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An Axiomatization of the Banzhaf Value

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Abstract: An axiomatization of the Banzhaf value is given. It is based on a version of three axioms, which are common to all the semi-values, and on an additional reduction axiom.

1 Introduction

A lot of effort had been devoted to the research of the semi values of games [2], especially to the Shapley-value. An axiomatization was given to the Shapley value [6], and to the Banzhaf value [3, 4].

The axiomatization given in [4] is based on the tool of the compound game and it does not determine the Banzhaf value uniquely. The axiomatization given in [3] treats a different approach. It is based on four axioms, three of them are standard and in fact determine the semi values [2]. We adopt here these three axioms.

We replace the fourth axiom of [3] by some reduction axiom, which has an intuitive meaning. This axiom says something about the value of a game in terms of another game, with smaller number of players.

It is worthwhile to mention that the use of such axioms is common in axiomatization of solution concepts (see for example: in [5] an axiomatization of the prekernel, and in [7] an axiomatization of the prenucleolus).

We define here for each coalition T two T-games derived from the original game by amalgamting all the players of T to one player called $\bar{T}$. Our additional axiom says that for any two-players-coalition $T = \{i, j\}$ the sum of the values of i and of j in the original game is less or equal to the value of $\bar{T}$ in the T-games. According to this axiom a unification of any two players is profitable.

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The main significance of the following axiomatization is the indication of the difference between the Banzhaf value and the Shapley value. For example take the three player weighted majority game $\left[1;\frac{1}{3},\frac{1}{3},\frac{1}{3}\right]$; its Banzhaf value is $\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$ while its Shapley value is $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$. After amalgamating players 1 and 2 to a new player we will get the game $\left[1;\frac{2}{3},\frac{1}{3}\right]=\left[1;\frac{1}{2},\frac{1}{2}\right]$. The Banzhaf value and the Shapley value coincide on the two player games and they are equal in this particular game to $\left(\frac{1}{2},\frac{1}{2}\right)$. Notice that $\frac{1}{4}+\frac{1}{4}\leqslant\frac{1}{2}$, while $\frac{1}{3}+\frac{1}{3}>\frac{1}{2}$. In other words, in this case according to the Shapley value unification is harmful while according to the Banzhaf value unification is never harmful, and in some cases it is profitable.

2 Definitions, the Axiom Systems, and Theorems

Before introducing the axioms we need several notations. A *game in coalitional function form* is a pair (N, v) where N is a finite set of players and v is a real valued function defined on the subsets of N and $v(\phi) = 0$. A *simple game* is a game (N, v) where v is ranged to $\{0, 1\}$. In a simple game all the coalitions $S \subset N$ with $v(S) = 1$ are called *winning* and all the rest *losing*.

A *monotonic* game is a game (N, v) which satisfies the condition whereby $S \subseteq T$ implies $v(S) \leqslant v(T)$.

Let $T \subseteq N$, $|N| = n$ the *T-unanimity* game denoted by $U_T^n = (N, u_T^n)$ is defined to be $u_T^n(S) = 1$ if $T \subseteq S$ and otherwise 0. Whenever $|T| = 1$ U_T^n is called *dictatorial*. Let $k \leqslant |T|$, the *k-T symmetric* game denoted by $S_T^{k,n} = (N, s_T^k)$ is defined to be $s_T^k(A) = 1$ if $k \leqslant |T \cap A|$ and otherwise 0.

We denote by G, SG and MSG the sets of all games, all the simple games and all the monotonic simple games respectively.

In order to simplify let us denote the game (N, v) by v whenever ambiguity cannot arise.

