TWO REMARKS ON BLACKWELL'S THEOREM

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Abstract. In a decision problem with uncertainty a decision maker receives partial information about the actual state via an information structure. After receiving a signal he is allowed to withdraw and get 0. We say that one structure is *better* than another when a withdrawal option exists, if it may never happen that the latter guarantees a positive profit while the former guarantees only 0. We characterize this order between information structures in terms that are different from those used by Blackwell's comparison of experiments.

We also treat the case of *malicious nature* that chooses a state in an adverse manner. It turns out that Blackwell's classical characterization holds also in this case.

1. Introduction

A decision problem is defined by a state space, a prior distribution, an action set and a utility function. Before taking an action a decision maker (DM) may obtain partial information about the true state of nature. The information is obtained through an information structure which chooses a signal with a probability that depends on the realized state. Comparing between information structures[1] has been the subject of many papers[2].

One possibility is to order information structures according to the expected utility they yield for the DM in a given decision problem. This ordering, however, is too specific, as one information structure may be better than another in a certain problem and worse in another. Blackwell [1] proposed a partial order that takes into account expected utility in the following way. It is said that one structure is better than another if *whatever the decision problem is*, the expected utility it guarantees is higher than that guaranteed by the other structure. This definition induces a partial order over information structures. Blackwell showed that this order can be equivalently defined in several different ways, some of them purely probabilistic. Blackwell characterized this partial order by three different means: stochastic matrices, expectation of convex functions, and mean preserving stochastic maps.

Although Blackwell's partial order is quite intuitive, it is too restrictive. It requires solid data about all possible decision problems. We propose another ordering between information structures. This ordering also induces a partial order, but it is defined on a wider range than Blackwell's order.

In this note we deal with a situation where a DM is always given a withdrawal option: to withdraw upon receiving a signal without any cost (i.e., getting the payoff 0). In the presence of such an option, the DM can always guarantee himself a non-negative payoff. It is said that a structure is *better* than another when a withdrawal option exists, if for every decision problem, it yields a positive payoff when the other structure does. That is, when a structure is better than another when a withdrawal option exists, it never happens that a positive profit is guaranteed when the DM gets his information via the latter structure, while a it is not guaranteed if the DM gets his information via the former structure.

[1]In the statistics literature, information structures are commonly referred to as *statistical experiments*.

[2]For a comprehensive survey of this literature see Torgersen [3]. A shorter review can be found in Le Cam [2].

Section 3 presents a characterization of the partial order "being better than when a withdrawal option exists" by means analogous to that of Blackwell's characterization. It turns out that one structure is better than another when a withdrawal option exists, if the latter results from the former by a multiplication with a non-negative matrix. In other words, the characterization is similar to that of Blackwell with non-negative matrixes replacing stochastic matrices.

In the same vein, in our characterization, non-negative convex functions replace convex functions and equality of two measures is replaced here by another relation between measures, absolute continuity.

In Section 4 we refer to a malicious nature that chooses a state in an adversary manner. Under theses circumstances, one can define an order between information structure in Blackwell's fashion: one structure is better than another when nature is malicious, if for every decision problem, when nature chooses a state in order to minimize payoffs, the expected utility it guarantees is higher than that guaranteed by the other. It is shown that this order coincides with 'being better than' defined by Blackwell.

2. Decision problems with incomplete information

Let K be a finite state set. The elements of K are called *states of nature*. An information structure provides an agent with partial information about the actual state. When the state of nature is k the agent receives a random signal s whose distribution depends on k. Formally, an *information structure* is a pair (S, σ), where S is a finite set of *signals* and $\sigma = \{\sigma_{k,s}\}_{k \in K, s \in S}$ is a stochastic matrix.[3] When the actual state is k, the agent receives the signal s with probability $\sigma_{k,s}$.

Upon getting a signal s the agent needs to take an action from a finite set A. If a is the action taken and k is the actual state, the agent receives the payoff $u(k, a)$. The *payoff matrix corresponding to* A, u, is the matrix $\big(u(k, a)\big)_{k,a}$ that has $|K|$ rows and $|A|$ columns.

