\mathbf{p}}(X)$$

for every $X \in \mathbb{R}^n_+$.

DOM states[12] that if an additive SVC $\mathbf{p}$ dominates $\mathbf{v}$, then the integral w.r.t. $\mathbf{v}$ is dominated by the expectation w.r.t. $\mathbf{p}$.

INCL. For every $S \subseteq N$, $\mathbf{v}(S) \subseteq \varphi(\mathbf{1}_S)$.

Theorem 1. *Fix an SVC* $\mathbf{v}$*. Let* φ *be a set-valued function defined on* $\mathbb{R}^n_+$*. The two following statements are equivalent:*
(i) φ *is homogeneous, concave, comprehensive, convex-closed and satisfies DOM and INCL;*
(ii) $\varphi(X) = \int^{cav} X d\mathbf{v}$ *for every* $X \in \mathbb{R}^n_+$*.*

Remark 3. *The concave integral possesses the monotonicity properties: for every* X, Y *where* $X \leq Y$*,* $\int^{cav} X d\mathbf{v} \subseteq \int^{cav} Y d\mathbf{v}$*.*

4 The Choquet integral

A list $A_1, A_2, ..., A_\ell$ of subsets of N is a *chain* if it is increasing w.r.t. inclusion, that is $A_1 \subseteq A_2 \subseteq ... \subseteq A_\ell$.

Definition 5. *Let* $X \in \mathbb{R}^n_+$*. The Choquet integral w.r.t.* $\mathbf{v}$ *is defines as follows:*

$$\int^{Ch} X d\mathbf{v} = \Bigg\{ \sum_{i=1}^{\ell} \alpha_i y_i; \ \sum_{i=1}^{\ell} \alpha_i \mathbf{1}_{A_i} \leq X, \quad y_i \in \mathbf{v}(A_i), \alpha_i \geq 0, i = 1, ..., \ell \text{ and } A_1, A_2, ..., A_\ell \text{ is a chain} \Bigg\}. \tag{8}$$

Remark 4. *It is obvious from the definitions that for every* X*,* $\int^{Ch} X d\mathbf{v} \subseteq \int^{cav} X d\mathbf{v}$*.*

The following lemma is analogous to Lemma 1, but its proof is a bit more involved and appears in the appendix.

[12] Note that only when the SVC, $\mathbf{v}$, has been fixed, DOM can be formulated. It is possible to be more explicit about this when defining the property, and refer to it as $\mathbf{v}$-DOM. There will be no confusion in what follows as to which SVC DOM refers to. Thus, we will use the abbreviated name DOM. This also holds for additional properties presented below.

Lemma 2. *For every* $\mathbf{v}$ *(convex-valued or not):*

1. $\int^{Ch} \cdot\, d\mathbf{v}$ *is convex-valued; and*

2. $\int^{Ch} \mathbf{1}_S d\mathbf{v} = \text{conv}(\mathbf{v}(S))$*, for every* $S \subseteq N$*.*

The following result provides conditions under which the Choquet integral can be written in a more standard fashion.

Lemma 3. $\mathbf{v}$ *is convex-valued if and only if for every* $X \in \mathbb{R}^n_+$

$$\int^{Ch} X d\mathbf{v} = \left\{ \sum_{i=1}^{n} (X_{\pi(i)} - X_{\pi(i-1)}) y_i;\ y_i \in \mathbf{v}\big(\{\pi(i), ..., \pi(n)\}\big), i = 1, ..., n \right\}, \tag{9}$$

where π *is a permutation on* N *such that* $0 =: X_{\pi(0)} \leq X_{\pi(1)} \leq ... \leq X_{\pi(n-1)} \leq X_{\pi(n)}$*.*

To see why Eq. (9) holds, note that it is sufficient to show that it holds for indicators (that is, $\mathbf{1}_S$ for every $S \subseteq N$). And indeed, Eq. (9) holds for indicators due to item 2 of Lemma 2 above.

Proposition 2. *A set-valued function* φ *defined on* $\mathbb{R}^k_+$ *coincides with* $\int^{Ch} \cdot\, d\mathbf{v}$ *if and only if* φ *satisfies,*

1. *For every* $X \in \mathbb{R}^n_+$*,* $\varphi(X)$ *is convex-closed and satisfies comprehensiveness; and*

2. *For every* $\lambda \in \mathbb{R}^k_+$*,*

$$\max_{y \in \varphi(X)} y \cdot \lambda = \int^{Ch} X dv_\lambda. \tag{10}$$

The proof of this proposition is similar to that of Proposition 1 and is therefore omitted.

4.1 Characterizing the Choquet integral

For the sake of completeness we provide a characterization of the Choquet integral. It follows the footsteps of Schmeidler's characterization in the classical case. The main concept needed is co-monotonicity.

