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Global Games[1]

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Abstract: Global games are real-valued functions defined on partitions (rather than subsets) of the set of players. They capture "public good" aspects of cooperation, i.e., situations where the payoff is naturally defined for all players ("the globe") together, as is the case with issues of environmental clean-up, medical research, and so forth.

We analyze the more general concept of lattice functions and apply it to partition functions, set functions and the interrelation between the two. We then use this analysis to define and characterize the Shapley value and the core of global games.

1 Introduction

Traditional cooperative game theory models a game by a set function which is interpreted as the payoffs that a coalition may guarantee itself. It is implicitly assumed that payoffs are defined for each player separately and the main question is what are the attainable payoff profiles for various possible coalitions. In a transferable utility game only one number is attached to each coalition, but it is typically interpreted as the maximum *total* payoff of a coalition, which is meaningful if the players may redistribute the "utilities" among themselves.

However, in many situations it seems more reasonable to say that the game's payoff – or utility – is simultaneously defined for all players. Consider, for instance, environmental problems such as air and water pollution, diminishing ozone layers, and other catastrophes. Although undoubtedly not perfectly precise, it seems safe to argue that the natural model for these problems is one in which the payoff is defined for *all players together.* (Or, if you will, that the utilities of the players coincide.)

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Questions of art and historical treasures preservation, a cure for cancer and AIDS, indeed, the progress of science and art in general, and many other issues – though not unrelated to nations' political interests – seem to be "global", at least as a first approximation.[3]

This paper models such games and tries to cope with the question of their "solution." Mathematically speaking, a global game is a real-valued partition function: for each partition, P, of the players, i.e., for every profile of cooperation, there is a value $h(P)$ describing the utility of all players ("the globe") should the structure of cooperation follow P.

It is this model of "games" for which we would like to propose solution concepts, that will be analogous to those applied to "ordinary" (transferable utility cooperative) games. Such concepts – as the Shapley value and the core – prescribe an allocation (or a set of allocations) describing how the surplus of cooperation is to be shared. The question which naturally arises at this point is: What is there to share? If the utility is identical to all players anyway, how does one share it?

The answer is that the global game does not describe the complete range of activities of the players. It is a "reduced form" model which captures a certain aspect of these activities – say, environmental clean-up or AIDS research – but does not deal with other aspects of interaction, in which each players's utility is well defined. Thus, we interpret $h(P)$ as a transferable-utility (or "monetary") value of P-cooperation, and the question is how to divide the surplus of full cooperation $h(\{N\})$ (where N is the set of players). All nations will benefit from a joint effort to clean up the atmosphere, but they still have to decide how to share the (positive) cost which exists even when they do cooperate. All nations will be better off once AIDS is cured, yet each of them also prefers to support the AIDS research to the minimal possible extent.

In a way, then, a global game may be considered as capturing the "public good" aspect of interaction, assuming there are no private goods (apart from the "monetary" transfers).[4]

The study of partition functions has shown some common features with that of set functions. Indeed, some of the results may be applied to a more general framework than both, namely, to real-valued functions on lattices.

To facilitate the reader's orientation, we first provide in Section 2 a brief formal definition of "ordinary" and global games, emphasizing the lattice structure. In Section 3 we deal with lattice functions in general and prove some results, most of which are known for the case of set functions. We find lattice functions to be of particular importance to cooperative game theory as they appear in a variety of models:

3 An additional example of "global" payoffs is the performance of a certain organization that depends on its internal structure but is not defined for separate coalitions.

4 Our concept differs from the cooperative games in which a coalition's payoff depends on the partition to which it belongs (Thrall and Lucas (1963)) since we focus on the "public good" aspect, i.e., on pure externality, where the "power" of the coalition is a less obvious concept.

(1) ordinary games (on sets);
(2) global games (on partitions);
(3) k-stage games (on chains of k sets) (see Beja-Gilboa) (1990));
(4) games defined on pairs (S, P) where S is a set and P is a partition containing it (see Thrall and Lucas (1963)).

In Section 4 we focus attention on partition functions again and study their relation to set functions defined by them. In particular we focus on properties such as additivity, monotonicity, convexity, and total positivity (defined in Section 3); these properties also shed some light on convex ("ordinary") games and study some properties which are stronger than convexity.

Finally, Section 5 deals with solution concepts for global games. We define and characterize the (unique) Shapley value which, somewhat surprisingly, turns out to depend only on all-or-none partitions, i.e., partitions in which some subset of players fully cooperate, while the rest do not cooperate at all. We also define the core and show that convex global games have a nonempty core which includes the Shapley value.

2 Games and Global Games – Definitions

In this section we formally define (transferable-utility cooperative) games and global games, emphasizing their common structure, i.e., the fact that both are real-valued functions on some lattice. For completeness' sake, we provide the formal definition of a lattice in Section 3; being standard and well-known, we do not find it merits repetition here.

Let N be a finite nonempty set of *players*. A *game* is simply a set-function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$.

Note that the power set of N is a lattice, with set-inclusion as a partial order. Obviously, every two subsets have "max" and "min" elements, namely, their union and intersection, respectively.

We now turn to define global games. A partition P is a set of nonempty pairwise disjoint subsets (coalitions) of N whose union is N. The set of all partitions is denoted $\mathscr{P}$.