A *decision problem* is given by (p, A, u) where $p \in \Delta(K)$ is a probability distribution over K, A is a finite set of actions and $u : K \times A \to \mathbf{R}$ is the utility function. Given an information structure $\mathcal{S} = (S, \sigma)$, the decision problem is described as follows: a state of nature $k \in K$ is randomly chosen according to p, then the agent receives

[3] A matrix $(\sigma_{k,s})$ is *stochastic* (resp. *sub-stochastic*) if $\sigma_{k,s} \geq 0$ for every k, s and $\sum_s \sigma_{k,s} = 1$ (resp. $\sum_s \sigma_{k,s} \leq 1$) for every k.

stochastic signal according to $\mathcal{S}$. Given the signal, the agent chooses an action $a \in A$ and receives payoff $u(k,a)$. Denote by $R(\mathcal{S};p,A,u)$ the best payoff the agent can receive in the decision problem.

Definition 1. We say that $\mathcal{S}$ is *better* than $\mathcal{T}$ if for every decision problem (p,A,u), $R(\mathcal{T};p,A,u)>0$ implies $R(\mathcal{S};p,A,u)>0$.

Remark 1. (i) If $R(\mathcal{S};p,A,u) \geq R(\mathcal{T};p,A,u)$ for every p,A,u, then $\mathcal{S}$ is better than $\mathcal{T}$. Conversely, assume that $\mathcal{S}$ is better than $\mathcal{T}$. If $R(\mathcal{S};p,A,u) < R(\mathcal{T};p,A,u)$ for some p,A,u, then define a new utility function $u' = u - R(\mathcal{S};p,A,u)$. One obtains, $R(\mathcal{S};p,A,u') = 0 < R(\mathcal{T};p,A,u)$, which is a contradiction. Thus, saying that $\mathcal{S}$ is better than $\mathcal{T}$ is equivalent to saying that $R(\mathcal{S};p,A,u) \geq R(\mathcal{T};p,A,u)$ for every p,A,u.
(ii) Let p_0 the uniform distribution over K. Note that if $R(\mathcal{S};p_0,A,u) \geq R(\mathcal{T};p_0,A,u)$ for every A,u, then $R(\mathcal{S};p,A,u) \geq R(\mathcal{T};p,A,u)$ for every p,A,u. Indeed, fix p,A,u and define, $u_0(k,a) = u(k,a)\frac{p(k)}{p_0(k)}$. Then, $R(\mathcal{S};p,A,u) = R(\mathcal{S};p_0,A,u_0) \geq R(\mathcal{T};p_0,A,u_0) = R(\mathcal{T};p,A,u)$.

Every information structure $\mathcal{S} = (S,\sigma)$ induces a probability measure m_σ over $\Delta(K)$ in the following way: Consider the probability space $K \times S$ equipped with the probability measure $p(k,s) = \frac{1}{|K|}\sigma_{k,s}$. Then the posterior distribution of k given s is a $\Delta(K)$-valued random variable defined over this probability space. We denote by m_σ the distribution of this random variable, and call it the *standard measure associated with* $\mathcal{S}$. This is a probability measure with finite support. Its atoms are the normalized columns of σ.

A *stochastic transformation* over $\Delta(K)$ is a function $T(x,E)$ defined for every $x \in \Delta(K)$ and a Borel subset E of $\Delta(K)$ such that $E \mapsto T(x,E)$ is a probability measure over $\Delta(K)$ for every $x \in \Delta(K)$, and such that $x \mapsto T(x,E)$ is measurable for every Borel subset E of $\Delta(K)$. For every probability measure m over $\Delta(K)$, the function $M(E) = \int T(x,E)\,\mathrm{d}m(x)$ is a probability measure over $\Delta(K)$. We denote $M = Tm$. T is called *mean preserving* if $\int y\; T(x,\mathrm{d}y) = x$ for every $x \in \Delta(K)$. The following theorem is due to Blackwell [1].