Definition 6. *We say that* $X, Y \in \mathbb{R}^n_+$ *are co-monotonic if for every* $i, j \in N$, $(X_i - X_j)(Y_i - Y_j) \geq 0$.

CO-MON. φ is co-monotonic additive, that is, $\varphi(X+Y) = \varphi(X) + \varphi(Y)$ for every two co-monotonic X, Y.

COINC. For every $S \subseteq N$, $\varphi(\mathbf{1}_S) = \mathbf{v}(S)$.

Theorem 2. *Fix a convex-valued SVC* $\mathbf{v}$. *Let* φ *be a set-valued function defined on* $\mathbb{R}^n_+$. *The two following statements are equivalent:*
(i) φ *is homogeneous, comprehensive, convex-closed and satisfies CO-MON and CO-INC;*
(ii) $\varphi(X) = \int^{Ch} X d\mathbf{v}$ *for every* $X \in \mathbb{R}^n_+$.

Given the existing theory on the Choquet integral, the proof is rather straightforward and is left for the reader. It is worth noting that Theorem 2 deals with convex-valued capacities while Theorem 1 does not impose this restriction. In case the values the capacity obtains are not convex, Choquet integral does not satisfy COINC. Indeed, let $\mathbf{v}$ be an SVC and define a new SVC $\hat{\mathbf{v}}^{Ch}$ by

$$\hat{\mathbf{v}}^{Ch}(S) = \int^{Ch} \mathbf{1}_S d\mathbf{v},$$

for every $S \subseteq N$. When $\mathbf{v}$ is convex-valued then $\mathbf{v} = \hat{\mathbf{v}}^{Ch}$, but $\mathbf{v} \subseteq \hat{\mathbf{v}}^{Ch}$ otherwise. It is immediate however that $\int^{Ch} X d\mathbf{v} = \int^{Ch} X d\hat{\mathbf{v}}^{Ch}$ for every $X \in \mathbb{R}^n_+$. And in particular,[13] $\int^{Ch} \mathbf{1}_S d\mathbf{v} = \text{conv}(\mathbf{v}(S))$. Thus, in the general case the characterization of the Choquet integral would take the following form.[14]

COINC-CONV. For every $S \subseteq N$, $\varphi(\mathbf{1}_S) = \text{conv}(\mathbf{v}(S))$.

[13] $\text{conv}(\mathbf{v}(S))$ denotes the convex hull of $\mathbf{v}(S)$.

[14] Parts of this hold for the concave integral as well. Note that the concave integral satisfies INCL, which is strictly weaker than COINC and even than $\varphi(\mathbf{1}_S) = \text{conv}(\mathbf{v}(S))$. In fact, it is easy to think of examples for which $\int^{cav} \mathbf{1}_S d\mathbf{v}$ strictly contains $\text{conv}(\mathbf{v}(S))$. Nevertheless, we can define $\hat{\mathbf{v}}^{cav}$ by $\hat{\mathbf{v}}^{cav}(S) = \int^{cav} \mathbf{1}_S d\mathbf{v}$ for every $S \subseteq N$, and it still holds that $\int^{cav} X d\mathbf{v} = \int^{cav} X d\hat{\mathbf{v}}^{cav}$ for every $X \in \mathbb{R}^n_+$.

Theorem 3. Fix an SVC $\mathbf{v}$. Let φ be a set-valued function defined on $\mathbb{R}^n_+$. The two following statements are equivalent:
(i) φ is homogeneous, comprehensive, convex-closed and satisfies CO-MON and COINC-CONV;
(ii) $\varphi(X) = \int^{Ch} X d\mathbf{v}$ for every $X \in \mathbb{R}^n_+$.

Remark 5. *Like the concave integral, the Choquet integral satisfies monotonicity. In addition, it satisfies translation-covariance: for every X and a constant $c > 0$, $\int^{Ch} X + c\mathbf{1}_N d\mathbf{v} = \int^{Ch} X d\mathbf{v} + c\mathbf{v}(N)$.*

4.2 Supermodular SVCs

In this section we provide a condition that is necessary and sufficient for the two integrals to coincide.

Definition 7. *An SVC $\mathbf{v}$ is* supermodular *if for every $S, T \subseteq N$,*

$$\mathbf{v}(S) + \mathbf{v}(T) \subseteq \mathbf{v}(S \cup T) + \mathbf{v}(S \cap T).$$

Theorem 4. *$\int^{Ch} \cdot \, d\mathbf{v} = \int^{cav} \cdot \, d\mathbf{v}$ if and only if $\mathbf{v}$ is supermodular.*

5 SVCs and Decision Making

In the sections above we introduced the notion of an SVC and proposed a theory of aggregation with respect to it. This section shows how these concepts can be applied in the context of decision making.

