Allocation Processes in Cooperative Games

by

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First version: December 1997
November 21, 2001

Abstract.
In this paper we attempt to introduce dynamics into the theory of cooperative games. A solution of the game is reached through an allocation process. At each stage of the allocation process of a cooperative game a budget of fixed size is distributed among the players. In the first part of this note we study a type of process that, at any stage, endows the budget to a player whose contribution to the total welfare, according to some measurements, is maximal. It is shown that the empirical distribution of the budget induced by each process of the family converges to a least square value of the game, one such value being the Shapley value. The two other allocation processes converge to the core or the least core.

* I am grateful to the Associate Editor and to the two anonymous referees of *International Journal of Game Theory*

1. Introduction

Existing cooperative game theory is static in nature. There are no temporal aspects in the existing studies of solution concepts, power indices, coalition structures, etc. This paper is a humble attempt to introduce dynamics into the field in general and to the study of solution concepts in particular.

At any period in time a budget of fixed size is distributed among one or more players. The decision as to which players are to receive which portion of the budget depends on the historical allocation and is specified by a predetermined rule. The rule according to which the budget is distributed induces the empirical distribution of the budget among the players at any period. Whether this distribution converges, and if so to what limit, depends on the specific rule applied. The study of some allocation rules is the central theme of this paper.

Suppose that a research fund is to be allocated between individuals and institutions. At each period, in order to determine the desirable allocation, the fund management examines the needs of each individual and institution and compares it with what each of them received in the past. An allocation rule takes into account both the needs and the historical allocation when determining what portion of the fund each participant, individual or institution, is entitled to receive.

Using the language of cooperative game theory, the situation is described as follows. Let N be a set of players and v be the characteristic function. For every coalition S in N, the value $v(S)$ is interpreted as the needs of S. A budget of fixed size B is to be distributed among the players at any period t. Let x_t^i be the portion that player i received at time t, that is, the allocation at time t is $x_t = (x_t^i)_i$. An allocation rule specifies how to allocate B, taking into consideration all needs and past allocations. In other words, an allocation rule determines the allocation at time T, x_T, as a function of v and of all x_t, $t < T$. The sequence of allocations x_t, $t = 1, 2, ...$ is the allocation process induced by the allocation rule.

The empirical distribution of the budget among the players is at any stage an allocation of the budget. Thus, an allocation process generates a sequence of allocations. The main theme of this paper is to present three types of allocation rules and processes. We prove that the processes we introduce generate converging sequences of allocations. Furthermore, the limit allocations are well-known solution concepts.

The first type of process is implicitly based on the idea that giving the budget to a player increases the total well-being of the entire group. The player whose marginal contribution to this well-being is maximal will receive the budget. The well-being is measured at any stage with respect to the needs of the coalitions on one hand, and the empirical distribution of the budget, on the other.

It is shown that each of the processes of the first type generates allocations that converge to some least square value (Ruiz, Valenciano and Zarzuelo 1995, 1998). A distinguished least square value is the Shapley value. In other words, we introduce an allocation process that converges to the Shapley value.

The second type of process converges either to the core of the game when the core is not empty or to the least core. Convergence in this context means that any accumulation point of the empirical allocations is in the core (or in the least core). In other words, the distance between the core and the empirical sequence of allocations shrinks to zero.

The proofs related to the first two types of allocation processes rely on a simple geometric principle that lies behind Blackwell's approachability (Blackwell (1956) and Lehrer (1997)). Blackwell's approachability theorem is not used here. A by-product of the proofs is a new proof of the Shapley-Bondareva theorem (Bondareva (1963) and Shapley (1967)) that does not resort to any duality argument.

The last type of allocation process is based on a well-known fictitious-play algorithm (Robinson, 1951). The linear programming problem of finding a point in the least core is translated to a zero-sum game. The fictitious-play algorithm is then applied to the constructed zero-sum game. This algorithm induces an allocation process which has an interesting interplay between coalitions and players. It also has the advantage of finding the least core without knowing the precise inequalities that define it.

2. Allocation Rules and Allocation Processes

Let N be a finite set of players, where the number of players $|N|$ is n. Consider a normalized cooperative game v (i.e., $v(N) = 1$). Let $Z = \{(z_1, ..., z_n);\ z_i \geq v(i)$ for all i and $\sum_i z_i =$

$v(N)\}$. Z is the set of allocations.

An allocation rule is a function that, at any time t, determines the allocation z_t as a function of past allocations. Formally,

Definition 1. An *allocation rule* R is a function $R : \cup_{t=0}^{\infty} Z^t \to Z$, where Z^t is the Cartesian product of Z with itself t times and Z^0 is a singleton that represents the empty history of allocations.