Theorem 1. (Blackwell [1]) *Let $\mathcal{S} = (S,\sigma)$ and $\mathcal{T} = (T,\tau)$ be two information structures. Then the following conditions are equivalent:*

(1) *For every p,A,u, if $R(\mathcal{T};p,A,u)>0$ then $R(\mathcal{S};p,A,u)>0$.*

(2) *There exists a stochastic matrix* $\varepsilon = \{\varepsilon_{s,t}\}_{s\in S, t\in T}$ *such that* $\tau = \sigma\varepsilon$, *the product of the matrices* σ *and* ε.
(3) *There exists a mean-preserving stochastic map* T *over* $\Delta(K)$ *such that* $Tm_\tau = m_\sigma$.
(4) *For every convex and continuous function* $h : \Delta(K) \to \mathbf{R}$, *if* $\int h \; dm_\tau > 0$ *then* $\int h \; dm_\sigma > 0$.

3. Decision problems with a withdrawal option

Assume that after having received a signal the agent is allowed to withdraw and to obtain the payoff 0. Formally, the decision problem (p, A, u) with a *withdrawal option* is the decision problem (p, A^0, u), where $A^0 = A \cup \{\underline{0}\}$ and $u(k, \underline{0}) = 0$ for every $k \in K$. Denote by $R^w(\mathcal{S}; p, A, u)$ the agent's optimal payoff, $R(\mathcal{S}; p, A^0, u)$.

A decision maker who needs to choose between obtaining information via $\mathcal{S}$ or via $\mathcal{T}$ before knowing the payoff function has no problem when $\mathcal{S}$ is better than $\mathcal{T}$. However, the "better than" order is not complete and quite often neither $\mathcal{S}$ is better than $\mathcal{T}$ nor $\mathcal{T}$ is better than $\mathcal{S}$. However, suppose that a withdrawal option is available. Moreover, suppose that the only information about the information structures is that whenever a positive profit is guaranteed when getting signals through $\mathcal{T}$, it is always the case when getting signals through $\mathcal{S}$. In the sense that it may never happen that getting signals through $\mathcal{T}$ ensures a positive profit while getting signals through $\mathcal{S}$ ensures only zero profit, $\mathcal{S}$ is better than $\mathcal{T}$. This order is formally defined as follows.

Definition 2. We say that $\mathcal{S}$ is *better* than $\mathcal{T}$ when a withdrawal option exists, if for every decision problem (p, A, u), $R^w(\mathcal{T}; p, A, u) > 0$, implies $R^w(\mathcal{S}; p, A, u) > 0$.

It is clear that when $\mathcal{S}$ is better than $\mathcal{T}$ when a withdrawal option exists, it is in particular better than $\mathcal{T}$. Theorem 1 states that $\mathcal{S}$ is better than $\mathcal{T}$ iff there exists a stochastic matrix ε such that $\tau = \sigma\varepsilon$. In the following theorem the fact that $\mathcal{S}$ is better than $\mathcal{T}$ when a withdrawal option exists, is characterized by weaker conditions than that of Theorem 1. For instance, $\tau = \sigma\varepsilon$, where ε is merely a matrix whose entries are nonnegative.

Example 1. Let the number of states of nature be 4 and σ, τ be given by

$$\sigma = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \text{ and } \tau = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

There exists no stochastic matrix ε such that $\tau = \sigma\varepsilon$. However, denote,

$$\varepsilon = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

one obtains $\tau = \sigma\varepsilon$. Thus, there exists a matrix ε with nonnegative entries such $\tau = \sigma\varepsilon$ and, by the following theorem, the information structure corresponding to σ is better than that corresponding to τ when a withdrawal option exists.

Condition (1) of Theorem 1 means that there exists a stochastic transformation (i.e., a linear transformation that maps *probability measures* over S to *probability measures* over T) which maps the k-th row of σ to the k-th row of τ. Our main result is analogous to Theorem 1. It characterizes when one information structure is better than another when a withdrawal option exists. However, instead of using terms of stochastic matrices, it uses terms of matrices that have nonnegative entries.