To that end fix an SVC, $\mathbf{v}$, and an integral operator $\int \cdot d\mathbf{v}$. The integral can be the Choquet integral, the concave integral, or perhaps another aggregation operator. The (chosen) integral with respect to the SVC $\mathbf{v}$ induces a natural ranking of all time-constraint profiles across the different individuals $1, ..., n$.

Formally, for any two time-constraint profiles $X, Y \in \mathbb{R}^n_+$ define a partial order $\succeq$ as follows:

$$X \succeq Y \quad \text{if and only if} \quad \int X d\mathbf{v} \supseteq \int Y d\mathbf{v}. \tag{11}$$

The interpretation is that a time-constraint profile X is preferred to a time-constraint profile Y if and only if the production possibilities given the SVC $\mathbf{v}$ under the profile X include all the production possibilities under the profile Y. Under comprehensiveness (see Definition 1) the condition in Eq. 11 is also equivalent to the Pareto frontier of $\int X d\mathbf{v}$ dominating that of $\int Y d\mathbf{v}$.

The preference relation $\succeq$ given in Eq. (11) is typically partial, that is, there might be two time-constraint profiles X and Y such that neither $X \succeq Y$ nor $Y \succeq X$. This brings us to the issue of selection out of $\int X d\mathbf{v}$ and the completion of $\succeq$.

As discussed in the introduction, unlike the classical theory of capacities, $\mathbf{v}$ and its extension to $\int X d\mathbf{v}$ explicitly model the tradeoffs of investing in different projects conditional on the time investment profile X. Nevertheless, it is quite intuitive to consider how the preferences[15] (over elements in $\mathbb{R}^k_+$) of a third party, say a manager who is responsible to the overall production conditional on market demand, prices, etc., shape the selection out of $\int X d\mathbf{v}$ for every time-constraint profile X. More formally, consider a utility function $U : \mathbb{R}^k_+ \to \mathbb{R}$ aggregating the different values across the k different agendas. Given a time-constraint profile X, the selection out of $\int X d\mathbf{v}$ given the utility function U will be[16] $\text{argmax}_{x \in \int X d\mathbf{v}} U(x)$.

Now, define a preference relation $\succeq^*$ over $\mathbb{R}^n_+$ as follows: for any two time-constraint profiles $X, Y \in \mathbb{R}^n_+$,

$$X \succeq^* Y \quad \text{if and only if} \quad \max_{x \in \int X d\mathbf{v}} U(x) \geq \max_{y \in \int Y d\mathbf{v}} U(y). \tag{12}$$

Notice, $\succeq^*$ is a complete preference relation and is a completion of $\succeq$. That is,

$$X \succeq Y \;\Rightarrow\; X \succeq^* Y.$$

It turns out that whenever U is linear (and monotone), that is there exists a $\lambda \in \mathbb{R}^n_+$ such that $U(x) = \lambda \cdot x$, for every $x \in \mathbb{R}^n_+$, the selection according to λ is equivalent to taking the integral with respect to the induced capacity v_λ. In other words, there is

[15] Note that these preferences over production possibilities are not the primitive. They do, however, together with the SVC and the integration method, determine the primitive (preferences over time-constraint profiles).

[16] Standard results in the analysis of multi-variate calculus (Berge's theorem of the maximum) guarantee that under "nice" properties of U over production possibilities, the selection will be well behaved.

invariance to changing the order of integration and maximization. We formalize this in the following proposition, which is implied directly by Proposition 1 for the concave integral, or Proposition 2 for the Choquet integral.

Proposition 3. *Let* $\mathbf{v}$ *be an SVC and* $\lambda \in \mathbb{R}^n_+$*. Then,*

$$\max_{x \in \int X d\mathbf{v}} \lambda \cdot x = \int X dv_\lambda$$

for every $X \in \mathbb{R}^n_+$*.*

Lastly, note that when U is not linear, a generalization of the proposition does not hold. That is, letting v_U be defined as $v_U(S) = \max_{y \in \mathbf{v}(S)} U(y)$ for every coalition S, it is not true that $\max_{x \in \int X d\mathbf{v}} U(x) = \int X dv_U$ for every X. A clear example for that is whenever U is strictly concave (or strictly convex).

We conclude with noting that the preference $\succeq$ over time constraint profiles could be considered subjective. It depends both on the ability of the different coalitions to produce across the different agendas (namely, the SVC $\mathbf{v}$), and on the integration method applied. In the context of production, it seems more natural to consider the concave integral, since it allows for more production possibilities than the Choquet integral (see Remark 4). Then, in the selection process (provided by the utility function U as discussed above), the decision maker can only benefit.

6 Final Comments

6.1 Integral of set-valued functions.

This paper deals with integration of functions w.r.t. set-valued capacities. This subject should not be confused with integration of set-valued function w.r.t. a (classical) measure, like Aumann (1965) integral.

