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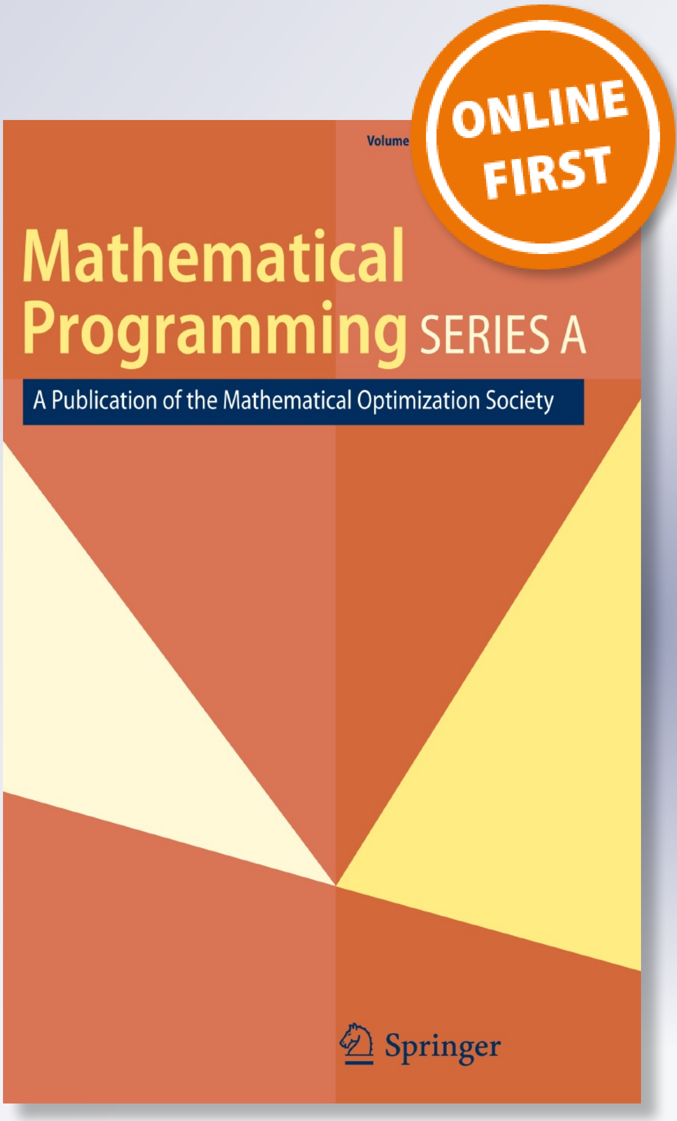

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Abstract The extension of set functions (or capacities) in a concave fashion, namely a concavification, is an important issue in decision theory and combinatorics. It turns out that some set-functions cannot be properly extended if the domain is restricted to be bounded. This paper examines the structure of those capacities that can be extended over a bounded domain in a concave manner. We present a property termed the *sandwich property* that is necessary and sufficient for a capacity to be concavifiable over a bounded domain. We show that when a capacity is interpreted as the product of any sub group of workers per a unit of time, the sandwich property provides a linkage between optimality of time allocations and efficiency.

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1 Introduction

Extensions of set functions in a concave fashion has proven to be an important issue in fields like decision theory, optimization, combinatorics, etc. It turns out that an extension may depend on the boundedness properties of the domain to which the set

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function is being extended. To see this we focus attention on the *concave integral* introduced by Lehrer [7] and axiomatized by Lehrer and Teper [8] in a decision theoretic context. The concave integral has a particularly interesting and natural interpretation in the context of production as well. Thus, the formal issue of extensions is deferred to the main text, where here we introduce an implication of this point to production.

To see the definition of the concave integral and the problem we are trying to address in this note, consider a group of individuals (workers), say N, where the productivity per unit of time of each sub-group S is $v(S)$. Let $x_i \in \mathbb{R}_+$ be the time worker i can invest. How would one measure the group's productivity when the time-investment profile is x? One natural way is the following. Time can be allocated across different sub-groups $\{\alpha_S : S \subseteq N\}$ as long as the time-investment constraint[1] $\sum \alpha_S \mathbf{1}_S = x$ is met. Each allocation $\{\alpha_S : S \subseteq N\}$ of this kind yields a productivity level of $\sum \alpha_S v(S)$. The concave integral of x is the productivity level of the time allocation that maximizes the productivity across all feasible time allocations (given the time-investment constraint x). A time allocation that maximizes productivity is called *optimal*.

Assume now that only one sub-group of workers can produce at any given point in time (that is, production is done sequentially). Among all optimal allocations, consider one that minimizes total production time $\sum \alpha_S$. This measures the minimal amount of time needed to complete the task while maintaining optimality. It is clear that the amount of time needed is bounded from below by the time available to the worker that can invest the most. However, production may be inefficient in the sense that it will last longer than that.

This point raises a question regarding the structure of set functions: under what assumptions on the capacity, optimality can be sustained together with time efficiency? It turns out that such capacities are characterized by a property we refer to as the *sandwich property*; a capacity satisfies the sandwich property if for every affine function that dominates it (that is, assigns higher value than the capacity to *every* sub-group of workers), there is a linear function that dominates the capacity but is dominated by the affine function.