An allocation rule can be considered as a strategy of an allocator in a repeated cooperative game.

An allocation rule R induces a sequence $z_1, z_2, ...$ of allocations as follows: z_1 is the first allocation which R attains on the set Z^0, $z_2 = R(z_1)$, $z_3 = R(z_1, z_2)$, etc. This sequence is called the *allocation sequence* induced by R.

3. Processes that Converge to the Least Square Value

3.1 The Least Square Value.

Denote by $\mathcal{L}$ the set of all additive games p (i.e., that satisfy $p(S) + p(T) = p(S+T)$ for any two disjoint coalitions S and T) such that $p(N) = 1$ and $p(i) \geq 0$ for every player i.

Let $\alpha = \{\alpha_S\}_{S \subseteq N}$ be a probability distribution over the set of coalitions. That is, α_S is the probability to choose the coalition S. Let $L_\alpha(v)$ be the game p in $\mathcal{L}$ that satisfies $p(N) = 1$ and achieves the minimum of

$$\sum_{S \subset N} \alpha_S (p(S) - v(S))^2.$$

So, the least square value, with respect to the probability distribution α, is the additive game that best approximates (in the sense of the Euclidean distance) the game v. In the language of linear algebra, the least square value is the projection (with respect to the probability distribution α) of v to the subspace of the additive games. For an elaboration on the subject, see Ruiz, Valenciano and Zarzuelo (1998).

3.2 The Approachability Principle.

The process we are about to describe converges to $L_\alpha(v)$. The convergence is guaranteed due to the following simple geometric observation. Let C be a closed and convex set in $\mathbb{R}^k$ and let $P(x)$ denote the closest point in C to any point x in $\mathbb{R}^k$. The set C is called the *target set*. Suppose that a sequence of uniformly bounded vectors x_t in $\mathbb{R}^k$ satisfies the condition

$$(1) \qquad \langle \overline{x}_t - P(\overline{x}_t), x_{t+1} - P(\overline{x}_t) \rangle \leq 0 \quad \text{for all} \quad \text{t},$$

where $\overline{x}_t$ is the average of $x_1, ..., x_t$ and $\langle \cdot, \cdot \rangle$ is the inner product. Then, the distance between $\overline{x}_t$ and $P(\overline{x}_t)$ converges to 0 as t goes to infinity. In other words, the average of the sequence x_t converges to the set C.

This is the geometric principle that lies behind Blackwell's Approachability Theorem (see Blackwell (1956) and Lehrer (1997)), not the Approachability Theorem itself. In this context, the principle bears no strategic aspect. It is purely geometric. Later on we will refer to it as the approachability principle.

Proposition 1. *Let D be a bounded, closed and convex set in $\mathbb{R}^k$. Suppose that a sequence of points x_t in D satisfies*

$$\langle \overline{x}_t, x_{t+1} \rangle \leq \langle \overline{x}_t, w \rangle \quad \text{for all} \quad \text{w} \in \text{D} \quad \text{and} \quad \text{for all} \quad \text{t}.$$

Then, $\overline{x}_t$ converges to $\operatorname{argmin}_{w \in D} \langle w, w \rangle$.

In words, if x_{t+1} is the lowest in D in the direction of the average $\overline{x}_t$ for every t, then $\overline{x}_t$ converges to the point in D whose Euclidean norm is minimal.

Proof: Let by $y = \operatorname{argmin}_{z \in D} \langle z, z \rangle$. Thus, y is the closest point (according to the Euclidean norm) in D to the origin. We will apply the approachability principle to the set $C = \{y\}$. Note that for any point z in D, $\langle y, y \rangle \leq \langle y, z \rangle$. In particular, $\langle y, x_{t+1} - y \rangle \geq 0$ for every t. On the other hand, by the assumption, $\langle \overline{x}_t, x_{t+1} \rangle \leq \langle \overline{x}_t, y \rangle$. Thus, $\langle \overline{x}_t, x_{t+1} - y \rangle \leq 0$. Subtracting the first inequality from the second, one obtains $\langle \overline{x}_t - y, x_{t+1} - y \rangle \leq 0$. Since $P(\overline{x}_t) = y$ for every t, the sequence x_t satisfies inequality (1). Hence $\overline{x}_t$ converges to the point y. ∎

3.3 The Allocation Rule and the Allocation Process.

We now use Proposition 1 to describe a process that converges to the additive game $L_\alpha(v)$. We start with the intuition.