6.2 Other integrals.

We discussed two types of integrals: the concave and Choquet. These two integrals are specific types of decomposition integrals (see Even and Lehrer [6]). Other important integrals of this family are Riemann (or Pan) and Shilkret [23]. Riemann integral has been introduced by Gavriluţ [7] (see also Wang et al. [26]) as an extension of Gould [9] integral in the case where the capacity is sub-additive (i.e., for every two disjoint sets $S, T \subseteq N$, $\mathbf{v}(S \cup T) \subseteq \mathbf{v}(S) + \mathbf{v}(T)$). The concept of decomposition integral, among which are Riemann and Shilkret integrals, can be generalized in a natural way to set-valued capacities.

Another type of integral that should be mentioned in this context is Sugeno [25] integral. The question of how to define the Sugeno integral w.r.t. set-valued games is left for the future.

Other approaches were proposed in the literature. One approach (see de Campos et al. [4]) unifies Choquet and Sugeno integrals through four essential properties. Another approach (see Klement et al. [13, 14]) building on Choquet, Sugeno [25] and Shilkret integrals, defines a universal integral. Both methods use different binary operations instead of the regular addition and multiplication.

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A Proofs

Proof of Lemma 1 . Fix X and let $\sum_{i=1}^{\ell_1} \alpha_i y_i \in \int^{cav} X d\mathbf{v}$, where $\sum_{i=1}^{\ell_1} \alpha_i \mathbf{1}_{A_i} \leq X, y_i \in \mathbf{v}(A_i), \alpha_i \geq 0, A_i \subseteq N,\ i = 1, ..., \ell_1$ and $\sum_{j=1}^{\ell_2} \beta_j z_j \in \int^{cav} X d\mathbf{v}$ where $\sum_{j=1}^{\ell_2} \beta_j \mathbf{1}_{B_j} \leq X, z_j \in \mathbf{v}(B_j), \beta_j \geq 0, B_j \subseteq N,\ j = 1, ..., \ell_2$. Let $\gamma \in (0, 1)$. Then, $\sum_{i=1}^{\ell_1} \gamma \alpha_i \mathbf{1}_{A_i} + \sum_{j=1}^{\ell 2} (1-\gamma) \beta_j \mathbf{1}_{B_j} \leq X$ and thus, $\sum_{i=1}^{\ell_1} \gamma \alpha_i y_i + \sum_{j=1}^{\ell_2} (1-\gamma) \beta_j z_j \in \int^{cav} X d\mathbf{v}$ which shows that $\int^{cav} X d\mathbf{v}$ is convex. □

Proof of Proposition 1. It is clear that $\varphi = \int^{cav} \cdot\, d\mathbf{v}$ satisfies (i). As for (ii), consider $\lambda \in \mathbb{R}_+^k$ and $X \in \mathbb{R}_+^n$. We show first that $\max_{y \in \varphi(X)} y \cdot \lambda \leq \int^{cav} X dv_\lambda$. Suppose that $\sum_{i=1}^{\ell} \alpha_i y_i \in \int^{cav} \cdot\, d\mathbf{v}$. It means that there exist $A_i \subseteq N,\ i = 1, ..., \ell$

such that $\sum_{i=1}^{\ell} \alpha_i \mathbf{1}_{A_i} \leq X, y_i \in \mathbf{v}(A_i)$ and $\alpha_i \geq 0, i = 1, ..., \ell$. In particular, $y_i \cdot \lambda \leq \max_{z \in \mathbf{v}(A_i)} z \cdot \lambda$. For $i = 1..., \ell$, denote $z_i := \text{argmax}_{z \in \mathbf{v}(A_i)} z \cdot \lambda$. Thus, (a) $\sum_{i=1}^{\ell} \alpha_i z_i \in \int^{cav} X d\mathbf{v}$; and (b) $(\sum_{i=1}^{\ell} \alpha_i y_i) \cdot \lambda \leq (\sum_{i=1}^{\ell} \alpha_i z_i) \cdot \lambda$. We conclude that for any $y \in \int^{cav} X d\mathbf{v}$, $y \cdot \lambda \leq \int^{cav} X dv_\lambda$, and hence the desired inequality. In order to show the converse inequality, let $\int^{cav} X dv_\lambda = \sum_{i=1}^{\ell} \alpha_i v_\lambda(A_i)$, where $\sum_{i=1}^{\ell} \alpha_i \mathbf{1}_{A_i} \leq X$ and $A_i \subseteq N, \alpha_i \geq 0, i = 1, ..., \ell$. In particular, $v_\lambda(A_i) = y_i \cdot \lambda$ for $y_i \in \mathbf{v}(A_i)$. Thus, $\sum_{i=1}^{\ell} \alpha_i y_i \in \int^{cav} X d\mathbf{v}$, which completes the proof of (ii).