Let v and u be two simple games. Define the games $v \vee u$ and $v \wedge u$ as follows:

For any coalition S

$$v \vee u(S) = \text{Max}\ \{v(S), u(S)\}$$

$$\text{and } v \wedge u(S) = \min\ \{v(S), u(S)\}. \tag{1}$$

The Banzhaf value η corresponds to each $(N, v) \in G$ a vector $\eta(v)$ in $\mathbb{R}^{|N|}$. $\eta(v)$ is defined as follows:

$$\eta^i(v) = \frac{1}{2^{|N|-1}} \sum_{S \subseteq N} v(S \cup \{i\}) - v(S) \qquad 1 \leqslant i \leqslant n.$$

If v is an n players monotonic simple game, $2^{n-1}\eta^i(v)$ counts the number of losing coalitions S that become winning after i joins them. Here we arrive at the definitions of the T-games. Let $(N, v) \in G$ and $T \subseteq N$. In case all the players in T are amalgamated into one player $\bar{T}$, two games can be derived:

$$(N - T \cup \{\bar{T}\}, v_T) \quad \text{and} \quad (N - T \cup \{\bar{T}\}, v_T^m),$$

where for each $S \subseteq N - T$

$$v_T(S) = v(S), \qquad v_T(S \cup \{\bar{T}\}) = v(S \cup T)$$

and[2] (2)

$$v_T^m(S) = v(S), \qquad v_T^m(S \cup \{\bar{T}\}) = \max_{\phi \neq B \subseteq T} v(S \cup B).$$

Notice that whenever v is monotonic, $v_T = v_T^m$.

We will use two axiom-systems, one for SG solely and the other for G. The two systems differ from one another only by the linearity[3] axiom used.

The first system is:

(D) If $v(S \cup \{i\}) = v(S) + v(\{i\})$ for every coalition S such that $i \notin S$ then $\varphi^i(v) = v(\{i\})$.

(ET) If for every coalition $S \subseteq N \backslash \{i, j\}$ $v(S \cup \{i\}) = v(S \cup \{j\})$ then $\varphi^i(v) = \varphi^j(v)$.

(UI)[4] $\varphi(v \vee u) + \varphi(v \wedge u) = \varphi(v) + \varphi(u)$.

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2 The m in v_T^m stands for max.

3 We call it linearity because additivity (or super additivity) is reserved for the fourth axiom.

4 For union-intersection.

(SA) $\varphi^i(v) + \varphi^j(v) \leqslant \varphi^{\bar{T}}(v_T)$ for every two-players-coalition $T = \{i, j\}$.

(SA^m) $\varphi^i(v) + \varphi^j(v) \leqslant \varphi^{\bar{T}}(v_T^m)$ for every two-players-coalition $T = \{i, j\}$.

(D) is the dummy axiom used by Shapley [6].

(ET) is the equal treatment axiom. It says that whenever v remains unchanged after interchanging i by j, $\varphi^i(v)$ equals to $\varphi^j(v)$.

(UI) is an axiom due to Dubey [1] and serves as the linearity axiom when the discussion concentrates on SG. Reasoning for (UI) can be found in [3][5].

If a question arises as for the amalgamtion of coalitions containing more than two players, one can argue that the amalgamation of "big" coalitions (according to (SA)) is done step by step. First two players are amalgamated to one player and then a third player is amalgamated to the new one, and so on. A technical answer could be given as follows: a general super additivity, namely $\sum_{i \in T} \varphi^i(v) \leqslant \varphi^{\bar{T}}(v_T)$ for all $T \subseteq N$, is not consistent with the other three axioms.

Theorem A: φ satisfies (D), (ET), (UI) and either (SA) or (SA^m) for every $v, u \in SG$ if and only if φ is the Banzhaf value.

In fact the following holds.

Proposition 1:

(i) If $v \in SG$ then

$$\eta^i(v) + \eta^j(v) \leqslant \eta^{\bar{T}}(v_T^m)$$

for every coalition $T = \{i, j\}$ and equality holds for every $T = \{i, j\}$ if and only if $v \in MSG$.

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5 The fourth axiom of [3] is $\sum_{i \in N} \varphi^i(v) = \sum_{i \in N} \eta^i(v)$.