The following section provides an example that demonstrates the issues raised in the introduction. Section 3 sets up the framework, formally describes the question we raise and presents the definition of the sandwich property. Section 4 presents the main characterization result and its proof, and provides additional insights as to how the sandwich property relates to standard notions in the theory of capacities. Lastly, Sect. 5 concludes. The rest of the proofs appear in an appendix.

2 An example

Consider a group of 6 workers and the least monotonic production function v (per unit of time) satisfying $v(1, 4, 5) = v(2, 5, 6) = v(3, 4, 6) = \frac{2}{3}$ and $v(N) = 1$. Assume that workers are time constrained: workers 1, 2 and 3 can work half an hour while workers 4, 5 and 6 can work a full hour. Also assume that only one sub-group of

[1] $\mathbf{1}_S$ is the indicator of S: it is an $|N|$-dimensional vector $(\mathbf{1}_S(1), \ldots, \mathbf{1}_S(|N|))$ such that $\mathbf{1}_S(i) = 1$ if $i \in S$ and $\mathbf{1}_S(i) = 0$ otherwise.

workers can produce at any given point in time. What is the optimal production level that can be extracted given the time constraints? Is it possible to obtain the level of optimal production within one hour (which is the time constraint of the worker that can invest the most)?

It is possible that all six workers work as one sub-group for half an hour, and then subgroup $\{4, 5, 6\}$ work as a subgroup for another half an hour. Subject to this assignment, all time constraints are met and the total productivity is $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2}$. However, if sub-groups $\{1, 4, 5\}$, $\{2, 5, 6\}$ and $\{3, 4, 6\}$ work consecutively for half an hour each, then the time constraints are still met but productivity increases to $3 \cdot \frac{1}{2} \cdot \frac{2}{3} = 1$. It turns out that this is the optimal time allocation and 1 is the optimal productivity level possible given the workers time constraints. However, according to this time allocation, production is inefficient in the sense that optimal production cannot be obtained within one hour. Rather, it takes one hour and a half. In the rest of this note we are going to show that any time allocation that achieves optimal production is inefficient in this sense.

The question is what is the property of v that prevents all optimal time allocations to be time inefficient?

3 Capacities and the sandwich property

3.1 Technical preliminaries

Let $N = \{1, \ldots, n\}$ be a set of players. A *capacity* defined on N is a function $v : 2^N \rightarrow \mathbb{R}_+$ such that $v(N) = 1$ and $v(\emptyset) = 0$, and $S \subseteq T$ implies $v(S) \leq v(T)$.[2] The *sub-capacity* v_S is the restriction of a capacity v to $S \subseteq N$. Denote by $\mathbb{R}^n_+$ the closed positive orthant of $\mathbb{R}^n$. A function h over $\mathbb{R}^n_+$ is (positive) *homogeneous* if $h(ax) = ah(x)$ for every $a \in \mathbb{R}_+$ and $x \in \mathbb{R}^n_+$. An *affine* function, $\ell(\cdot)$, over $\mathbb{R}^n_+$ is characterized by $a \in \mathbb{R}$ and $(b_1, \ldots, b_n) \in \mathbb{R}^n$ and defined $\ell(x_1, \ldots, x_n) = a + \sum_{i=1}^n b_i x_i$ for every $(x_1, \ldots, x_n) \in \mathbb{R}$. It is *linear* if $a = 0$ (that is, it is affine and homogeneous). The characteristic function of $S \subseteq N$, denoted $\mathbf{1}_S$, is the vector in $\mathbb{R}^n_+$ whose i-th coordinate is 1 whenever $i \in S$ and is 0 otherwise.

3.2 The sandwich property

Let $Q = [0, 1]^n$ be the n-dimensional unit cube. A capacity v can be viewed also as a real function defined over the extreme points of Q, $\{\mathbf{1}_S : S \subseteq N\}$. Let f and g be functions defined on Q and v a capacity. We say that f *dominates* g if $f(x) \geq g(x)$ for every $x \in Q$, and that a function f *dominates* v if $f(\mathbf{1}_S) \geq v(S)$ for every $S \subseteq N$.

Definition 1 A capacity v has the *sandwich property* if for every affine function f over $\mathbb{R}^n_+$ that dominates v, there is a linear function ℓ that is dominated by f and dominates v.

[2] We show in Sect. 5 that the results hold for non-monotonic set functions that may obtain negative values as well.

Example 1 Consider $N = \{1, 2, 3\}$ and a capacity v defined over N such that $v(S) = 1$ if $\{2\} \subsetneq S$ (and $v(S) = 0$ otherwise). Suppose that $f(x_1, x_2, x_3) = a + b_1x_1 + b_2x_2 + b_3x_3$ is an affine function that dominates v. Define ℓ as $\ell(x_1, x_2, x_3) = b_1x_1 + (a + b_2)x_2 + b_3x_3$ if $a + b_2 \leq 1$ and as $\ell(x_1, x_2, x_3) = x_2$ otherwise. Since $a, a + b_2 \geq 0$, the function ℓ is dominated by f. Moreover, ℓ is dominating v. Indeed, when $a + b_2 \leq 1$, since $a + b_1 + b_2 \geq 1$, we obtain, $b_1 \geq 0$. For the same reason $b_3 \geq 0$. Therefore, in this case and in the case when $a + b_2 > 1$, the function ℓ is non-negative. Furthermore, $\ell(\mathbf{1}_{12})$ and $\ell(\mathbf{1}_{23})$ are at least 1 (in the first case it is because $\ell(\mathbf{1}_{12}) = f(\mathbf{1}_{12})$ and $\ell(\mathbf{1}_{23}) = f(\mathbf{1}_{23})$). Hence v has the sandwich property.