The set of partitions is partially ordered by the "coarser" than relation defined as follows: P is *coarser than* Q, denoted $P \geq Q$, if for every $A \in Q$ there is $B \in P$ such that $A \subseteq B$. Note that $(\mathscr{P}, \geq)$ is a lattice where $P \vee Q$ is the finest partition coarser than both (the *meet*) and $P \wedge Q$ is the coarsest partition finer than both (the *join*). Note that these notations are not entirely conventional. However, with the interpretation of global games, it is more intuitive to define monotonicity with respect to the "coarser than" relation, which means "more cooperation," whence the rest of the notations follow.

We will also use the terms "P is finer than Q", "$P \leq Q$," "P is a refinement of Q," and "Q is a coarsening of P."

We extend the definitions above (and below) to subsets of N. Thus, if P^A is a partition of A, P^B is a partition of B and $A \cap B = \emptyset$, $P^A \cup P^B$ is a well-defined partition of $A \cup B$.

For a partition P (of a subset A) we denote by $\mathscr{B}(P)$ the algebra (of subsets of A) generated by it. If P is a partition of A, and $B \in \mathscr{B}(P)$, P^B will denote the induced partition of B. Let P_f^A and P_c^A denote the finest and coarsest partitions of A, respectively, i.e., $P_f^A = \{\{i\} | i \in A\}$, $P_c^A = \{A\}$.

A *global game* is simply a partition function $h: \mathscr{P} \to \mathbb{R}$. For simplicity we normalize all global games (by subtracting $h(P_f)$) and assume $h(P_f) = 0$.

Thus, global games, like "ordinary" ones, are real-valued lattice functions. We therefore devote the next section to lattice functions in general, and continue the more specific analysis in Section 4.

3 Lattice Functions

A poset (partially ordered set) is a pair $(X, \geq)$ where X is a set and $\geq$ is a binary relation on it satisfying:

(i) reflexivity: $x \geq x$ for all $X \in X$.
(ii) anti-symmetry: $x \geq y$ and $y \geq x$ implies $x = y$ for all $x, y \in X$.
(iii) transivity: $x \geq y$ and $y \geq z$ implies $x \geq z$ for all $x, y, z \geq X$.

In this paper we restrict our attention to the case of a finite X.

An element $x \in X$ is said to be a supremum (an infinum) of a set $A \subseteq X$ if the following hold:

(i) $x \geq y$ $(x \leq y)$ for all $y \in A$;
(ii) if $z \geq y$ $(z \leq y)$ for all $y \in A$, then $z \geq x$ $(z \leq x)$.

A supremum of a set A is denoted by $\vee A$, and infimum by $\wedge A$. If a supremum (infimum) of A belongs to A, it is called a maximum (minimum). In view of anti-symmetry, suprema and infima are unique. If $A = \{x, y\}$ its supremum and infimum will be denoted by $x \vee y$ and $x \wedge y$, respectively.

A poset $(X, \geq)$ is a *lattice* if for every $x, y \in X$ there exist $x \vee y$ and $x \wedge y$. We will refer to $\wedge$ and $\vee$ as binary operations. We will also use the notations $x_1 \vee x_2 \vee \ldots \vee x_n, x_1 \wedge x_2 \wedge \ldots \wedge x_n, \vee_{1 \leq i \leq n} x_i$ and $\wedge_{1 \leq i \leq n} x_i$ with their obvious meanings.

Obviously a lattice has a (unique) maximum and minimum, denoted by x^* and x_*, respectively. A *lattice function* is a real-valued function on X. It is 0-normalized if $f(x_*) = 0$. The linear space of lattice functions on X will be identified with $\mathbb{R}^{|X|}$, but also denoted by $F(X)$ when this notation will be more suggestive. The subspace of 0-normalized lattice functions will be denoted by $F_0(X)$.

A lattice function $f \in F(X)$ is *monotone* if $x \geq y$ implies $f(x) \geq f(y)$. A function $f \in F(X)$ is *convex* if

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$$

for all $x, y \in X$. It is *additive* if equality holds for all $x, y \in X$.

A lattice function f is *totally positive* if for every $x_1, x_2, \ldots x_n \in X$

$$f(x_1 \vee x_2 \vee \ldots \vee x_n) \geq \Sigma_{\{I \mid \varnothing \neq I \subseteq \{1,\ldots,n\}\}} (-1)^{|I|+1} f(\wedge_{i \in I} x_i)$$

(This definition coincides with the Dempster-Shafer definition of belief functions for the lattice of subsets of a given set. See Dempster (1967), Shafer (1976).)

We note without proof that every totally positive function is convex, though the converse is false.

For $x, y \in X$ we write $x > y$ if $x \geq y$ and $x \neq y$. We also use an additional binary relation, denoted $>^*$, and defined by $x >^* y$ if $x > y$ but for no $z \in X$ $x > z > y$. (Put differently, $>^*$ is the minimal relation whose transitive closure is $>$.)

For $x \in X$ we define $g_x \in F(X)$ by $g_x(y) = 1$ if $y \geq x$ and $g_x(y) = 0$ otherwise.

Proposition 3.1: $\{g_x\}_{x \in X}$ is a linear basis for $F(X)$, as is $\{g_x\}_{x \in X, x \neq x_*}$ for $F_0(X)$.