(ii) If $v \in SG$ then

$$\eta^i(v) + \eta^j(v) = \eta^{\bar{T}}(v_T)$$

for every coalition $T = \{i, j\}$.

It means that (SA) together with the other axioms give additivity instead of super additivity.

Remarks:

1. The Banzhaf value satisfies (UI) on SG as it will be shown below, while the Shapley value satisfies it on MSG and not on SG.

2. The axiom (SA) distinguishes between the Shapley value and the Banzhaf value. In terms of v_T the efficiency axiom can be written as

$$\sum_{i \in T} \varphi^i(v) = \varphi^{\bar{T}}(v_T) \text{ for every } |T| = |N|,$$

while Banzhaf value satisfies it for every $|T| = 2$.

3. Given that φ is equal to the Banzhaf value in all the two players games (which coincide with the Shapley value), the "2-efficiency" axiom:

$$\sum_{i \in T} \varphi^i(v) = \varphi^{\bar{T}}(v_T) \text{ for every } |T| = 2$$

determines the Banzhaf value uniquely (the proof appears in the Appendix).

Finally, we can use the standard linearity axiom of Shapley in order to axiomize the Banzhaf values on G.

(LI) $\quad \varphi(v) + \varphi(u) = \varphi(u + v)$.

Theorem B: φ satisfies (D), (ET), (LI) and either (SA) or (SA^m) for every $v, u \in G$ if and only if φ is the Banzhaf value on G.

3 Proofs

Proposition 1 Proof: Let $(N, v) \in SG$, where $|N| = n$, then by the definition of η

$$2^{n-1}(\eta^i(v) + \eta^j(v)) = \sum_{S \subseteq N} [v(S \cup \{i\}) - v(S)] + \sum_{S \subseteq N} [v(S \cup \{j\})$$

$$- v(S)] = \sum_{S \subseteq N \backslash \{i,j\}} [v(S \cup \{i\}) - v(S) + v(S \cup \{i,j\}) - v(S \cup \{j\})]$$

$$+ \sum_{S \subseteq N \backslash \{i,j\}} [v(S \cup \{j\}) - v(S) + v(S \cup \{i,j\}) - v(S \cup \{i\})]$$

$$= 2 \sum_{S \subseteq N \backslash \{i,j\}} [v(S \cup \{i,j\}) - v(S)]$$

The last term is equal to $2 \cdot 2^{n-2} \eta^{\bar{T}}(v_T)$ for $T = \{i,j\}$ (this gives the proof for (ii)) and furthermore it is less or equal to

$$2 \cdot 2^{n-1} \eta^{\bar{T}}(v_T^m) = 2 \sum \operatorname*{Max}_{\phi \neq B \subseteq \{i,j\}} v(S \cup B) - v(S).$$

$v \in MSG$ if and only if for every $\{i,j\}$ and $S \subseteq N \backslash \{i,j\}$

$$v(S \cup \{i,j\}) = \operatorname*{Max}_{\phi \neq B \subseteq \{i,j\}} v(S \cup B).$$

So equality holds iff $v \in MSG$ as desired at (i).

Theorems A and B Proofs: It is known that η satisfies (D), (ET) and (LI). Proposition 1 shows that η satisfies also (SA) and (SA^m). Thus in order to show that η satisfies all the axioms it remains to show that for every $v, u \in SG$

$$\varphi(v \vee u) + \varphi(v \wedge u)) = \varphi(v) + \varphi(u).$$

Let $S \subseteq N$ and $i \in N$

$$2^{n-1}(\eta^i(v \vee u) + \eta^i(v \wedge n) = \sum_{S \subseteq N} \text{Max}\,\{v(S \cup i), u(S \cup i)\}$$

$$- \text{Max}\,\{v(S), u(S)\} + \sum_{S \subseteq N} \text{Min}\,\{v(S \cup i), u(S \cup i)\}$$

$$-\operatorname{Min}\{v(S), u(S)\} = \sum_{S \subseteq N} [v(S \cup i) + u(S \cup i) - v(S) - u(S)]$$

$$= 2^{n-1}(\eta^i(v) + \eta^i(u)).$$

The first step in order to prove the other direction of Theorem A will be to show that the axiom system determines the values of φ on the unanimity games. This step is proved through two inductions simultaneously, the first one is on the players number and the second is on the number of the non-dummy players in the unanimity game. We give here the induction step.