The following example demonstrates a capacity for which the sandwich property does not hold.

Example 2 Consider the example described in Sect. 2. We show that v does not have the sandwich property. Consider the affine function $f(x_1, \ldots, x_6) = \frac{1}{2} + \frac{1}{6}x_1 + \frac{1}{6}x_2 + \frac{1}{6}x_3$. It is easy to check that f dominates v. In order to verify that indeed v does not have the sandwich property, suppose that $\ell(x_1, \ldots, x_6) = \sum a_i x_i$ is a linear function dominated by f and dominates v. In particular

$$\frac{1}{2} = f(\mathbf{1}_{456}) \geq \ell(\mathbf{1}_{456}) = a_4 + a_5 + a_6. \tag{1}$$

Furthermore, $\ell(\mathbf{1}_{145}) \geq v(1, 4, 5), \ell(\mathbf{1}_{256}) \geq v(2, 5, 6)$ *and* $\ell(\mathbf{1}_{346}) \geq v(3, 4, 6)$. Summing up these inequalities results in,

$$a_1 + a_2 + a_3 + 2(a_4 + a_5 + a_6) \geq 3 \cdot \frac{2}{3} = 2. \tag{2}$$

Due to Eq. (1), the LHS of Eq. (2) is smaller than or equal to $a_1 + a_2 + a_3 + 1$. Thus, $a_1 + a_2 + a_3 \geq 1$. Since $1 = f(\mathbf{1}_{123i}) \geq \ell(\mathbf{1}_{123i})$ for every $i = 4, 5, 6$, it must be that $a_4 = a_5 = a_6 = 0$. Due to Eq. (2), $a_1 + a_2 + a_3 \geq 2$, which implies that $\ell(\mathbf{1}_N) \geq 2 > f(\mathbf{1}_N)$. This contradicts the assumption that f dominates ℓ. We conclude that v does not have the sandwich property.

Note that the argument uses the facts that f dominates v and that $v(1, 4, 5) = v(2, 5, 6) = v(3, 4, 6) = \frac{2}{3}$ and $v(N) = 1$. Any v that satisfies these conditions does not have the sandwich property.

3.3 Concave extensions of v

3.3.1 The concave integral

In this section we consider two concave extensions of v to Q. We start with the concave integral introduced and axiomatized in Lehrer [7] and Lehrer and Teper [8]. For every $x \in Q$, the concave integral is defined as

$$\int^{cav} x \, dv = \max \left\{ \sum_{S \subseteq N} \alpha_S v(S); \ \sum_{S \subseteq N} \alpha_S \mathbf{1}_S = x, \alpha_S \geq 0 \right\}. \tag{3}$$

Remark 1 The maximum here is well defined due to compactness of the constraints and the continuity of the objective function $\sum_{S\subseteq N}\alpha_S v(S)$.

The concave integral can be thought of as a generalization of the *totally balanced cover*;[3] while the totally balanced cover is defined on the same domain as v, the extreme points of Q, the concave integral is an extension of v to all of Q.

Proposition 1 *Fix a capacity v. Then,*

(i) $\int^{cav}\cdot\, dv$ *is the least concave and homogeneous function that dominates v.*
(ii) $\int^{cav} x\, dv = \min\{\ell(x);\ \ell$ *is linear and dominates* $v\}$ *for every* $x \in Q$.
(iii) There is a compact and convex set C of linear functions dominating v, such that

$$\int^{cav} x\, dv = \min_{\ell\in C}\ell(x)$$

for every $x \in Q$.

Proof The proofs of (i) and (ii) are standard and appear in Lehrer and Teper [9]. (iii) is an immediate consequence of (ii) due to the compactness of Q. □

We say that $\sum_{S\subseteq N}\alpha_S\mathbf{1}_S$ is a *decomposition of* x if $\sum_{S\subseteq N}\alpha_S\mathbf{1}_S = x$.

Definition 2 A decomposition $\sum_{S\subseteq N}\alpha_S\mathbf{1}_S$ of x is *optimal* if it attains the maximum for Eq. (3), that is, $\int^{cav} x\, dv = \sum_{S\subseteq N}\alpha_S v(S)$.

Denote,

$$e_v(x) = \min\left\{\sum_{S\subseteq N}\alpha_S;\ \sum_{S\subseteq N}\alpha_S\mathbf{1}_S = x,\ \int^{cav} x\, dv = \sum_{S\subseteq N}\alpha_S v(S)\right\}.$$

Definition 3 An optimal decomposition of x, $\sum_{S\subseteq N}\alpha_S\mathbf{1}_S$, is *sharp* if the sum of its coefficients, $\sum_{S\subseteq N}\alpha_S$, is equal to $e_v(x)$.