Proof: First we show that $\{g_x\}_{x \in X}$ are linearly independent. Assume

$$\Sigma_{x \in X} \alpha_x g_x = 0.$$

Considering x_*, we obtain $\Sigma_{x \in X} \alpha_x g_x(x_*) = \alpha_{x_*} = 0$. Next consider y such that $y >^* x_*$. Obviously, $\alpha_y = 0$ follows, and the proof continues by induction.

Since there are $|X|$ functions in$\{g_x\}_{x \in X}$, they have to constitute a basis of $F(X)$. Similarly, $\{g_x\}_{x \neq x_*} \subseteq F_0(X)$ are independent and of the appropriate dimension to be a basis for $F_0(X)$. □

Given $f \in F(X)$ let $\{\alpha_x(f)\}_{x \in X}$ be the unique set of coefficients such that

$$f = \Sigma_x \alpha_x(f) g_x.$$

Note that $(f - f(x_*)g_{x_*}) \in F_0(X)$. Hence for $x \neq x_*$, $\alpha_x(f - f(x_*)g_{x_*}) = \alpha_x(f)$.

Theorem 3.2: Let $f \in F(X)$ be given. Then f is totally positive and monotone iff for all $x \neq x_*$, $\alpha_x(f) \geq 0$.

Proof: Since f is totally positive and monotone iff $(f - f(x_*)g_{x_*})$ is, we assume w.l.o.g. (without loss of generality) that $f \in F_0(X)$.

First consider the "if" part. Since the monotonicity of g_x is trivial, it suffices to show that g_x is totally positive for every $x \in X$, and the conclusion will follow as the set of totally positive and monotone functions is a cone.

Let $x \in X$ and $x_1,...,x_n \in X$ be given. We wish to show that

$$g_x(\vee_{i\leq n} x_i) \geq \Sigma_{\emptyset\neq I\subseteq\{1,...,n\}} (-1)^{|I|+1} g_x(\wedge_{i\in I} x_i).$$

Let $J = \{1 \leq j \leq n | x \leq x_j\}$. If $J = \emptyset$ the right side vanishes and the inequality holds. Otherwise, the inequality may be reduced to

$$1 \geq \Sigma_{\emptyset\neq I\subseteq J} (-1)^{|I|+1}.$$

However,

$$1 + \Sigma_{\emptyset\neq I\subseteq J} (-1)^{|I|} = \Sigma_{I\subseteq J} (-1)^{|I|} = (1-1)^{|J|} = 0$$

Hence, g_x is totally positive for all x, and the "if" part is proved.

For the "only if" part we first prove the following.

Claim: Let $f \in F_0(X)$ and $x \in X$. Assume that $\{x_1,...,x_n\} = \{y \in X | x >^* y\}$. Then

$$f(x) - \Sigma_{\emptyset\neq I\subseteq\{1,...,n\}} (-1)^{|I|+1} f(\wedge_{i\in I} x_i) = \alpha_x(f).$$

Proof: The case $x = x_*$ is trivial. Assume, then, that $x > x_*$ and $n \geq 1$. Recall that for all $y \in X$.

$$f(y) = \Sigma_{z\in X} \alpha_z(f) g_z(y) = \Sigma_{z\leq y} \alpha_z(f).$$

Hence, we have to show that

$$\Sigma_{z\leq x} \alpha_z(f) - \Sigma_{\emptyset\neq I\subseteq\{1,...,n\}} (-1)^{|I|+1} \Sigma_{z\leq\wedge_{i\in I}x_i} \alpha_z(f) = \alpha_x(f)$$

or

$$\Sigma_{z<x} \alpha_z(f) - \Sigma_{\emptyset\neq I\subseteq\{1,...,n\}} (-1)^{|I|+1} \Sigma_{z\leq\wedge_{i\in I}x_i} \alpha_z(f) = 0$$

The expression on the left side equals

$$\Sigma_{z<x} \alpha_z[1 - \Sigma_{\emptyset\neq I\subseteq\{1,...,n\};z\leq\wedge_{i\in I}x_i} (-1)^{|I|+1}]$$

It is sufficient (and necessary) to show that the expression in brackets vanishes for all $z < x$. However, this is proven almost identically to the "if" part of the theorem. We therefore consider the claim proved. □

To complete the proof of the theorem, let $f \in F_0(X)$ be totally positive and let $x > x_*$. Let $\{x_1,\ldots,x_n\} = \{y \in X \mid x >^* y\}$. If $n = 1$, then

$$\alpha_x(f) = f(x) - f(x_1)$$

and nonnegativity follows from monotonicity of f. Otherwise, i.e., $n > 1$, we have, since $x >^* x_i$ for all $i \leq n$,

$$\vee_{1 \leq i \leq n}\, x_i = x$$

and nonnegativity follows from the claim and the fact that f is totally positive. □

Remark 3.3: Proposition 3.1 is well-known for the case of subsets (of a given set) ordered by inclusion. For this case a version of Theorem 3.2 was also proved by Dempster (1967), Shafer (1976). In their theorem, nonnegativity of $f \in F_0(X)$ replaces monotonicity. Obviously, monotonicity is a stronger requirement in general, the two coincide for totally positive functions on subset-lattices, but they do not coincide in general. In fact, nonnegativity and total positivity do not suffice for nonnegativity of all coefficients, as is shown in the following example:

$$X = \{x,y,z\},\, x > y > z$$
$$f(z) = 0,\, f(y) = 2,\, f(x) = 1.$$

This example seems somewhat anomalous since the lattice involved is rather "thin." To formalize this notion, define, for $x \in X$, the *level* of $x > x_*$ to be $\ell(x) = |\{z_1,\ldots,z_k\}|$ such that $x >^* z_1 >^* z_2 >^* \ldots >^* z_k = x_*$ and $\ell(x_*) = 0$. Further define the *degree* of x to be $d(x) = |\{y \mid x >^* y\}|$.