A matrix $\varepsilon = \{\varepsilon_{s,t}\}_{s\in S, t\in T}$ with nonnegative entries induces a linear transformation that maps *measures* (not necessarily probability) over S to *measures* over T. The fact that $\tau = \sigma\varepsilon$, with ε being a matrix with nonnegative entries, means that there exists such a transformation that maps the k-th row of σ (which is a probability measure of S) to the k-th row of τ (a probability measure over T).

Theorem 2. *The following conditions are equivalent:*

(1) *$\mathcal{S}$ is better than $\mathcal{T}$ when a withdrawal option exists.*
(2) *There exists a matrix ε with nonnegative entries such that $\tau = \sigma\varepsilon$.*
(3) *For every convex, continuous and nonnegative function $h : \Delta(K) \to \mathbf{R}$, if $\int h \; dm_\tau > 0$ then $\int h \; dm_\sigma > 0$.*
(4) *There exists a mean-preserving stochastic map T over $\Delta(K)$ such that Tm_τ is absolutely continuous with respect to m_σ.*

Proof. (1) $\Rightarrow$ (2): Assume that there exists no matrix ε with non-negative entries such that $\tau = \sigma\varepsilon$. In particular, there exists some column τ_{*t} of τ that is not a conic combination[4] of the columns of σ. By the separation theorem there exists a $|K| \times 1$ matrix α such that[5] $\langle \tau_{*t}, \alpha\rangle > 0$ but $\langle \sigma\varepsilon, \alpha\rangle \leq 0$ for every $|S| \times 1$ matrix ε

[4]A *conic combination* of vectors is a linear combination with nonnegative coefficients

[5]For matrices X,Y of the same dimension, we denote $\langle X, Y\rangle = \mathrm{tr}(X^tY)$= the sum of the entries along the main diagonal of the matrix X^tY.

with nonnegative coefficients. (Note that the action set A corresponding to α is a singleton.) Let p be the uniform distribution over K.

Consider the strategy (in the game played under τ) that prescribes playing the action a when obtaining the signal t, and withdrawing otherwise. The expected payoff of this strategy is $\frac{\sum_k \tau_{kt}}{|K|}\langle \tau_{*t}, \alpha\rangle > 0$. Thus, $R^w(\mathcal{T}; p, A, u) > 0$. However, the expected payoff of any strategy γ is $\langle \sigma\gamma, \alpha\rangle \leq 0$. This means that $R^w(\mathcal{S}; p, A, u) = 0$, which proves the desired assertion.

(3) $\Rightarrow$ (2): If there exists no matrix ε with non-negative entries such that $\tau = \sigma\varepsilon$, let α be as in the previous paragraph and define, $h(x) = \max(x\alpha, 0)$ (α is a $|K|$ dimensional vector and $x\alpha$ the inner product of x and α). The function h is non-negative and as a maximum of two linear function, it is convex. Finally, the properties of α imply that $\int h \ \mathrm{d}m_\tau > 0$ but $\int h \ \mathrm{d}m_\sigma = 0$, contrary to (3).

(2) $\Rightarrow$ (1): let A be the set of actions. A *strategy* of the agent is given by a sub-stochastic matrix $\gamma = \{\gamma_{s,a}\}_{s\in S, a\in A}$: if the signal is s the player takes action a with probability $\gamma_{s,a}$. Assume w.l.o.g. that p is the uniform distribution over K. Then the expected payoff to the player is given by $\frac{1}{|K|}\langle \tau\gamma, \alpha\rangle$, where α is the payoff matrix corresponding to A, u. Assume that this is strictly greater than 0. By assumption, $\tau = \sigma\varepsilon$ for some matrix ε with nonnegative entries. Thus, $\varepsilon\gamma = C\gamma'$ for some sub-stochastic matrix γ' and a constant $C > 0$. If the agent uses the strategy γ' in the $\mathcal{S}$-game, then his payoff is

$$R^w(\mathcal{S}; p, A, u) = \frac{1}{|K|}\langle \sigma\gamma', \alpha\rangle = \frac{1}{C|K|}\langle \tau\gamma, \alpha\rangle = \frac{1}{C}R^w(\mathcal{T}; p, A, u) > 0.$$