Lemma 1 *For any capacity v, e_v is homogeneous and satisfies $e_v(x) \geq \max x$ for every $x \in Q$.*

Fix $x \in Q$ and permute N by $\pi : N \to N$ such that $\pi(i) \geq \pi(j)$ if $x_i \geq x_j$. That is, $x_{\pi^{-1}(i)}$ is increasing with i. In particular, it is easy to see that $x = x_{\pi^{-1}(1)}\mathbf{1}_{\{\pi^{-1}(1),\ldots,\pi^{-1}(n)\}} + \sum_{i=2}^{n}(x_{\pi^{-1}(i)} - x_{\pi^{-1}(i-1)})\mathbf{1}_{\{\pi^{-1}(i),\ldots,\pi^{-1}(n)\}}$ and that $x_{\pi^{-1}(1)} + \sum_{i=2}^{n}(x_{\pi^{-1}(i)} - x_{\pi^{-1}(i-1)}) = \max x$. We refer to such a decomposition as the *Choquet decomposition*. The reason we refer to such a decomposition as the Choquet decomposition is that the Choquet integral (Choquet [5]) of x with respect to v is $x_{\pi^{-1}(1)}v(\{\pi^{-1}(1),\ldots,\pi^{-1}(n)\}) + \sum_{i=2}^{n}(x_{\pi^{-1}(i)} - x_{\pi^{-1}(i-1)})v(\{\pi^{-1}(i),\ldots,\pi^{-1}(n)\})$. Note however, that the Choquet decomposition

[3] The totally balanced cover of a capacity v is a capacity $\hat{v}$ defined by $\hat{v}(S) = \int^{cav}\mathbf{1}_S\, dv$, for every $S \subseteq N$. A capacity v is *totally balanced* if $v = \hat{v}$.

need not be optimal for every capacity v and $x \in Q$. The next example shows that it is possible to find a capacity v such that the Choquet decomposition is not optimal for some x and that $e_v(x) > \max x$.

Example 3 (Example 2 continued) Consider $x = (0.5, 0.5, 0.5, 1, 1, 1)$. The Choquet decomposition of x is $\frac{1}{2}\mathbf{1}_{\{123456\}} + \frac{1}{2}\mathbf{1}_{\{456\}}$ with value $\frac{1}{2}v(1,2,3,4,5,6) + \frac{1}{2}v(4,5,6) = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 0 = \frac{1}{2}$. However, it can also be decomposed as $x = \frac{1}{2}\mathbf{1}_{\{145\}} + \frac{1}{2}\mathbf{1}_{\{256\}} + \frac{1}{2}\mathbf{1}_{\{346\}}$, which implies that $\int^{cav} x\, dv \geq 3\cdot\frac{1}{2}\cdot\frac{2}{3} = 1$. Thus, the Choquet decomposition is not optimal. We show that $e_v(x) > 1$. Assume that $\sum \alpha_S \mathbf{1}_S$ is an optimal decomposition of x and $\sum \alpha_S = 1$. Note that $v(E) \leq \frac{2}{3}$ whenever $E \neq N$. Thus, $\sum_{S\subseteq N}\alpha_S v(S) = \sum_{S\subsetneq N}\alpha_S v(S) + a_N v(N) \leq \frac{2}{3}\sum_{S\subsetneq N}\alpha_S + \alpha_N = \frac{2}{3}(1-\alpha_N) + \alpha_N = \frac{2}{3} + \frac{\alpha_N}{3} \leq \frac{5}{6} < \int^{cav} x\, dv$ (the inequality is because $\sum \alpha_S \mathbf{1}_S$ is decomposition of x and therefore, α_N cannot exceed $\frac{1}{2}$). This is a contradiction.

3.3.2 A non-homogeneous concave extension

For every $x \in Q$, define

$$\psi_v(x) = \max\left\{\sum_{S\subseteq N}\alpha_S v(S);\ \sum_{S\subseteq N}\alpha_S \mathbf{1}_S = x, \alpha_S \geq 0 \text{ and } \sum_{S\subseteq N}\alpha_S \leq 1\right\}.$$

The definition of ψ_v is different than that of the concave integral appearing in Eq. (3) in that it restricts the sum of weights (and hence the weights α_S themselves) to be bounded by 1. We say that a decomposition $\sum_{S\subseteq N}\alpha_S v(S)$ is ψ_v *-feasible* if $\sum_{S\subseteq N}\alpha_S \leq 1$.

The main question we are interested in is whether, or under what conditions, $\psi_v = \int^{cav} \cdot dv$. It is clear that $\psi_v(x) \leq \int^{cav} x\, dv$ for every $x \in Q$, however the inequality can be strict. For instant, in Example 2 above, since $e_v(x) > 1$ it must be that $\psi_v(x) < \int^{cav} x\, dv$.

The following is a dual characterization of ψ_v; since it is not a homogeneous extension, duality should not take into account only homogeneous functions.

Proposition 2 *Fix a capacity v. Then,*

(i) *$\psi_v(\cdot)$ is the least concave function ϕ that dominates v.*
(ii) *$\psi_v(x) = \min\{f(x);\ f$ is affine and dominates $v\}$ for every $x \in Q$.*
(iii) *There is a compact and convex set C of affine functions that dominate v, such that*

$$\psi_v(x) = \min_{f\in C} f(x)$$

for every $x \in Q$.