We assume that φ is already determined on the simple games with n players and on the games of the type U_T^{n+1} where $|T| \leqslant k$. Furthermore, we assume that

$$\varphi^i(U_T^{n+1}) = 1/2^{|T|-1} \text{ if } i \in T \text{ and otherwise } 0.$$

Let $T \subseteq \{1, \ldots, n+1\}$ where $|T| = k+1$, and $i \in T$. Denote

$$T' = T \setminus \{i\}.$$

By (UI):

$$\varphi(U_{T'}^{n+1} \vee U_{\{i\}}^{n+1}) + \varphi(U_{T'}^{n+1} \wedge U_{\{i\}}^{n+1}) = \varphi(U_{T'}^{n+1}) + \varphi(U_{\{i\}}^{n+1}). \tag{3}$$

By (D) and (ET) there are constants a, b and c s.t.

$$\varphi^j(U_T^{n+1}) = \begin{cases} a & j \in T \\ 0 & j \notin T \end{cases} \tag{4a}$$

and

$$\varphi^j(U_{T'}^{n+1} \vee U_{\{i\}}^{n+1}) = \begin{cases} b & j \in T' \\ c & j = i \\ 0 & \text{otherwise.} \end{cases} \tag{4b}$$

By the induction hypothesis and by (4a), (4b)

$$a + b = 1/2^{k-1} \tag{5a}$$

and with (D) we get

$$a + c = 1. \tag{5b}$$

Now, in order to apply (SA) (or (SA^m)) take two players i, j from T and amalgamate them to one player. The game which derives is $U_{T'}^{n}$ (after identifying $j \in T'$ with the new player $\overline{ij}$).

Hence by (SA) (or by (SA^m))

$$2a \leqslant 1/2^{|T'|-1} = 1/2^{k-1}. \tag{6a}$$

Secondly, amalgamate the player i with any player from T' in the game $U_{T'}^{n+1} \vee U_{\{i\}}^{n+1}$. The game which derives is a dictatorial game so,

$$b + c \leqslant 1. \tag{6b}$$

Add (6a) to (6b) to get

$$2a + b + c \leqslant 1 + \frac{1}{2^{k-1}} \tag{7}$$

(5a), (5b) and (7) bring us to conclude that the inequalities in (6a) and (6b) become equalities. In particular

$$2a = \frac{1}{2^{k-1}}.$$

This gives the value of φ on U_T^{n+1}.

Theorem B's proof has the same procedure and we'll give here the induction step: Let $|T| = k + 1$ by (LI)

$$\varphi((|T| - 1)U_T^{n+1}) + \varphi(S_T^{k,n+1}) = \sum_{\substack{T' \subseteq T \\ |T'| = k}} \varphi(U_{T'}^{n+1}) \tag{8}$$

By (ET) and (D) there are constants a and b s.t.

$$\varphi^i(U_T^{n+1}) = \begin{cases} a & i \in T \\ 0 & i \notin T \end{cases}$$

and

$$\varphi^i(S_T^{k,n+1}) = \begin{cases} b & i \in T \\ 0 & i \notin T. \end{cases}$$

By (8) and by the induction hypothesis

$$ka + b = (|T| - 1)/2^{k-1} = k/2^{k-1}. \tag{9}$$

Amalgamate i to j $(i, j \in T)$ in U_T^{n+1} and derive the game $U_{T'}^n$ (as above $T' = T \backslash \{i\}$). Hence by (SA) (or (SA^m))

$$2a \leqslant \frac{1}{2^{k-1}}. \tag{10a}$$

Similarly by amalgamating i to j in $S_T^{k,n+1}$ (for some i and j in T) we will derive the game

