A COMMENT ON AN EXAMPLE BY MACHINA

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Abstract:

Machina (2007) demonstrates an example where the Choquet utility maximization theory is inconsistent with a natural symmetry requirement. The comment shows that this requirement is consistent with two models: expected utility maximization w.r.t. the concave integral (Lehrer, 2005) and utility maximization w.r.t. a partially-specified probability (Lehrer, 2007).

Keywords: Choquet utility maximization, partially-specified probability, concave integral for capacities

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1. The example

In a recent paper Machina (2007, Table 2, p. 11) introduced the following example. An urn contains twenty balls of four different colors a,b,c, and d. Ten balls are either a or b and the other ten are either c or d. A decision maker (DM) chooses an act, then a ball is randomly drawn and a reward is given to DM according to the color of the ball and the chosen act. The following table summarizes the rewards related to four acts. We will later discuss two decision problems: f_1 vs. f_2 and f_3 vs. f_4.

	a	b	c	d
f_1	0	200	100	100
f_2	0	100	200	100
f_3	100	200	100	0
f_4	100	100	200	0

One can see that f_4 is a mirror image of f_1 and f_3 is a mirror image of f_2. Thus, if DM strictly prefers f_2 to f_1 (i.e. $f_2 \succ f_1$) we would expect that he should strictly prefers f_3 to f_4 (i.e. $f_3 \succ f_4$). This is inconsistent with Choquet expected utility maximization (Schmeidler, 1989), as explained in Machina (2007). This explanation is replicated in the next section.

The aim of this comment is to show that '$f_2 \succ f_1$ and $f_3 \succ f_4$' is consistent with two models. The first is that of expected utility maximization w.r.t the concave integral for capacities (Lehrer, 2005). The second model is that of expected utility maximization w.r.t partially-specified probabilities (Lehrer, 2007).

2. Why Choquet expected utility maximization does not work here?

Let Ω be a finite state space. A capacity (or a non-additive probability) is a function $v : \Omega \rightarrow [0, 1]$ such that $v(\emptyset) = 0$, $v(\Omega) = 1$ and is monotonic w.r.t. inclusion.

An act is a function defined over Ω and ranged to the set of rewards, say W. Suppose that the preference order on W is represented by a (non-negative) utility function U.

We denote by $\mathbb{1}_E$ the indicator of a subset E of Ω. Let f be an act and v be a capacity. A decomposition of $U \circ f$ is a positive combination of indicators that equals $U \circ f$. Formally, *a decomposition of* $U \circ f$ is $\sum_{i=1}^{k} \alpha_i \mathbb{1}_{E_i}$ that is equal to $U \circ f$ and $\alpha_i \geq 0$, $i = 1, ..., k$. The *Choquet decomposition* is the one that satisfies $\Omega = E_1 \supsetneqq E_2 \supsetneqq ... \supsetneqq E_k$.

The capacity v and the utility function U induce an order over the set of acts as follows. Suppose that e and g are acts, then

$$e \succ_{Cho} g \text{ if and only if } \sum_{i=1}^{k} \alpha_i v(E_i) > \sum_{i=1}^{\ell} \beta_i v(G_i),$$

where $\sum_{i=1}^{k} \alpha_i \mathbb{1}_{E_i}$ and $\sum_{i=1}^{\ell} \beta_i \mathbb{1}_{G_i}$ are the Choquet decompositions of $U \circ e$ and $U \circ g$, respectively.

Back to Machina's example. The Choquet decomposition of $U \circ f_1$ is $U(0)\mathbb{1}_{\Omega} + U(100)\mathbb{1}_{\{bcd\}} + \big(U(200) - U(100)\big)\mathbb{1}_{\{b\}}$ and that of $U \circ f_2$ is $U(0)\mathbb{1}_{\Omega} + U(100)\mathbb{1}_{\{bcd\}} + \big(U(200) - U(100)\big)\mathbb{1}_{\{c\}}$. Suppose that $f_2 \succ_{Cho} f_1$ as induced by some capacity v defined over $\Omega = \{a, b, c, d\}$ and a utility function U defined over $W = \{0, 100, 200\}$ (without loss of generality $U(0) = 0$). It implies that $U(100)v(bcd) + \big(U(200) - U(100)\big)v(b) < U(100)v(bcd) + \big(U(200) - U(100)\big)v(c)$, which implies that $v(b) < v(c)$. However, the fact that $v(b) < v(c)$ implies, by a similar calculation, that $f_4 \succ_{Cho} f_3$.

In short, $f_2 \succ_{Cho} f_1$ implies $f_4 \succ_{Cho} f_3$. In other words, $f_2 \succ_{Cho} f_1$ and $f_3 \succ_{Cho} f_4$ together, are inconsistent with Choquet utility maximization theory.

3. Explanation with the concave integral for capacities

Lehrer (2005) proposed and axiomatized a different integral for capacities than the Choquet integral. This is the concave integral, which in turn induces another order over acts.

As before, suppose that e and g are acts. Define an order $\succ_{cav}$ as follows:

$$e \succ_{cav} g \text{ if and only if } \max \sum_{i=1}^{k} \alpha_i v(E_i) > \max \sum_{j=1}^{\ell} \beta_j v(G_j), \tag{1}$$

where the maxima are taken over all decompositions $\sum_{i=1}^{k} \alpha_i \mathbb{1}_{E_i}$ of e and $\sum_{j=1}^{\ell} \beta_j \mathbb{1}_{G_j}$ of g.