Note that $d(x_*) = 0$ and that $d(x) = 1$ for every $x \in X$ with $\ell(x) = 1$. If, however, for every $x \in X$ with $\ell(x) > 1$ we have $d(x) > 1$, the lattice is called *rich*.

Observation 3.4: Assume X is a rich lattice. Then $f \in F_0(X)$ is totally positive and nonnegative iff for every $x \neq x_*$ $\alpha_x(f) \geq 0$.

Proof: In view of the proof of Theorem 3.3, it suffices to note that if f is totally positive and nonnegative, $\alpha_x(f) \geq 0$ follows from total positivity for x with $\ell(x) > 1$, and for x with $\ell(x) = 1$,

$$\alpha_x(f) = f(x) - f(x_*) = f(x) \geq 0. \quad \square$$

It is immediate that for every set N the lattice of partitions $\mathscr{P}$ is rich. Hence, for global games, as for ordinary ones, nonnegativity of totally positive games suffices for the nonnegativity of all corresponding coefficients.

Remark 3.5: It is also important to note that our theorem also applies to non-distributive lattices. (A lattice is *distributive* if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \forall\, x,y,z \in X$$

or, equivalently,

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \forall\, x,y,z \in X;$$

see, e.g., Graetzer (1971).)

Indeed, while subset-lattices (with inclusion) are distributive, partition-lattices (with "coarser than" relation) are not. Consider the following example:

$$N = \{1,2,3,4\}$$

$$P = \{\{1,2\},\{3,4\}\},\ Q = \{\{1,3\},\{2\},\{4\}\},\ R = \{\{2,4\},\{1\},\{3\}\}$$

in which

$$(P \vee Q) \wedge (P \vee R) = \{N\} \neq P = P \vee (Q \wedge R).$$

4 On Global and Ordinary Games

In this section we focus again on global games and study their relationship to ordinary ones.

For a global game $h \in F_0(\mathscr{P})$ define the *associated game* $v_h \in F_0(2^N)$ by

$$v_h(A) = h(P_c^A \cup P_f^{A^c}) \text{ for } A \neq \emptyset$$

(and $v_h(\emptyset) = 0$).

That is, the payoff to a coalition A is the global payoff if all members of A cooperate, but all the other players do not.

On the other hand, given $v \in F_0(2^N)$ define the *associated global game* $h_v \in F_0(\mathscr{P})$ by

$$h_v(P) = \Sigma_{A \in P}\, v(A).$$

Obviously, for every game v there are global games h such that $v = v_h$. However, not every global game h is the associated global for some v. If there is a game v such that $h = h_v$, h will be called *partially additive*. Denote by PA the subspace of partially additive global games. See the Appendix for a characterization of PA.

We now turn to study some relations between properties of global games and their associated (ordinary) games and vice versa. Beginning with properties of h inherited by v_h we have:

Proposition 4.1: If $h \in F_0(\mathscr{P})$ is monotone and convex, so is $v_h \in F_0(2^N)$.

Proof: To see that v_h is monotone let $A \subseteq B$ be given. Then $v_h(A) = h(P_c^A \cup P_f^{A^c}) \leq h(P_c^B \cup P_f^{B^c}) = v_h(B)$ follows from monotonicity of h w.r.t. (with respect to) coarsening.

Next consider convexity. For arbitrary $A, B \subseteq N$ we have to show that

$$v_h(A \cup B) + v_h(A \cap B) \geq v_h(A) + v_h(B).$$

First assume that $A \cap B \neq \varnothing$. Then

$$(P_c^A \cup P_f^{A^c}) \wedge (P_c^B \cup P_f^{B^c}) = (P_c^{A \cap B} \cup p_f^{(A \cap B)^c})$$

and

$$(P_c^A \cup P_f^{A^c}) \vee (P_c^B \cup P_f^{B^c}) = (P_c^{A \cup B} \cup P_f^{(A \cup B)^c})$$

If $A \cap B = \varnothing$, however, we obtain

$$(P_c^A \cup p_f^{A^c}) \wedge (P_c^B \cup P_f^{B^c}) = P_f$$

and

$$\begin{aligned}(P_c^A \cup P_f^{A^c}) \vee (P_c^B \cup P_f^{B^c}) &= (P_c^A \cup P_c^B \cup P_f^{(A \cup B)^c}) \\ &\leq (P_c^{A \cup B} \cup P_f^{(A \cup B)^c}),\end{aligned}$$

whence the desired inequality follows in both cases from the monotonicity and convexity of h. □

Similarly, one obtains the following result (the proof of which is omitted).