Now suppose that φ satisfies (i) and (ii), then $\varphi = \int^{cav} \cdot \, d\mathbf{v}$. Suppose to the contrary that there is X such that $\varphi(X) \neq \int^{cav} X \ d\mathbf{v}$. Suppose that there exists $y \in \varphi(X) \setminus \int^{cav} X \ d\mathbf{v}$. Since both $\varphi(X)$ and $\int^{cav} X \ d\mathbf{v}$ are closed, convex and satisfy comprehensiveness there is a separating $\lambda \in \mathbb{R}_+^k$ which satisfies: $y \cdot \lambda > \max_{z \in \int^{cav} X \ d\mathbf{v}} z \cdot \lambda$. However, from the first part of the proof we know that $\max_{z \in \int^{cav} X \ d\mathbf{v}} z \cdot \lambda = \int^{cav} X dv_\lambda$. We therefore obtain that $y \cdot \lambda > \int^{cav} X dv_\lambda$, which contradicts (ii). The case where there exists $y \in \int^{cav} X \ d\mathbf{v} \setminus \varphi(X)$ is handled in a similar fashion and the proof is complete. □

Proof of Theorem 1. We show first that (ii) implies (i). From Lemma 1 and Proposition 1 we get that the set-valued function $\int^{cav} \cdot d\mathbf{v}$ is comprehensive and convex-closed valued. Homogeneity and INCL are obvious. It remains to show that $\int^{cav} \cdot d\mathbf{v}$ is concave and satisfies DOM. Due to homogeneity, in order to show concavity of $\int^{cav} \cdot d\mathbf{v}$ it is sufficient to prove that $\int^{cav} X d\mathbf{v} + \int^{cav} Y d\mathbf{v} \subseteq \int^{cav} X + Y d\mathbf{v}$. Indeed, let $x \in \int^{cav} X d\mathbf{v}$ and $y \in \int^{cav} Y d\mathbf{v}$. Then, $x \in \sum_S \alpha_S \mathbf{v}(S)$, where $\sum_S \alpha_S \mathbf{1}_S = X$ and $\alpha_S \geq 0$ for every S. Similarly, $y \in \sum_T \beta_T \mathbf{v}(T)$, where $\sum_T \beta_T \mathbf{1}_T = Y$ and $\beta_S \geq 0$ for every T. Thus, $x + y \in \sum_S \alpha_S \mathbf{v}(S) + \sum_T \beta_T \mathbf{v}(T)$ while $\sum_S \alpha_S \mathbf{1}_S + \sum_T \beta_T \mathbf{1}_T = X + Y$ and all the coefficient are non-negative. We therefore obtain that $x + y \in \int^{cav} X + Y d\mathbf{v}$ as desired.

The proof that $\int^{cav} \cdot d\mathbf{v}$ satisfies DOM is also easy. Let $X \in \mathbb{R}_+^n$ and suppose that $\mathbf{v}(S) \leq \mathbf{p}(S)$. Let $x \in \int^{cav} X d\mathbf{v}$. Then, $x \in \sum_S \alpha_S \mathbf{v}(S)$, where $\sum_S \alpha_S \mathbf{1}_S = X$ and $\alpha_S \geq 0$ for every S. However, $\sum_S \alpha_S \mathbf{v}(S) \subseteq \sum_S \alpha_S \mathbf{p}(S)$ and since $\mathbf{p}$ is comprehensive, $x \in \sum_S \alpha_S \mathbf{p}(S) = \mathbb{E}_{\mathbf{p}}(X)$ (where the last equality is due to the additivity of $\mathbf{p}$).

We now turn to the less trivial direction: (i) implies (ii). Due to INCL and the fact that φ is concave, it is clear that $\int^{cav} X d\mathbf{v} \subseteq \varphi(X)$ for every $X \in \mathbb{R}_+^n$. In order to show the converse inclusion, for every $X \in \mathbb{R}_+^n$ and $\lambda \in \mathbb{R}_+^k$ denote $c_\lambda(X) := \text{argmax}_{y \in \int^{cav} X d\mathbf{v}} y \cdot \lambda$. Since $\int^{cav} X d\mathbf{v}$ is closed, $c_\lambda(X)$ is well defined.