Proof The proofs of (i) and (ii) are standard. (i) appears in Azrieli and Lehrer [2] and (ii) in Lehrer and Teper [9]. Again, (iii) is an immediate consequence from (ii) due to the compactness of Q. □

4 Main result

4.1 The characterization result

Example 2 shows v that does not have the sandwich property. Example 3 shows that for the same capacity v there is $x \in Q$ such that $e_v(x) > \max x$, implying $\psi_v(x) < \int^{cav} x dv$. These three properties are not coincidental, as the following theorem states.

Theorem 1 *Fix a capacity v. The following are equivalent:*

(i) *v has the sandwich property;*
(ii) *$e_v(x) = \max x$ for every $x \in Q$;*
(iii) *$\psi_v(x) = \int^{cav} x dv$ for every $x \in Q$; and*

Proof We begin by showing that (ii) is equivalent to (iii). Indeed, $\psi_v(x) \leq \int^{cav} x\, dv$ for every $x \in Q$, and there is equality for x if and only if the sharp decomposition of x is feasible in the definition of ψ_v. Now if $e_v(x) = \max x$ for $x \in Q$, then the optimal decomposition of x is ψ_v-feasible, and thus $\psi_v(x) = \int^{cav} x\, dv$. If this holds for every $x \in Q$ then $\psi_v = \int^{cav} \cdot\, dv$. As for the inverse implication, assume that there exists $x \in Q$ for which $e_v(x) > \max x$. Since e_v is homogeneous, $e_v(\frac{x}{\max x}) > 1$, implying that the optimal decomposition of $\frac{x}{\max x}$ is not ψ_v-feasible. Therefore, $\psi_v(\frac{x}{\max x}) < \int^{cav} \frac{x}{\max x}\, dv$.

Next we show that (i) is equivalent to (iii). Suppose $\psi_v = \int^{cav} \cdot\, dv$. Thus, by Proposition 1 (iii) and Proposition 2 (ii), there is a compact and convex set C of linear functions that dominate v such that

$$\min_{\ell \in C} \ell(x) = \min \left\{ f(x);\ f \text{ is affine and dominates } v \right\}.$$

It implies that for every affine function f that dominates v and every $x \in Q$ there is a linear $\ell \in C$ that satisfies $f(x) \geq \ell(x)$.

Fix an affine function f that dominates v and consider the following zero-sum game. The action set of player 1 (PI) – the maximizer– is Q, while that of player 2 (PII) is C. The actions sets are both compact and convex. When PI is choosing x and PII $\ell \in C$, the payoff is $\ell(x) - f(x)$. Note that the payoff function is linear in each player's actions. Moreover, by the previous paragraph, we obtain that for every action of PI there is an action of PII that guarantees that the payoff is non-positive. The minmax theorem implies that the value of the game is non-positive and that there is one action, say ℓ^*, such that $\ell^*(x) - f(x) \leq 0$ for every $x \in Q$. In particular, f dominates ℓ^*. This shows that v has the sandwich property.

As for the inverse direction, suppose that v has the sandwich property. Fix $x \in Q$. By Proposition 2 (ii) there is a affine function f that dominates v and satisfies $\psi_v(x) = f(x)$. Due to the sandwich property there is a linear function ℓ dominated by f and dominates v. In particular $f(x) \geq \ell(x)$. By Proposition 1 (ii), $\ell(x) \geq \int^{cav} x\, dv$ and therefore, $\psi_v(x) = f(x) \geq \int^{cav} x\, dv$ implying $\psi_v(x) = \int^{cav} x\, dv$. This completes the proof. □

4.2 Reformulating the sandwich property

An alternative way to look at the issue raised in the Introduction is the following. While ψ_v is a concave extension of v like the concave integral, it is not homogeneous. Thus, asking when $\psi_v = \int^{cav} \cdot dv$ is equivalent to asking under what assumptions on v, the extension ψ_v is homogeneous. Put differently, when is the least concave function over Q that dominates v homogeneous?

Theorem 1 provides a clear answer. It is immediate that point (iii) of the theorem is equivalent to the homogeneity of ψ_v, implying the following corollary.

Corollary 1 *Fix a capacity v. Then, v has the sandwich property if and only if ψ_v is homogenous.*

Considering this point of view, we obtain a Shapley-Bondareva like condition (see Bondareva [4] and Shapley [14]) for the sandwich property. For every $x \in Q$, let

$$\mathcal{D}(x) = \left\{ (\mathcal{S}, (\alpha_S)_{S\in\mathcal{S}})\,;\ \mathcal{S} \subseteq 2^N, \sum_{S\in\mathcal{S}} \alpha_S \mathbf{1}_S = x, \alpha_S \geq 0 \text{ and } \sum_{S\in\mathcal{S}} \alpha_S \leq 1 \right\}.$$

When $\psi_v(x) = \sum_{S\in\mathcal{S}} \alpha_S v(S)$ we say that $(\mathcal{S}, (\alpha_S)_{S\in\mathcal{S}}) \in \mathcal{D}(x)$ is a *ψ_v-optimal* (decomposition of x). The following result follows Corollary 1.