For any time t denote by $\overline{z}_t$ the historical distribution of the budget up to time t. That is, $\overline{z}_t^i$ is the frequency of the stages up to time t, where player i received the budget. For any $S \subset N$, let $\overline{z}_t(S)$ be $\sum_{i\in S} \overline{z}_t^i$. At time t, the figure $\overline{z}_t(S) - v(S)$ measures the extra benefit of the coalition S up to time t, beyond its needs ($v(S)$). This is the weight given to coalition S. The smaller $\overline{z}_t(S)$ is, the greater the weight assigned to S. A coalition whose relative accumulation is big is assigned a negative weight, while a coalition whose relative accumulation is small is assigned a positive weight.

When a player, say i, is chosen to receive the budget at time $t+1$, the benefit to coalition S is $\mathbb{1}_{i\in S} - v(S)$. That is, the benefit to S is $1 - v(S)$ if i is in S, and $-v(S)$ otherwise. The weighted extra benefit of coalition S is therefore $(\overline{z}_t(S) - v(S))(\mathbb{1}_{i\in S} - v(S))$.

Suppose that α_S is the probability of choosing coalition S. The expected (with respect to the probability distribution over all the coalitions) weighted **extra** benefit if i receives the budget is therefore $\sum_S \alpha_S(\overline{z}_t(S) - v(S))\big(\mathbb{1}_{i\in S} - v(S)\big)$. In order to minimize this figure a player related to the minimal weighted extra benefit is chosen at time $t+1$ and is given the entire budget. In other words, at time $t+1$ a player who minimizes the expected weighted **extra** benefit receives the budget. A formal description of this process follows.

For any $i \in N$ let h_i be a vector in $I\!R^{2^n-1}$ whose coordinate corresponding to the coalition S is $\sqrt{\alpha_S}\big(\mathbb{1}_{i\in S} - v(S)\big)$. Define D to be the convex hull of $\{h_i\}_{i\in N}$. Note that every point in D corresponds to a convex combination of the h_i's and therefore to a specific allocation (recall, $v(N) = 1$). In particular, the closest point in D to the origin corresponds to the allocation $L_\alpha(v)$.

Let e_i be the i^{th} vector of the standard basis of $I\!R^n$.

Define the allocation rule R_1 inductively as follows. The first allocation chosen by R_1, z_1, is an arbitrary standard basis vector. That is, $z_1 = e_{i_1}$ for some arbitrary i_1. Suppose that the allocations $z_1, z_2, z_3, ..., z_{t-1}$, all standard basis vectors, have been chosen. In other words, for every $r \le t$ there is a player i_r such that $z_r = e_{i_r}$. Denote $x_r = h_{i_r}$ for every $r \le t$ and $\overline{x}_t$ the average of $x_1, ..., x_t$. Let $h_{i_{t+1}}$ be an extreme point in D that minimizes $\langle \overline{x}_t, \cdot \rangle$ (if there is more than one extreme point in D that achieves the minimum, one may choose either one).

Note that $\overline{x}_t(S) = \sqrt{\alpha_S}\big(\overline{z}_t(S) - v(S)\big)$. Therefore, $\langle \overline{x}_t, h_i \rangle = \sum_S \alpha_S(\overline{z}_t(S) - v(S))\big(\mathbb{1}_{i \in S} - v(S)\big)$. Thus, the extreme point of D, h_i, that minimizes the inner product on the left side of this equation indeed corresponds to the player that minimizes the expected extra benefit (i.e., the sum on the right side), meaning that the allocation rule R_1 bears the intuition provided earlier.

Define $R_1(z_1, z_2, z_3, ..., z_{t-1}) = e_{i_t}$ for every t.

Theorem 1. *Let $z_1, z_2, ...$ be the allocation process induced by R_1. Then, $\overline{z}_t$ converges to the least square value of the game that corresponds to the weights α_S, $S \subseteq N$.*

Proof: Denote $x_t = h_{i_t}$ for every period t. Since D is the convex hull of the h_i's, $h_{i_{t+1}} = \mathrm{argmin}_{w \in D} \langle \overline{x}_t, w \rangle$. R_1 is so defined as to obtain $\langle \overline{x}_t, x_{t+1} \rangle \leq \langle \overline{x}_t, w \rangle$ for all w ∈ D and for all t. Therefore, the sequence x_t satisfies the condition of Proposition 1. This ensures that the sequence of averages $\overline{x}_t$ converges to $\mathrm{argmin}_{w \in D} \langle w, w \rangle$. Hence, $\overline{z}_t$ converges to $L_\alpha(v)$. ∎

A dual interpretation of this process is the following. A coalition is chosen randomly according to the distribution $(\alpha_S)_S$. At time $t+1$ the coalition S is assigned a weight proportional to the excess corresponding to the allocation $\overline{z}_t$, $\overline{z}_t(S) - v(S)$. At any time a player whose contribution to the expected weighted welfare of society $\sum_S \alpha_S(\overline{z}_t(S) - v(S))\big(\mathbb{1}_{i \in S} - v(S)\big)$ is maximal, is chosen. It is the **expected** welfare due to the random selection of coalitions (the α_S component) and it is the **weighted** one due to the weights, $\overline{z}_t(S) - v(S)$ that appear in this expression. This player receives the entire budget and is denoted player i_{t+1}.