Assume there is an $X \in \mathbb{R}_+^k$ for which $\varphi(X) \setminus \int^{cav} X d\mathbf{v} \neq \emptyset$ and let $x \in \varphi(X) \setminus$

$\int^{cav} X d\mathbf{v}$. Since $\int^{cav} X d\mathbf{v}$ is comprehensive and convex, there exists a $\lambda \in \mathbb{R}^k_+$ such that $\max_{y\in\int^{cav} X d\mathbf{v}} y\cdot\lambda < x\cdot\lambda$. In addition, due to DOM, if $\mathbf{p}$ is an additive SVC such that $\mathbf{v}(S) \subseteq \mathbf{p}(S)$ for every S, we have that $x\cdot\lambda \le \max_{y\in\mathbb{E}_{\mathbf{p}}(X)} y\cdot\lambda$, implying that $\max_{y\in\int^{cav} X d\mathbf{v}} y\cdot\lambda < \max_{y\in\mathbb{E}_{\mathbf{p}}(X)} y\cdot\lambda$. Thus, in order to show that $\int^{cav} X d\mathbf{v} \supseteq \varphi(X)$, it is sufficient to show that for every $X \in \mathbb{R}^n_+$, $\lambda \in \mathbb{R}^k_+$ and $x \in c_\lambda(X)$ there is an additive $\mathbf{p}$ such that $\mathbf{v}(S) \subseteq \mathbf{p}(S)$ for every $S \subseteq N$, where $x \in \mathrm{argmax}_{y\in\mathbb{E}_{\mathbf{p}}(X)} y\cdot\lambda$.

Indeed, consider the function $\int^{cav} \cdot dv_\lambda$. It is homogeneous and concave. As such, it has a vector $p \in \mathbb{R}^n$ such that $\int^{cav} Y dv_\lambda \le Y\cdot p$ for every $Y \in \mathbb{R}^n_+$, while $\int^{cav} X dv_\lambda = X\cdot p$. Denote

$$\mathbf{p}(S) = \{x \in \mathbb{R}^k_+;\ x\cdot\lambda \le \mathbf{1}_S\cdot p\}.$$

Note that $\mathbb{E}_{\mathbf{p}}(X) = \{x \in \mathbb{R}^k_+;\ x\cdot\lambda \le X\cdot p\}$.

Due to Remark 1, $\mathbf{v}(S) \subseteq \{y \in \mathbb{R}^k_+; y\cdot\lambda \le v_\lambda(S)\}$ and since $v_\lambda(S) \le \int^{cav} \mathbf{1}_S dv_\lambda$, we obtain, $\mathbf{v}(S) \subseteq \{y \in \mathbb{R}^k_+; y\cdot\lambda \le \int^{cav} \mathbf{1}_S dv_\lambda\} \subseteq \{y \in \mathbb{R}^k_+; y\cdot\lambda \le \mathbf{1}_S\cdot p\} = \mathbf{p}(S)$ implying that $\mathbf{p}$ dominates $\mathbf{v}$.

Now consider $x \in c_\lambda(X)$. Since, $x \in \int^{cav} X d\mathbf{v}$, $x = \sum_S \alpha_S x_S$, where $x_S \in \mathbf{v}(S)$, $\sum_S \alpha_S \mathbf{1}_S = X$ and $\alpha_S \ge 0$ for every S. Moreover, since $x \in c_\lambda(X)$, for every S with $\alpha_S > 0$, $x_S \in c_\lambda(\mathbf{1}_S)$, meaning that $x_S\cdot\lambda = \int^{cav} \mathbf{1}_S dv_\lambda$. Thus, $x\cdot\lambda = \int^{cav} X dv_\lambda = X\cdot p$. Thus, $x \in \mathbb{E}_{\mathbf{p}}(X)$ and moreover, $x \in \mathrm{argmax}_{\mathrm{y}\in\mathbb{E}_{\mathbf{p}}(\mathrm{X})} \mathrm{y}\cdot\lambda$. This completes the proof. □

Proof of Lemma 2. We start with the first item of the lemma. Fix X and consider $y = \sum_{i=1}^{\ell_1} \alpha_i y_i \in \int^{Ch} X d\mathbf{v}$, where $Y := \sum_{i=1}^{\ell_1} \alpha_i \mathbf{1}_{A_i} \le X, y_i \in \mathbf{v}(A_i), \alpha_i \ge 0,\ i = 1, ..., \ell_1$ and $A_1, ..., A_{\ell_1}$ is a chain. We show the following claim.

Claim: There exists $w \in \int^{Ch} X d\mathbf{v}$ and a permutation π of $\{1, ..., n\}$ such that (a) $w \ge y$; (b) $w = \sum_{j=1}^n \delta_j w_j$ where $\delta_j \ge 0$, $j = 1, ..., n$; and (c) $w_j \in \mathbf{v}(\{\pi(j), ..., \pi(n)\})$.

Proof of Claim. Let Z be such that (1) $X \ge Z \ge Y$, (2) there is $z \in \int^{Ch} Z d\mathbf{v}$ such that $z \ge y$; and (3) Z is a maximal of those satisfying (1) and (2). That is, if there is $X \ge Z' \ge Y$ satisfying (2) and $Z' \ge Z$, then $Z' = Z$.