As for the example above, suppose that DM believes that at least one of the balls must be b or c. The minimal estimation of the probability of the event $\{bc\}$ is therefore $\frac{1}{20}$. Let v represent the minimal estimation of each event. It implies that $v(a) = v(b) = v(c) = v(d) = 0$, $v(bc) = \frac{1}{20}$ and $v(ab) = v(cd) = \frac{1}{2}$. The rest can be completed in a monotonic manner.

The best decomposition (i.e. the one that maximizes the right-hand side of eq. (1)) of $U \circ f_1$ is the Choquet decomposition, $U(0)\mathbb{1}_{\Omega} + U(100)\mathbb{1}_{\{bcd\}} + \big(U(200) - U(100)\big)\mathbb{1}_{\{b\}}$.

However the best decomposition of $U \circ f_2$ is no long the Choquet one. The best decomposition is $U(100)\mathbb{1}_{\{cd\}} + \big(U(200) - U(100)\big)\mathbb{1}_{\{bc\}} + \big(2U(100) - U(200)\big)\mathbb{1}_{\{b\}}$.

It turns out that due to eq. (1), $f_2 \succ_{cav} f_1$. The reason is that $U(0)v(\Omega)+U(100)v(bcd)+\big(U(200) - U(100)\big)v(b) = \frac{U(100)}{2} < \frac{U(100)}{2} + \frac{U(200)-U(100)}{20} = U(100)v(cd) + \big(U(200) - U(100)\big)v(bc) + \big(2U(100) - U(200)\big)v(b)$.

For a similar calculation $f_3 \succ_{cav} f_4$.

4. Explanation with partially-specified probabilities

Here the model used is that of partially-specified probabilities (Lehrer, 2007). It is known that[1] $\mathbb{P}(a, b) = \mathbb{P}(c, d) = \frac{1}{2}$. Equivalently, there are two non-trivial random variables with known expectation: $X_1 = [1, 1, 0, 0]$ whose expectation is $\frac{1}{2}$, and $X_2 = [0, 0, 1, 1]$ whose expectation is $\frac{1}{2}$.

It will be shown that 'f_2 preferred to f_1' is supported by the belief that $\mathbb{P}(b, c)$ is positive, which implies that $\mathbb{P}(b, c) \geq \frac{1}{20}$. Indeed, suppose that the expectation of the random variable $[0(a), 1(b), 1(c), 0(d)]$ (i.e the random variable that takes the value 0 on a, 1 on b, 1 on c, 0 on d) is at least $\frac{1}{20}$. Denote this random variable by $X_3 = [0, 1, 1, 0]$. So, $\mathbb{E}(X_3) \geq \frac{1}{20}$. In order to simplify the explanation assume that expectation of X_3 is known to be precisely $\frac{1}{20}$.

The probability $\mathbb{P}$ is only partially known. For instance, the probability $\mathbb{P}(a)$ is unknown. However, the expectation of some, but not all, random variables is known. Here, the expectation of any linear combination of X_1, X_2 and X_3 is known. Denote by $\mathcal{Y}$ the set of random variables whose expectation is known. In other words, DM knows $\mathbb{E}(X)$ for every $X \in \mathcal{Y}$. The pair $(\mathbb{P}, \mathcal{Y})$ is a *partially-specified probability*, which induces an order over acts.[2]

Suppose that e and g are acts. Define an order $\succ_{PSP}$ as follows:

$$e \succ_{PSP} g \text{ if and only if } \max \sum_{i=1}^{k} \alpha_i \mathbb{E}(X_i) > \max \sum_{j=1}^{\ell} \beta_j \mathbb{E}(X'_j), \tag{2}$$

[1] $\mathbb{P}$ denotes the probability function and $\mathbb{E}$ stands for expectation w.r.t $\mathbb{P}$.

[2] Lehrer (2007) provides an axiomatization and introduces the corresponding solution concepts for strategic models (Nash-like and correlated-like).

where the left-hand side maximum is taken over all linear combinations $\sum_{i=1}^{k} \alpha_i X_i$ that are less than or equal to e with $X_i \in \mathcal{Y}$, and the right-hand side maximum is taken over all linear combinations $\sum_{j=1}^{\ell} \beta_j X'_j$ that are less than or equal to g with $X'_i \in \mathcal{Y}$.

Note that $U \circ f_1$ corresponds to the random variable $Y = [0, U(200), U(100), U(100)]$ and $U \circ f_2$ corresponds to the random variable $Z = [0, U(100), U(200), U(100)]$. Denote, $m = \min\big(U(100), U(200) - U(100)\big)$. Since, $Z \geq mX_3 + [0, 0, U(100), U(100)]$, $\mathbb{E}(Z) \geq m\mathbb{E}(X_3) + U(100)\mathbb{E}\big(X_2\big) = \frac{m}{20} + \frac{U(100)}{2}$.

The best approximation from below (in the sense of eq. (2)) to Y with linear combinations of X_1, X_2 and X_3 is $U(100)X_2$, whose expectation is $\frac{U(100)}{2}$. Thus, according to eq. (2)), f_2 is preferred to f_1 (because $\frac{m}{20} + \frac{U(100)}{2} > \frac{U(100)}{2}$).

From similar arguments, f_3 is preferred to f_4 (in this calculation X_1 and X_3 are used). Thus, $f_2 \succ_{PSP} f_1$ and at the same time $f_3 \succ_{PSP} f_4$.

5. References

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