Proposition 4.2: If $h \in F_0(\mathscr{P})$ is monotone and totally positive, so is v_h.

We now turn to study properties of a game v which may or may not be inherited by h_v :

Remark 4.3: If a game v is monotone and convex, the global game h_v has to be monotone but need not be convex.

Proof: Convexity of v implies its super-additivity, that is, that

$$v(A \cup B) \geq v(A) + v(B)$$

for all $A, B \subseteq N$ with $A \cap B = \emptyset$, which implies monotonicity of h_v. To show that h_v need not be convex, let $N = \{1,2,3,4\}$, define

$$v(A) = \begin{cases} |A| - 1 & A \neq \emptyset \\ 0 & A = \emptyset \end{cases}$$

Thus, v is monotone and convex. However, for $P = \{\{1,2\},\{3,4\}\}$ and $Q = \{\{1,3\},\{2,4\}\}$ we get $h_v(P) + h_v(Q) > h_v(P \wedge Q) + h_v(P \vee Q)$. □

Proposition 4.4: If a game v is monotone and totally positive, so is the global game h_v.

Proof: Let $v \in F_0(2^N)$ be given. For $A \subseteq N$ let u_A denote the unanimity game on A (g_A in Section 3's notation). Let $\alpha_A(v)$ be the unique coefficients such that

$$v = \Sigma_{\emptyset \neq A \subseteq N} \alpha_A(v) u_A$$

Claim:

$$h_v = \Sigma_{\emptyset \neq A \subseteq N} \alpha_A(v) g_{(P_c^A \cup P_f^{A^c})}$$

Proof of Claim: Given $P \in \mathscr{P}$,

$$\begin{aligned} h_v(P) &= \Sigma_{B \in P} v(B) = \Sigma_{B \in P} \Sigma_{A \subseteq B} \alpha_A(v) \\ &= \Sigma_{\{A \mid P \geq (P_c^A \cup P_f^{A^c})\}} \alpha_A(v) \\ &= \Sigma_{\{A \mid \emptyset \neq A \subseteq N\}} \alpha_A(v) g_{(P_c^A \cup P_f^{A^c})}(P) \end{aligned}$$

Hence, the (unique) coefficients $\{\alpha_P(h_v)\}_{P\in\mathscr{P}}$ such that $h_v = \Sigma_{P\in\mathscr{P}} \alpha_P(h_v)g_P$ are given by

$$\alpha_P(h_v) = \begin{cases}\alpha_A(v) & \text{if } P = (P_c^A \cup P_f^{A^c}) \\ 0 & \text{otherwise}\end{cases}$$ □

Theorem 3.2 may be now invoked twice – once for the subset lattice to deduce $\alpha_A(v) \geqslant 0$ for all $A \subseteq N$, $A \neq \emptyset$, and then for the partition lattice to deduce that h_v is totally positive and monotone. □

We summarize these results as follows:

$\{v \mid h_v$ is totally positive and monotone$\}$ =
$\{v \mid v$ is totally positive and monotone$\}$ $\subset$
$\{v \mid h_v$ is convex and monotone$\}$ $\subset$
$\{v \mid v$ is convex and monotone$\}$,

where the inclusions are strict. (An example showing the second inclusion was given in Remark 4.3 above; as for the first inclusion, let $N = \{1,2,3\}$ and $v = u_{\{1,2\}} + u_{\{2,2\}} + u_{\{1,3\}} - (1/2)u_{\{1,2,3\}}$, (where u_A is the unaminity game on A) for which h_v is convex and monotone).

It is not too surprising that additivity of $h \in F_0(\mathscr{P})$ will not always be inherited by v_h. Indeed, the inequality

$$v_h(A \cup B) + v_h(A \cap B) \leq v_h(A) + v_h(B)$$

has to hold whenever $A \cap B \neq \emptyset$, but may be violated otherwise. More specifically:

Remark 4.5: $h \in F_0(\mathscr{P})$ may be monotone and additve without v_h being additive.

Proof: Let $N = \{1,2,3\}$, and define

$$v(A) = \begin{cases}|A| - 1 & \text{if } A \neq \emptyset \\ 0 & \text{otherwise}\end{cases}$$

Define $h = h_v$ (so that $v = v_h$ also holds). It is easily verified that h is additive. However,

$$v(\{1,2,3\}) > v(\{1,2\}) + v(\{3\})$$

so that v is not. □

However, for a large enough set of players, additivity of h is inherited by v_h:

Proposition 4.6: Let $|N| \geq 4$ and assume that $h \in F_0(\mathscr{P})$ is additive. Then $h \equiv 0$, and $v_h \equiv 0$.

Proof: Let h be given and let v_h be its associated game. We first prove that $v_h(A) = v_h(B)$ whenever $|A| = |B|$. The proof is by induction on $k = |A| = |B|$. For $k = 1$ there is nothing to prove as $v_h(\{i\}) = 0$ for all $i \in N$. We therefore assume the claim for k and prove for $(k + 1)$.