Proposition 3 *Fix a capacity v. Then, v has the sandwich property if and only if for every $x \in Q$, $\gamma \in (0, 1)$, ψ_v-optimal $(\mathcal{S}, (\alpha_S)_{S\in\mathcal{S}}) \in \mathcal{D}(x)$, and $(\mathcal{T}, (\beta_T)_{T\in\mathcal{T}}) \in \mathcal{D}(\gamma x)$, one has*

$$\sum_{S\in\mathcal{S}} \gamma \alpha_S v(S) \geq \sum_{T\in\mathcal{T}} \beta_T v(T).$$

4.3 Other implications

Lemma 2 *Let v be capacity. Then, v has the sandwich property, if and only if each of its sub-capacities has the sandwich property.*

The *core* (Bondareva [4] and Shapley [14]) of a capacity v is the set of linear functions that dominate v and attain the same value as v on $\mathbf{1}_N$. Such a linear function is called a *core allocation* of v. A capacity is *totally balanced* if for every $S \subseteq N$ the core of the sub-capacity v_S is non-empty.

Claim 1 $e_v(\mathbf{1}_N) = 1$ *if and only if v has a non-empty core.*

Following the proof of Claim 1, it can be shown that $e_v(\mathbf{1}_S) = 1$ if and only if v is totally balanced. This leads to the following corollary.

Corollary 2 $e_v(\mathbf{1}_S) = 1$ *for every coalition $S \subseteq N$ if and only if v is totally balanced.*

The following corollary is an immediate consequence of Claim 1 and Theorem 1.

Corollary 3 *If v has the sandwich property, then it is totally balanced.*

Claim 2 *Any totally balanced capacity with three players has the sandwich property.*

A capacity v is *convex* (Shapley [15]) if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$, for every $S, T \subseteq N$. The following claim follows Lovasz [10], which shows that the concave integral coincides with Choquet integral if and only if the capacity is convex.

Claim 3 *If v is convex, then it has the sandwich property.*

Note that the capacity in Example 1 is totally balanced but not convex. Thus, Claim 3 does not imply Claim 2.

A capacity v is *exact* (Schmeidler [12]) if for every $S \subseteq N$ there is a core allocation p such that[4] $p(S) = v(S)$. A capacity has a *large core* (Sharkey [16]; See also Estévez-Fernández [6]) if for every linear function ℓ that dominates v, there is a core allocation that is dominated by ℓ and dominates v.

Remark 2 There are two differences between the definition of the sandwich property and large core. First, the definition of the sandwich property relates to every affine function f that dominates v, while that of a large core relates to every linear function ℓ that dominates v. Second, while the definition of large core requires that the linear function dominated by ℓ and dominating v be a core allocation, the sandwich property does not require this.

The following is an example of a capacity that has the sandwich property but is neither exact (and therefore not convex (Schmeidler [12]), implying that the inverse of Claim 3 is incorrect), nor has a large core.

Example 4 (Example 1 continued) Note that the core of v is non-empty and it consists of a single point $(0, 1, 0)$. In particular, there is no core allocation p that satisfies $p(2) = v(2)$. Thus, v is not exact. Finally, consider the linear function $\ell'(x_1, x_2, x_3) = x_1 + x_3$. This function dominates v, but dominates no core allocation of v. Therefore, v does not have a large core.

Example 1 shows that there are capacities with the sandwich property that have a core that is not large. The next example shows that inverse implication does not hold either: largeness of the core does not imply the sandwich property.

Example 5 Consider the set $N = \{1, \ldots, 6\}$ and v defined over it, as in Example 2. Recall that v does not have the sandwich property. Consider $N' = N \cup \{7\}$. That is, N' contains the additional player 7. Define w over N' as follows. $w(S) = v\,(S \setminus \{7\})$ if $S \neq N'$ and $w(N') = 7$. (In words, player 7 contributes nothing to S, unless $S = N$, in which case his contribution amounts to $w(N') - v(N) = 7 - 1 = 6$.) By definition $w_N = v$. Recall that v does not have the sandwich property and therefore, by Lemma 2, w does not have this property either.

We now show that w has a large core. Assume that $\ell(x_1, \ldots, x_7) = \sum_{i=1}^{7} a_i x_i$ is a linear function that dominates w. It implies that $a_i \geq 0$ for every $i \in N'$ and $\sum_{i=1}^{7} a_i \geq 7$.

[4] For a vector $x \in \mathbb{R}^N$ and a coalition $S \subseteq N$, by $x(S)$ we mean the inner product $\langle x, \mathbf{1}_S \rangle = \sum_{i \in S} x_i$.

Table 1 5 vectors whose sum is 2

	1	2	3	4	5	6	7
x_1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	0	0	0	0
x_2	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1
x_3	$\frac{1}{3}$	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{5}{6}$
x_4	0	0	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$
x_5	0	$\frac{1}{3}$	0	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{5}{6}$

Define, $a_i(\beta) = \min[a_i, \beta \cdot a_i + (1-\beta)], i = 1, \ldots, 7$. When $\beta = 0$, $\sum_{i=1}^{7} a_i(\beta) = \sum_{i=1}^{7} \min(a_i, 1) \leq 7$ and when $\beta = 1$, $\sum_{i=1}^{7} a_i(\beta) = \sum_{i=1}^{7} a_i \geq 7$. Thus, there is $\beta^* \in [0, 1]$ such that $\sum_{i=1}^{7} a_i(\beta^*) = 7$. Define $p(x_1, \ldots, x_7) = \sum_{i \in N} a_i(\beta^*) x_i$. Note that $a_i(\beta^*) \leq a_i$ $(i = 1, \ldots, 7)$ with equality only when $a_i \leq 1$. Clearly, $w(N') = 7 = p(\mathbf{1}_{N'})$ and p is dominated by ℓ.