Suppose first that $Z = X$. In the chain decomposition[17] of Z there might be multiple appearances of the same set. Consider its most concise version, where each set in the chain appears only once. The proof of Proposition 1 of Even and Lehrer

[17] By a *decomposition* of X we mean ℓ sets $A_1, ..., A_\ell$ and ℓ non-negative coefficients $\alpha_1, ..., \alpha_\ell$ such that $\sum_{i=1}^{\ell} \alpha_i \mathbf{1}_{A_i} = X$. When $A_1, ..., A_\ell$ is a chain, we refer to it as a *chain decomposition*.

[6] shows that the only chain decompositions of X has the form $\Big(\{\pi(i), ..., \pi(n)\}\Big)_{i=1}^{n}$, where π is a permutation on N such that $X_{\pi(1)} \leq ... \leq X_{\pi(n-1)} \leq X_{\pi(n)}$. In this case, $X = \sum_{i=1}^{n}(X_{\pi(i)} - X_{\pi(i-1)})\mathbf{1}_{\{\pi(i),...,\pi(n)\}}$. Note that in this case (i.e., where $Z = X$) the claim is correct.

Suppose, otherwise, that $Z \neq X$. It means that there is $i \in N$ such that $X(i) > Z(i)$. Let the chain decomposition of Z be $Z = \sum_{i=1}^{\ell} \delta_j \mathbf{1}_{C_j}$. Suppose that $C_1 \subseteq ... \subseteq C_\ell$ and let ℓ' be the index at which $i \in C_{\ell'+1} \setminus C_{\ell'}$. We set $\ell' = 0$ if i is not a member of any C_j and $\ell' = \ell$ if i is a member of all C_j's. Since $X(i) > Z(i)$, there is an $\varepsilon > 0$ such that $Z' := \sum_{j=1}^{\ell'-1} \delta_j \mathbf{1}_{C_j} + (\delta_{\ell'} - \varepsilon)\mathbf{1}_{C_{\ell'}} + \varepsilon \mathbf{1}_{C_{\ell'}\cup\{i\}} + \sum_{j=\ell'+1}^{\ell} \delta_j \mathbf{1}_{C_j}$ (with all coefficients being non-negative) satisfies (1) and (2) (due to monotonicity, $z \in \int^{Ch} Z d\mathbf{v}$ implies $z \in \int^{Ch} Z' d\mathbf{v}$) $Z' \geq Z$ and $Z' \neq Z$. This is a contradiction to the choice of Z. We conclude that $Z = X$ and the claim is proven.

We now let $z := \sum_{j=1}^{\ell_2} \beta_j z_j \in \int^{Ch} X d\mathbf{v}$ where $\sum_{j=1}^{\ell_2} \beta_j \mathbf{1}_{B_j} \leq X, z_j \in \mathbf{v}(B_j), \beta_j \geq 0, j = 1, ..., \ell_2$ and $B_1, ..., B_{\ell_2}$ is a chain. Let $\gamma \in (0, 1)$. By the previous argument y and z can be produced by the same chain whose sets are of the type $\{\pi(i), ..., \pi(n)\}$. Now we can construct the convex combination, as in the proof of Lemma 1, and conclude that $\gamma y + (1-\gamma)z \in \int^{Ch} X d\mathbf{v}$, implying that $\int^{Ch} X d\mathbf{v}$ is convex.

We now prove the second item of the lemma. First, $\int^{Ch} \mathbf{1}_S d\mathbf{v} \supseteq \text{conv}(\mathbf{v}(S))$ since the integral is convex-valued (by the first item of this lemma), and since $\int^{Ch} \mathbf{1}_S d\mathbf{v} \supseteq \mathbf{v}(S)$. Now, to see that the other containment, $\int^{Ch} \mathbf{1}_S d\mathbf{v} \subseteq \text{conv}(\mathbf{v}(S))$, also holds, take any chain $A_1 \subseteq A_2 \subseteq ... \subseteq A_\ell \subseteq S$ and corresponding nonnegative scalars, $\alpha_1, ..., \alpha_\ell$, such that $\sum_{i=1}^{l} \alpha_i \mathbf{1}_{A_i} \leq \mathbf{1}_S$. Since $A_1, A_2, ..., A_\ell$ is a chain, $\sum_{i=1}^{l} \alpha_i \leq 1$. W.l.o.g.[18] $\sum_{i=1}^{l} \alpha_i = 1$. That is, $\sum_{i=1}^{l} \alpha_i v(A_i)$ is a convex combination of $\mathbf{v}(A_1), ..., \mathbf{v}(A_\ell)$, each of which is a subset of $\mathbf{v}(S)$ due to monotonicity. Hence, $\sum_{i=1}^{l} \alpha_i v(A_i) \subseteq \text{conv}(\mathbf{v}(S))$. Since this is true for arbitrary $A_1 \subseteq A_2 \subseteq ... \subseteq A_\ell$ and corresponding nonnegative scalars, $\alpha_1, ..., \alpha_\ell$, this is true also for $\int^{Ch} \mathbf{1}_S d\mathbf{v}$.