If $(k + 1) = |N|$ there is nothing to prove. Hence, assume $(k + 1) < |N|$ and let C be a subset of cardinality $k + 2$. Consider any $A, B \subseteq C$ with $A \neq B$ and $|A| = |B| = k + 1$. Using additivity of h for $P = (P_c^A \cup P_f^{A^c})$ and $Q = (P_c^B \cup P_f^{B^c})$ we obtain

$$v_h(C) + v_h(A \cap B) = v_h(A) + v_h(B)$$

Note that $|A \cap B| = k \geq 1$. Furthermore, $|C| = k + 2 > 2$. Hence, we can choose a set $D \subseteq C$ with $D \neq A$, $D \neq B$ and $|D| = k + 1$. Then

$$v_h(C) + v_h(A \cap D) = v_h(A) + v_h(D)$$

also holds. However, $v_h(A \cap B) = v_h(A \cap D)$ as $|A \cap D| = k = |A \cap B|$, so we obtain $v_h(B) = v_h(D)$. Similarly, we get $v_h(A) = v_h(B) = v_h(D)$. Since this holds for subsets (of size $k + 1$) of *any* set C (of size $k + 2$), it also holds for any A, B with $|A| = |B| = k + 1$.

(Note that so far we have not used the fact $|N| \geq 4$.)

Hence, there exists a function d: $\mathbb{N} \to \mathbb{R}$ such that $v_h(A) = d(|A|)$ (which implies $d(1) = 0$). Denote $\alpha = d(2)$. Assume w.l.o.g. $N = \{1,2,3,4\} \cup M$ (with $M \cap \{1,2,3,4\} = \emptyset$).

As in the first part of the proof of Proposition 4.1 above, we know that whenever $A \cap B \neq \emptyset$, the following holds

$$v_h(A \cup B) + v_h(A \cap B) = v_h(A) + v_h(B)$$

For $A = \{1,2\}$ and $B = \{2,3\}$ we obtain $v_h(\{1,2,3\}) = d(3) = 2\alpha$.

Next consider $P_1 = \{\{1,2\}, \{3,4\}\} \cup \{\{i\}\}_{i \in M}$ and $Q_1 = \{\{1,2,3\},\{4\}\} \cup \{\{i\}\}_{i \in M}$. The equality

$$h(P_1 \wedge Q_1) + h(P_1 \vee Q_1) = h(P_1) + h(Q_1)$$

implies

$$d(4) + d(2) = 2d(2) + d(3)$$

or

$$d(4) = d(3) + d(2) = 3\alpha.$$

However, for $Q_2 = \{\{1,4\},\{2,3\}\} \cup \{\{i\}\}_{i \in M}$ the equality

$$h(P_1 \wedge Q_2) + h(P_1 \vee Q_2) = h(P_1) + h(Q_2)$$

yields

$$d(4) = 2d(2) + 2d(2) = 4\alpha.$$

Together we obtain $\alpha = 0$, whence it follows by induction that $h \equiv 0$, and $v_h \equiv 0$. $\square$

5 Solution Concepts

A solution concept for a global game – as for any game – should "solve" it, i.e., prescribe a certain possible outcome of it. In the context of global games, however, it is not entirely clear what do "outcome" or "solution" mean. More precisely, one has to decide whether global games should be "solved" directly, or should they first be reduced to "ordinary" games? The indirect strategy would rely on the assumption that we know how to solve games (whatever "solve" may mean), and the only problem with global games is that we do not know the "value" of each coalition. Hence, all we need to ask of a solution concept for global games is to translate the global game to an ordinary one, to which standard solution concepts may be applied.

In this paper we follow the direct solution strategy. It seems to us that this strategy eliminates arbitrary choices which will be required (using the indirect strategy) to specify an ordinary game (or set of such) for a global one.

We confine ourselves to the Shapley value and the core. A subsection is devoted to each.

5.1 The Shapley Value

We first introduce the following definitions. For $h \in F_0(\mathscr{P})$ and $i \in N$, i is a *dummy-player* in h if for all $P \in \mathscr{P}$

$$h(P) = h(P \wedge (\{\{i\}\} \cup P_c^{N\setminus\{i\}})).$$

Two players $i,j \in N$ are *interchangeable* in h if for all $P \in \mathscr{P}$

$$h(P \wedge (\{\{i\}\} \cup P_c^{N\setminus\{i\}})) = h(P \wedge (\{\{j\}\} \cup P_c^{N\setminus\{i\}}))$$

An operator $\psi : F_0(\mathscr{P}) \rightarrow \mathbb{R}^N$ is a *Shapley-value* (for global games) if it satisfies the following axioms.

1. Linearity;
2. Dummy: for all $h \in F_0(\mathscr{P})$ and $i \in N$, if i dummy in h, then $(\psi h)(i) = 0$.
3. Interchangeable players: for all $h \in F_0(\mathscr{P})$ and $i,j \in N$, if i and j are interchangeable in h, then $(\psi h)(i) = (\psi h)(j)$.
4. Efficiency:

$$\Sigma_{i \in N}(\psi h)(i) = h(\{N\}).$$

Let us comment briefly on the interpretation of these axioms. Linearity has its usual meaning: suppose that the players in the global game h (say, environmental clean-up) are also involved in a different global game g (e.g., art treasures preservation). It is desirable that one will be able to solve each game separately and obtain the same outcome that would result from considering the two global issues together $(h + g)$. Similarly, homogeneity (that is, $\psi(\alpha h) = \alpha(\psi h)$) simply means scale invariance.