We proceed to show that p is a core member of w. Fix $S \neq N'$. Either ***(a)*** $p(\mathbf{1}_S) \geq 1$ (which occurs, for instance, when for one $i \in S$, $a_i(\beta^*) \geq 1$), in which case $p(\mathbf{1}_S) \geq 1 \geq w(S)$; or ***(b)*** $1 > p(\mathbf{1}_S)$. This happens only when $1 > a_i(\beta^*) = a_i$ for every $i = 1, \ldots, 7$, and therefore, $p(\mathbf{1}_S) = \ell(\mathbf{1}_S)$. Since $\ell(\mathbf{1}_S) \geq w(S)$ we obtain, $p(\mathbf{1}_S) \geq w(S)$. This completes the proof that p dominates w. We conclude that indeed, p is a core member of w and is dominated by ℓ, showing that w has a large core.

The example suggests the following claim.

Claim 4 *If any sub-capacity of v has a large core, then v has the sandwich property.*

The following example shows that in the spirit of the example above, we can extend v to be exact, showing that exactness does not imply the sandwich property.

Example 6 Consider Table 1.

Define $w(S) = \min_i x_i(S)$ for every $S \subseteq N'$. For instance, when $S = \{1, 2, 3\}$ the minimum is obtained at x_2 where $x_2(S) = 0$, and when $S = \{1, 4, 5\}$, the minimum is obtain at all x_i's, and $x_i(S) = \frac{2}{3}$. The purpose of x_3, x_4 and x_5 is to ensure that for any S that contains all members of N but one from $\{1, 2, 3\}$, $w(S) \leq \frac{5}{6}$ (e.g., when $S = \{2, \ldots, 6\}$, $w(S) = x_3(S) = \frac{5}{6}$). As a minimum of homogeneous functions that attain the same value at $\mathbf{1}_{N'}$, w (defined on N') is exact.

Consider w_N. As in Example 2, $f(x_1, \ldots, x_6) = \frac{1}{2} + \frac{1}{6}x_1 + \frac{1}{6}x_2 + \frac{1}{6}x_3$ dominates w_N. Moreover, in Example 2 we showed that v does not have the sandwich property. The argument was using f as a dominating affine function, and the facts that $v(1, 4, 5) = v(2, 5, 6) = v(3, 4, 6) = \frac{2}{3}$ and $v(N) = 1$. These equalities are also satisfied by w_N. We therefore conclude that w_N also does not have the sandwich property, and by Lemma 2, w neither. To sum up, w is exact and does not have the sandwich property.

5 A final comment

In this note we address the issue of the structure of a set function v for which the concave integral, as an extension of v to a bounded domain, abides to the boundedness assumptions. When described formally, we ask when is the concave integral coincides with the more restrictive extension ψ_v. We show that a property termed the sandwich property is a necessary and sufficient condition for that to occur.

We have discussed above monotonic set functions (or, capacities). However one can think of an example in line with the introduction, for which $v(S) = p(S) - c(S)$, where $p(S)$ is the (gross) productivity of coalition S per unit of time, and $c(S)$ the cost of convening S per unit of time. That is, $v(S)$ captures the net gain from coalition S producing during a unit of time. In this case v is no longer a capacity as it is not necessarily monotonic or non-negative.

However, our analysis holds for general set functions if we consider one modification. In this case the definition of the concave integral should be slightly amended as follows (compare with Eq. (3)):

$$\int^{cav,\leq} x\, dv = \max\left\{\sum_{S\subseteq N} \beta_S v(S);\ \ \sum_{S\subseteq N} \beta_S \mathbf{1}_S \leq x, \beta_S \geq 0\right\}. \tag{4}$$

The difference between Eqs. (3) and (4) is that in the former the decomposition $\sum_{S\subseteq N} \beta_S \mathbf{1}_S$ is forced to be equal to x while in the latter it is allowed to be smaller or equal to x. In other words, Eq. (4) allows all possible *sub-decompositions* in which x does not have to be fully exhausted.

One can also consider an alternative definition of an integral with respect to a (non-monotonic) set function. For a set function v let v_* be a capacity defined by $v_*(S) = \max\{v(T) : T \subseteq S\}$, for every $S \subseteq T$ (see Murofushi et al. [11]). It is easy to see that v_* is monotonic. Now it is possible to consider the definition of the concave integral as in Eq. (3), with the difference that it is taken with respect to v_*. However, the two definitions above coincide. That is, $\int^{cav,\leq} \cdot\, dv = \int^{cav} \cdot\, dv_*$. The proof is similar to the one of point (ii) of Lemma 1 in Lehrer and Teper [9] and is omitted.