□

Proof of Theorem 4. We assume first that v is not supermodular and show that $\int^{Ch} \cdot\, d\mathbf{v} \neq \int^{cav} \cdot\, d\mathbf{v}$. If $\mathbf{v}$ is not supermodular, then there are S and T such

$$\mathbf{v}(S) + \mathbf{v}(T) \not\subseteq \mathbf{v}(S \cup T) + \mathbf{v}(S \cap T). \tag{13}$$

[18]Otherwise, define $\alpha_0 = 1 - \sum_{i=1}^{l} \alpha_i$, and associate it with $A_0 = \emptyset$. Recalling that $\mathbf{v}(A_0) = \{0\}$, one can then continue the argument with $A_0, ..., A_\ell$.

Let $X = \mathbf{1}_S + \mathbf{1}_T$. By definition, $\int^{Ch} X d\mathbf{v} = \mathbf{v}(S \cup T) + \mathbf{v}(S \cap T)$. However, $\int^{cav} X d\mathbf{v} \supseteq \mathbf{v}(S) + \mathbf{v}(T)$ (since $X = \mathbf{1}_S + \mathbf{1}_T$). Eq. (13) implies that $\int^{Ch} \cdot \, d\mathbf{v} \neq \int^{cav} \cdot \, d\mathbf{v}$, a contradiction.

Now assume that v is supermodular. We show in three steps that for every X, $\int^{Ch} X d\mathbf{v} = \int^{cav} X d\mathbf{v}$.

Step 1: We start with a simple case where $X = \alpha \mathbf{1}_A + \beta \mathbf{1}_B$ and $\int^{cav} X d\mathbf{v} = \alpha \mathbf{v}(A) + \beta \mathbf{v}(B)$. W.l.o.g. $\alpha \leq \beta$. Due to supermodularity,

$$\alpha \mathbf{v}(A) + \beta \mathbf{v}(B) = \alpha \mathbf{v}(A) + \alpha \mathbf{v}(B) + (\beta - \alpha)\mathbf{v}(B) \subseteq \alpha \mathbf{v}(A \cup B) + \alpha \mathbf{v}(A \cap B) + (\beta - \alpha)\mathbf{v}(B).$$

Since the RHS is a chain decomposition of X, we obtain that $\int^{cav} X d\mathbf{v} \subseteq \int^{Ch} X d\mathbf{v}$ and therefore $\int^{cav} X d\mathbf{v} = \int^{Ch} X d\mathbf{v}$. The main purpose of this simple case is to show that $\alpha \mathbf{v}(A) + \beta \mathbf{v}(B)$ is contained in a sum of sets formed by a chain whose largest set is the union $A \cup B$. We turn now to the general case.

Step 2: Let $\sum_{i=1}^{\ell} \alpha_i \mathbf{1}_{A_i} \leq X$ and consider $\sum_{i=1}^{\ell} \alpha_i \mathbf{v}(A_i)$. We show that $\sum_{i=1}^{\ell} \alpha_i \mathbf{v}(A_i) \subseteq \sum_{i=1}^{\ell'} \beta_j \mathbf{v}(B_j)$ where the B_j's form a chain. We now use the above argument over and over again. We start with A_1 and A_2 and show that $\alpha_1 \mathbf{v}(A_1) + \alpha_2 \mathbf{v}(A_2)$ is contained in a sum formed by a chain whose largest set is $A_1 \cup A_2$. We then incorporate A_3 and show that $\alpha_1 \mathbf{v}(A_1) + \alpha_2 \mathbf{v}(A_2) + \alpha_3 \mathbf{v}(A_3)$ is contained in a sum formed by sets, the largest of which is the union, $A_1 \cup A_2 \cup A_3$. Using this argument successively shows that $\sum_{i=1}^{\ell} \alpha_i \mathbf{v}(A_i)$ is contained in a sum formed by sets, the largest of which (w.r.t. inclusion) is $\cup_{i=1}^{\ell} A_i$. That is, all other sets are contained in $\cup_{i=1}^{\ell} A_i$.

Step 3: Consider a sum formed by sets, the largest of which (w.r.t. inclusion) is $\cup_{i=1}^{\ell} A_i$ and that the latter has the largest coefficient. Such a sum exists due to the Closeness condition in the definition of an SVC (see Definition 1). All other sets in this sum are subsets of $\cup_{i=1}^{\ell} A_i$. using the argument in the previous step, we conclude that the union of all other sets is a strict subset of $\cup_{i=1}^{\ell} A_i$. Thus, we use the same argument again and again. We thereby obtain a chain whose respective sum contains $\sum_{i=1}^{\ell} \alpha_i \mathbf{1}_{A_i}$, which concludes the proof. □