Next consider the dummy axiom. A player i is a "dummy" in a global game h if the payoff is independent of i's cooperative behavior. As formulated, we only require that for every partition P, $h(P)$ will equal the payoff of the partition obtained from P by player i's desertion. Obviously, this also means that player i may decide to join another set in P but will still not affect the payoff. It seems reasonable that such a player will have no share in the surplus of cooperation $h(\{N\})$.

As for Axiom 3, two players i and j are "interchangeable" if for every partition P the desertion of i from his/her current coalition to form a separate coalition $\{i\}$ has the same impact on $h(P)$ as j would have. (Notice that in the formulation given above the term $h(P)$ was cancelled on both sides of the equality.) The requirement that i and j will get the same payoff according to ψh has a flavor of "symmetry" or "fairness" (though, as we shall see later, it is stronger than the traditional meaning of "symmetry").

Finally, the efficiency axiom simply requires that the overall surplus of cooperation, $h(\{N\})$, will be shared among the players.

We recall that the counterparts of these axioms for the context of ordinary games characterize the Shapley value for set functions (introduced in Shapley (1953)). Let it be denoted by ϕ.

Theorem 5.1.1: There is a unique Shapley value ψ for the space of global games and it equals the Shapley value of the induced (ordinary) game, i.e.,

$$\psi h = \phi(v_h) \text{ for all } h \in F_0(\mathscr{P}).$$

Proof: First we show that $\phi(v_h)$ is a Shapley value. Linearity and efficiency are immediate. It is also easily verified that if i is dummy in h (i,j are interchangeable in h), then i is also dummy in v_h (i,j are interchangeable in v_h) in the usual sense. I.e., $v_h(S) = v_h(S \setminus \{i\}), \forall S \subseteq N (v_h(S \setminus \{i\}) = v_h(S \setminus \{j\}), \forall S \subseteq N, i,j \in S)$, which implies that $\phi(v_h)(i) = 0$ $(\phi(v_h)(i) = \phi(v_h)(j)$.) Hence, $\phi(v_h)$ is a Shapley value.

Next we show uniqueness. Let ψ be a Shapley value on $F_0(\mathscr{P})$. It suffices to show that ψ is uniquely determined on $\{g_P\}_{P \in \mathscr{P}, P \neq P_f}$ since this set is a basis for $F_0(\mathscr{P})$. Consider, then, g_P for some $P \neq P_f$. Assume $P = \{A_1, A_2, \ldots, A_k\} \cup P_f^{A_{k+1}}$ where $|A_\ell| > 1$ for $1 \leq \ell \leq k$. (That is, $A_1, \ldots, A_k$ are the non-singleton coalitions in P.) Obviously, $i \in A_{k+1}$ is a dummy player in g_P. Similarly, every $i,j \in \cup_{\ell=1}^k A_\ell$ are interchangeable since for every $Q \in \mathscr{P}$

$$g_P(Q \wedge (\{\{i\}\} \cup P_c^{N \setminus \{i\}})) = g_P(Q \wedge (\{\{j\}\} \cup P_c^{N \setminus \{j\}})) = 0.$$

Hence, ψ has to satisfy

$$(\psi g_P)(i) = \begin{cases} 1/\Sigma_{\ell=1}^k |A_\ell| & \text{for } i \in \cup_{\ell=1}^k A_\ell \\ 0 & \text{otherwise} \end{cases}$$

Thus a Shapley value, if such exists, is unique and the formulae above may be used to compute it via the coefficients $\{\alpha_P(h)\}_P$. Since existence was established earlier, $\psi h = \phi(v_f)$ is proved. (Notice that this equality is also simple to verify directly: for g_P with $P = \{A_1, \ldots, A_k\} \cup P_f^{A_{k+1}}$ as above,

$$v_{g_P} = u_{\cup_{\ell=1}^k A_\ell}$$

where u_S is the unanimity game on S.) $\square$

Remark 5.1.2: It may seem surprising that the Shapley value oh f does not depend on all of the numbers $\{h(P)\}_{P \in \mathscr{P}}$. As a matter of fact, the (small) subset

$$\{h(P_c^A \cup P_f^{A^c})\}_{A \subseteq N}$$

i.e., the value of f on "all-or-none" partitions alone determines ψh, while the value of h on partitions which are not of this form is immaterial.

An attempt to understand this phenomenon may be the following: the axiom which should be held responsible for it is the interchangeability axiom (3): it focuses on the damage that a player may cause by deserting his/her coalition, and should two such players have the same "threat" power, they are given the same payoff. In a way, this axiom simply distinguishes between those players who do cooperate in some way (i.e., in some nontrivial coalition) and those who do not (singletons). The former have a viable threat, the latter do not. The precise way in which the "cooperative" players cooperate – i.e., via which coalition – does not matter; it only matters that they do. Hence, the payoff depends only on the best that the "cooperative" players may obtain – $h(P_c^A \cup P_f^{A^c})$ – where A is the set of "cooperative" ones.

Whether this property is desirable or not is debatable. We believe that in some situations it will be quite intuitive and will capture the essence of the cooperative global game, while in others it may well be inappropriate. Since axiom (3) seems innocent, yet guarantees uniqueness, we chose it to define "*the* Shapley value." However, one may certainly wish to consider other solution concepts, as suggested below.