(2) $\Rightarrow$ (4): Let x be an atom of μ_τ, which corresponds to a column of τ. By (2), this column is a conic combination of some columns of σ. It follows that x is in the convex hull of the atoms of m_σ. Therefore there exists a probability measure, μ_x, over $\Delta(K)$ which is absolutely continuous w.r.t. m_σ such that $x = \int y \ \mathrm{d}\mu_x$. We let $T(x, E) = \mu_x(E)$ for every atom x of m_τ and $T(x, E) = \delta_x(E)$ for every x outside the support of m_τ. Here δ_x is Dirac's atomic measure at x. Then T is mean preserving and $T\tau$ is absolutely continuous w.r.t. σ.

(4) $\Rightarrow$ (3): Let $h : \Delta(K) \to \mathbf{R}$ be convex, continuous and nonnegative function over $\Delta(K)$, such that $\int h \ \mathrm{d}m_\tau > 0$. Since h is convex and T is mean-preserving it

follows that $\int h \, \mathrm{d}Tm_\tau \geq \int h \, \mathrm{d}m_\tau > 0$. Since h is nonnegative and Tm_τ is absolutely continuous w.r.t. m_σ it follows that $\int h \, \mathrm{d}m_\sigma > 0$.

□

Remark 2. Unlike the case without a withdrawal option, Condition (2) of Theorem 2 does not imply that for every decision problem (p, A, u), $R^w(\mathcal{T}; p, A, u) \geq R^w(\mathcal{S}; p, A, u)$. The latter is equivalent to $R(\mathcal{T}; p, A, u) \geq R(\mathcal{S}; p, A, u)$ for every decision problem (p, A, u). Indeed, by adding a large positive constant M to u, the withdrawal option becomes irrelevant. Therefore, if for every decision problem (p, A, u), $R^w(\mathcal{T}; p, A, u) \geq R^w(\mathcal{S}; p, A, u)$, then $R(\mathcal{T}; p, A, u + M) = R^w(\mathcal{T}; p, A, u + M) \geq R^w(\mathcal{S}; p, A, u + M) = R(\mathcal{S}; p, A, u + M)$ for every (p, A, u). The latter implies that $R(\mathcal{T}; p, A, u) \geq R(\mathcal{S}; p, A, u)$ for every (p, A, u).

4. A Malevolent Nature

In the previous section we let the decision have an extra withdrawal option. In this section we let Nature have an extra power. Consider a situation in which Nature chooses her state strategically to minimizes the agent payoff. For an information structure $\mathcal{S} = (s, \sigma)$ and a set of actions A, the agent (the maximizer) and Nature (the minimizer) play a zero-sum game. Nature chooses a state k, then a signal s is chosen according to $\mathcal{S}$ and informed to the agent, who then chooses an action a. The payoff is $u(k, a)$. Let

$$R_m(\mathcal{S}; A, u) = \min_{p \in \Delta(K)} R(\mathcal{S}; p, A, u)$$

be the value of this game. The value exists since each player has finitely many strategies. It turns out that the partial order induced over information structures when Nature is malevolent coincides with that of Theorem 1. Formally,

Theorem 3. *Let $\mathcal{S} = (S, \sigma)$ and $\mathcal{T} = (T, \tau)$ be two information structures. Then the following conditions are equivalent:*

1. *For every A, if $R_m(\mathcal{T}; A, u) > 0$ then $R_m(\mathcal{S}; A, u) > 0$.*
2. *There exists a stochastic matrix $\varepsilon = \{\varepsilon_{s,t}\}_{s \in S, t \in T}$ such that $\tau = \sigma\varepsilon$.*