$((N \backslash \{i, j\}) \cup \{\overline{ij}\}, v')$ where

$$v'(A) = \begin{cases} 1 & \overline{ij} \in A \text{ and } k - 2 \leqslant |(A \backslash \{\overline{ij}\}) \cap T| \\ 0 & \text{otherwise.} \end{cases}$$

By the induction hypothesis (v' is a game with a smaller number of players),

$$\varphi^{\bar{ij}}(v') = k/2^{k-1}, \quad \text{thus}$$

$$2b \leqslant k/2^{k-1}. \tag{10b}$$

From (10a) and (10b) we get

$$ka + b \leqslant k/2^k + k/2^k = k/2^{k-1} \tag{11}$$

(9) and (11) give equalities in (10a) and in (10b) instead of inequalities. In particular $a = 1/2^k$ as desired. By the same way $\varphi^i(c \cdot U_T^n) = c/2^{|T|-1}$ whenever $i \in T$ and otherwise 0, for any $c > 0$ and $T \subseteq N$.

From now on, both proofs are matters of linearity. As shown in [3] (*UI*) determine together with the value of φ on the unanimity games the values of φ on *MSG*. To conclude the proof of Theorem A it will be noticed first that a simple game (N, v) can be represented as $\bigvee_{v(T)=1} W_T^n$ where $W_T^n(S) = 1$ only when $S = T$, and second that by (*UI*)

$$\varphi(U_T^n) = \varphi(U_T^n - W_T^n \vee W_T^n) = \varphi(U_T^n - W_T^n \vee W_T^n) + \varphi(U_T^n - W_T^n \wedge W_T^n)$$

$$= \varphi(U_T^n - W_T^n) + \varphi(W_T^n)$$

so,

$$\varphi(W_T^n) = \varphi(U_T^n) - \varphi(U_T^n - W_T^n).$$

$\varphi(U_T^n)$ and $\varphi(U_T^n - W_T^n)$ are already determined because U_T^n and $U_T^n - W_T^n$ are both in *MSG*.

From the point in which φ is determined on the set $\{c \cdot U_T^{|N|} \mid c > 0, T \subseteq N\}$ on, the proof of Theorem B is essentially standard (see for example [4]).

Appendix

Proof of Remark 3: The proof is through induction on the players number. By the assumption φ is already determined on the two-players-games. Let $V = (\{1\}, v)$ a game with one player, then $v(1) = c$, to get $\varphi(v)$ take the following two players game u: $u(S) = c$ when $S = \{1, 2\}$ and 0 otherwise. After amalgamating 1 to 2 in u we will get v. By the assumption, $\varphi(u)$ is known ($\varphi(u) = \eta(u)$) so $\varphi^1(v) = \eta^1(u) + \eta^2(u)$ by "2-efficiency" axiom.

In order to establish that the Banzhaf value is determined uniquely on all the games, assume inductively that the Banzhaf value is already determined on all the games with n players or less. Take a game $V = (\{1, \ldots, n+1\}, v)$ of $n + 1$ players. By amalgamating i and j in V to a new player $\overline{ij}$ we will get the game $V_{\{i,j\}}$, which is of n players. By "2-efficiency" axiom we get $x_i + x_j = \varphi^{\overline{ij}}(V_{\{i,j\}})$ where $x_i = \varphi^i(V)$. We have got $\binom{n+1}{2}$ equations with $n + 1$ variables; an equation for each pair $i, j \in \{1, \ldots, n+1\}$. Obviously this equations system has degree $n + 1$. It is consistent because $\varphi^{\overline{ij}}(V_{\{i,j\}}) = \eta^i(V) + \eta^j(V)$, by Proposition 1, and by the induction hypothesis. Thus it has a unique solution which is $x_i = \eta^i(V)$.

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