Remark 5.1.3: One obvious alternative to the interchangeability axiom is the good old-fashioned symmetry: for a permutation $\pi: N \to N$ and $h \in F_0(\mathscr{P})$ define $\pi h \in F_0(\mathscr{P})$ by $(\pi h)(\{A_1,...,A_k\}) = h(\{\pi A_1,..., \pi A_k\})$, and for $x \in \mathbb{R}^N$ define $\pi x \in \mathbb{R}^N$ by $\pi x(i) = x(\pi i)$. Then we may define

3′. Symmetry: for every permutation $\pi: N \to N$, $\psi(\pi h) = \pi(\psi h)$.

It is easy to check that, in the presence of (1), (2) and (4), this axiom is strictly weaker than (3). More specifically, when defining ψ_{g_P}, one is restricted to assign $(\psi_{g_P})(i) = (\psi_{g_P})(j)$ if $|P(i)| = |P(j)|$ (where $P(i)$ is the member of P containing i) but players in coalitions of different sizes may get different payoffs.

5.2 The Core

For ordinary games, the core is a set of allocations $x \in \mathbb{R}^N$ no coalition may (unilaterally) improve upon. For global games, however, it is not entirely clear what is meant by "improve upon", since a coalition cannot act alone.

However, suppose that in a global environment clean-up game a certain set of countries is assigned a share in the total cost which exceeds what it would cost this set to perform the clean-up on its own. That is, suppose that

$$\Sigma_{i \in A}\, x(i) < h(P_c^A \cup P_f^{A^c}).$$

In such a case it would make sense for the coalition A to undertake the whole project by itself, and the allocation x is not "stable."

Next, suppose that two disjoint coalitions may get more if they operate alone, while the others do not cooperate. I.e.,

$$\Sigma_{i \in A}\, x(i) + \Sigma_{i \in B}\, x(i) < h(P_c^A \cup P_c^B \cup P_f^{(A \cup B)^c}), \text{ for } A \cap B = \emptyset.$$

And a similar argument excludes such an allocation x. Thus we are led to the following definition: $x \in \mathbb{R}^N$ is in the *core* of $h \in F_0(\mathscr{P})$ iff for every $P = \{A_1,\ldots,A_k\} \cup P_f^{A_{k+1}} \in \mathscr{P}$ with $|A_\ell| > 1$, $\ell = 1,\ldots,k$, the following condition holds:

$$\Sigma_{i \in \cup_{\ell=1}^k A_i}\, x(i) \geq h(P)$$

with equality for $P = \{N\}$.

Observation 5.2.1: If $h \in F_0(\mathscr{P})$ is monotone, then

$$\text{core}(h) = \text{core}(v_h)$$

Proof: The inclusion $\text{core}(h) \subseteq \text{core}(v_h)$ is immediate, while the converse inclusion is trivial (in the presence of monotonicity of h). □

Thus, the Shapley-Bondareva conditions for non-emptiness of the core of ordinary games (Bondareva (1963), Shapley (1967)), also characterize non-emptiness of the core of global games. Moreover, a convex global game has a nonempty core which includes its Shapley value (see Shapley (1971)).

Appendix

In Gilboa-Lehrer (1989) we introduced the following reflexive and symmetric binary relation on partitions: $P, Q \in \mathscr{P}$ are *non-intersecting* if for every $A \in P$ either (i) there is $B \in Q$ such that $A \subseteq B$, or (ii) there are $\{B_i\}_{i=1}^n \subseteq Q$ such that $A = \bigcup_{i=1}^n B_i$.

Observation A.1: For $P, Q \in \mathscr{P}$ the following are equivalent:

(i) P and Q are non-intersecting;
(ii) there exists $A \in \mathscr{B}(P) \cap \mathscr{B}(Q)$ such that $P^A \leq Q^A$ and $P^{A^c} \geq Q^{A^c}$;
(iii) $P \cup Q = (P \wedge Q) \cup (P \vee Q)$.

We quote the following from Gilboa-Lehrer (1989):

Fact A.2: A global game h is partially additive iff

$$h(P) + h(Q) = h(P \wedge Q) + h(P \vee Q)$$

for every non-intersecting partitions P and Q.

Finally, note that if h is partially additive, the condition $h = h_v$ does not define v uniquely. Indeed, $h_v = h_w$ iff $(v - w)$ is an additive game with $(v - w)(N) = 0$. (Recall that by the definition in Section 3, a game is additive iff

$$v(A) + v(B) = v(A \cup B) + v(A \cap B)$$

for every $A, B \subseteq N$, which is equivalent to

$$v(A) + v(B) = v(A \cup B)$$

whenever $A \cap B = \emptyset$).

In particular, given a partially additive $h \in F_0(\mathscr{P})$, there is a unique game v such that $h = h_v$ and $v(\{i\}) = 0$ for all $i \in N$. It is easily verifiable that this v coincides with v_h. Hence, h is partially additive iff $h = (h)_{v_h}$.

On the other hand, every game v with $v(\{i\}) = 0$ for all $i \in N$ satisfies $v = (v)_{h_v}$.

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