Proof. (2) ⇒ (1): Suppose that $\tau = \sigma\varepsilon$ for a stochastic matrix ε, and let A be a finite set of actions, such that $R_m(\mathcal{T}; A, u) > 0$. If $p \in \Delta(K)$ is an optimal strategy for Nature in the $\mathcal{S}$-game, then $R(\mathcal{T}; p, A, u) \geq R_m(\mathcal{T}; A, u) > 0$. By Theorem 1,

$$R_m(\mathcal{S}; A, u) = R(\mathcal{S}; p, A, u) > 0.$$

(1) $\Rightarrow$ (2): Assume that there exists no stochastic matrix $\varepsilon = \{\varepsilon_{s,t}\}_{s\in S,t\in T}$ such that $\tau = \sigma\varepsilon$. Let Q_1 be the set of all matrices of the form $\sigma\varepsilon$ where ε ranges over all stochastic matrices $\varepsilon = \{\varepsilon_{s,t}\}_{s\in S,t\in T}$. Let Q_2 be the set of all matrices of the form $\delta\tau$ where δ is a diagonal $|K| \times |K|$-matrix with nonnegative entries. Q_1 is a compact convex set and Q_2 is a convex cone in the vector space of all $|K| \times |T|$ matrices.

Assume first that $Q_1 \cap Q_2 \neq \emptyset$. Let $\delta\tau \in Q_2$ belongs also to Q_1. As any matrix in Q_1, $\delta\tau$ is stochastic. This may happen only if δ is the identity matrix, in which case $\tau \in Q_1$, that is $\tau = \sigma\varepsilon$ and (2) is satisfied.

Now assume that $Q_1 \cap Q_2 \neq \emptyset$. By the separation theorem there exists a $|K| \times |T|$ matrix α that strictly separates Q_1 and Q_2. That is, $\langle \alpha, x\rangle < 0$ for every $x \in Q_1$ and $\langle \alpha x\rangle > 0$ for every $x \in Q_2$. Define A to be the set of the columns of α and let u be the utility function that turns α to be the payoff matrix that corresponds to A, u.

Suppose first that the game is played under $\mathcal{S}$, and that Nature's mixed strategy is uniform over K. Let an agent's strategy be given by a stochastic matrix $\varepsilon = \{\varepsilon_{s,t}\}_{s\in S,t\in T}$ (that is, when he receives the signal s he chooses the t-column of α with probability $\varepsilon_{s,t}$.) The agent's payoff is then, $\frac{1}{|K|}(\sigma\varepsilon)\cdot\alpha$, which is strictly smaller than 0 since $\sigma\varepsilon \in Q_1$. It follows that $R_m(\mathcal{S}; A) < 0$.

Suppose now that the game is played under $\mathcal{T}$, and that the agent's strategy prescribes him to play the signal he received. If Nature picks state k (it is a pure strategy of Nature), then the payoff is $\langle \eta_k\tau, \alpha\rangle$, where η_k is the $|K| \times |K|$-matrix whose only non-zero entry is the (k, k)-th entry which is 1. Since $\eta_k\tau \in Q_2$ for every k, it follows that the agent has a positive payoff against every pure strategy of Nature and therefore against any of its mixed strategies. Thus, $R_m(\mathcal{T}; A) > 0$. $\square$

5. Final remarks

5.1. **Different withdrawal options.** Assume that, instead of a constant zero payoff, the withdrawal option yields a payoff $b \in \mathbf{R}^K$ that depends on the realized state of nature. For every decision problem (p, A, u) and information structure $\mathcal{S}$, denote by $R^w(\mathcal{S}; p, A, u; b)$ the DM's optimal payoff if he is allowed the withdrawal option b. Then it turns out that the Theorem 2 can be stated as follows:

Theorem 4. (1) *There exists a matrix ε with nonnegative entries such that $\tau = \sigma\varepsilon$. Implies*

(2) *For every decision problem* (p, A, u)*, if* $R^w(\mathcal{T}; p, A, u; b) > R(\mathcal{T}; b)$*, then* $R^w(\mathcal{S}; p, A, u; b) > R(\mathcal{S}; b)$*.*

We do not know the analogue condition in the case of several withdrawal options.

5.2. **General measures and signal spaces.** The results above can be stated in more general terms, not necessarily with finite signal spaces. In order to keep this note concise, we choose to omit the details.

References

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