Appendix

Proof of Lemma 1 Fix a capacity v. The homogeneity of e_v is implied by the definition of $\int^{cav} \cdot dv$. Now, fix $x \in Q$ and consider a decomposition of x, $\sum_{S\subseteq N} \alpha_S \mathbf{1}_S = x$. Letting $i^* = \arg\max_{i\in\{1,\ldots,n\}} x_i$, we get that $\sum_{S\subseteq N} \alpha_S \geq \sum_{S\subseteq N, i^*\in S} \alpha_S = x_{i^*} = \max x$. Since $e_v(x)$ is the minimum over all such decomposition, we obtain $e_v(x) \geq \max x$. □

Proof of Lemma 2. Let v be a capacity defined over N and v_S its sub-capacity. If v_S does not have the sandwich property, then there is an affine function $f_S(x) = a + \sum_{i\in S} a_i x_i$ (where $x \in [0, 1]^S$) that dominates v_S, and there is no linear function dominated by f_S and that is dominating v_S. Denote,

$$b = \max_{\substack{A\subseteq S\\ B\subseteq N\setminus S}} [v(A \cup B) - v(A)].$$

b is the maximal contribution of a coalition B in $N\backslash S$ to a coalition A in S. Define the following affine function over Q, $f(x) = a + \sum_{i\in S} a_i x_i + \sum_{i\in N\backslash S} bx_i$. It is clear that f dominates v. Suppose, to the contrary of the assumption that v does have the sandwich property and that there exists a linear function $\ell(x) = \sum_{i\in N} c_i x_i$, dominated by f and is dominating v. Then, $\ell_S(x) = \sum_{i\in S} c_i x_i$ is dominated by f_S and is dominating v_S. This is a contradiction.

On the other hand, if v has the sandwich property, let f_S be an affine function that dominates v_S. Define f in the same way as defined above. By assumption, there is a linear function ℓ that is dominating v and is dominated by f. The restriction of ℓ to S, ℓ_S, is dominating v_S and is dominated by f_S, implying that v_S has the sandwich property. □

Proof of Claim 1. Suppose that the core of v is non-empty and let $\sum \alpha_S \mathbf{1}_S$ be a decomposition of $\mathbf{1}_N$, then by the Shapley-Bondareva theorem (Bondareva [4] and Shapley [14]), $\sum \alpha_S v(S) \leq v(N)$. Thus, the decomposition $\mathbf{1}_N$ of itself is optimal, implying that $e_v(\mathbf{1}_N) \leq 1$, and by Lemma 1, $e_v(\mathbf{1}_N) = 1$.

Now suppose that $e_v(\mathbf{1}_N) = 1$. It implies that any sharp decomposition of $\mathbf{1}_N$, $\sum \alpha_S \mathbf{1}_S$, satisfies $S = N$ whenever $\alpha_S > 0$. Thus, the decomposition is in fact $\mathbf{1}_N$ itself. □

Proof of Claim 2. Let $x \in Q$ and $\sum \alpha_S \mathbf{1}_S$ be its sharp decomposition. W.l.o.g, $\alpha_{\{12\}} \leq \alpha_{\{13\}}, \alpha_{\{23\}}$. Thus we can replace $\alpha_{\{12\}}\mathbf{1}_{12}+\alpha_{\{13\}}\mathbf{1}_{13}+\alpha_{\{23\}}\mathbf{1}_{23}$ by $2(\alpha_{\{12\}}\mathbf{1}_{123}+(\alpha_{\{13\}}-\alpha_{\{12\}})\mathbf{1}_{13} + (\alpha_{\{23\}} - \alpha_{\{12\}})\mathbf{1}_{23})$.

We conclude that one may assume that in a sharp decomposition the coefficient of $\mathbf{1}_{12}$ is 0. Furthermore, based on the total balancedness of v, a similar argument would imply that no two coalitions with positive coefficients in the decomposition are disjoint. Thus, at most one singleton has a positive coefficient and the one that has a positive coefficient, if exists, is included in the other coalitions whose coefficients are positive.

We now show that there is $i \in N$ that is a member of all coalitions whose coefficients are positive. This shows that $e(x) = \max x$ and with the help of Theorem 1 completes the proof.

Case 1: $\alpha_{\{1\}} > 0$. Then, $\alpha_{\{2\}} = \alpha_{\{3\}} = \alpha_{\{23\}} = 0$, which means that $1 \in S$ when $\alpha_{\{S\}} > 0$. Case 2: $\alpha_{\{2\}} > 0$. Then, $\alpha_{\{1\}} = \alpha_{\{3\}} = \alpha_{\{13\}} = 0$ implying that $2 \in S$ if $\alpha_{\{S\}} > 0$. Case 3: $\alpha_{\{3\}} > 0$. Similar to the previous case. Finally, Case 4: $\alpha_{\{i\}} = 0$ for every $i \in N$. Since $\alpha_{\{12\}} = 0$, $3 \in S$ whenever $\alpha_S > 0$. This completes the proof. □

Proof of Claim 3. By Lovazc [10] (pp. 246–249), when v is convex for any $x \in Q$, the Choquet decomposition is optimal. Thus, $e(x) = \max x$. □

Proof of Claim 4. Assume that any sub-capacity of v has a large core. This implies that v is totally balanced. Thus, every sub-capacity of v is totally balanced and has a large core. As such, any sub-capacity is exact (Sharkey [16]), implying that v is convex (Biswas et al. [3]). By Claim 3, v has the sandwich property. □

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