3.4 The Process that Converges to the Shapley Value.

Keane (1969) proved that if coalition S is chosen with probability $\alpha_S = c\frac{(|S|-1)!(|N|-|S|-1)!}{(|N|-2)!}$, where c, the normalization factor, is equal to $\frac{1}{n-1}(\sum_{j=1}^{(|N|-1} \frac{(|N|}{j(|N|-j)})$, then $L_\alpha(v)$ is the Shapley value. Understanding the logic behind these probabilities, which is the purpose of this subsection, may enhance the intuition behind the allocation process that converges to the Shapley value.

In determining, at any specific period, which player is entitled to get the entire budget, it is enough to compare only pairs of players. When players, say i and j, are compared with each other, only coalitions that contain one and not the other matter. The reason is that as far as coalition S is concerned, when a player receives the budget, all players of S benefit equally. Thus, i and j benefit equally whether both i and j are in S, or not. In other words, only coalitions that contain only one of i and j make a difference.

Let S be a coalition that contains either i or j (but not both). Subtracting i, j from this coalition, one obtains a coalition of size $|S|-1$ in $N \setminus \{i,j\}$. Thus, selecting a coalition of size s that contains, say i and not j, is equivalent to choosing a coalition of size $s-1$ from the set $N \setminus \{i,j\}$.

The α_S's are determined by the following procedure according to which coalitions from $N - \{i,j\}$ are selected. An order of the players in $N - \{i,j\}$ is randomly chosen and then a cutoff point of the order is randomly selected. Both selections (of the order and of the cutoff point) are taken according to the uniform distribution. Thus, coalitions of the same size have the same probability. Furthermore, the total probability of all coalitions of a certain size is constant across sizes. Therefore, the probability of selecting a coalition of size $|S|-1$ from the set $N - \{i,j\}$ is $\alpha_S = c\frac{(|S|-1)!(|N|-|S|-1)!}{(|N|-2)!}$.

4. Processes that Converge to the Core and to the Least Core

4.1 The Process that Converges to the Core.

The process described here also utilizes the approachability principle. Let the target set C be the non-negative orthant of $I\!R^k$. When applied to C, the approachability principle obtains the following form. If a sequence of uniformly bounded vectors x_t, $t = 1, 2...$ in $I\!R^k$ satisfies the condition[1]

$$\langle \min(\overline{x}_t, 0), x_{t+1} \rangle \leq 0 \quad \text{for} \quad \text{all} \quad t,$$

then $\min(\overline{x}_t, 0)$ converges to 0.

Let $S_1, ..., S_k$, where $k = 2^n - 1$, be the list of all non-empty coalitions. Denote by y_i the vector in $I\!R^k$ whose ℓ^{th} coordinate is $\mathbb{1}_{i \in S_\ell} - v(S_\ell)$.

[1] For two vectors $x, y \in I\!R^n$, $\min(x,y)$ denotes the coordinate-wise minimum of the two.

Definition 2. We say that v is balanced if for every k-dimensional vector, $\alpha = (\alpha_\ell)_{\ell=1}^k$, with α_ℓ being a non-negative number for every ℓ, there is a player i such that $\langle \alpha, y_i \rangle \geq 0$.

The Shapley-Bondareva theorem (Bondareva (1963) and Shapley (1967)) states that the core of the cooperative game v is non-empty if and only if v is balanced.

We assume that v is balanced and construct a process the generates at least one point in the core. The argument involves the approachability principle alone. No separation theorem is needed. In other words, what follows establishes a proof of the non-trivial part of the Shapley-Bondareva theorem that does not rely on any duality argument.

We define the allocation rule R_2 inductively. Alongside this definition two sequences will be defined: the allocation process $z_1, z_2, ...$ of vectors in $I\!R^n$ and an auxiliary sequence $x_1, x_2, ...$, of vectors in $I\!R^k$. Let i_1 be an arbitrary player. Set $z_1 = e_{i_1}$ and $x_1 = y_{i_1}$. Suppose that the sequences $z_1, ..., z_t$ and $x_1, ..., x_t$ have been defined so that for every $r \leq t$ there is a player i_r such that $z_r = e_{i_r}$ and $x_r = y_{i_r}$. Furthermore,

$$\langle \min(\overline{x}_r, 0), x_{r+1} \rangle \leq 0 \quad \text{for} \quad \text{all} \quad r < t.$$

By the assumption that v is balanced, there is a player i_{t+1} such that

$$\langle \min(\overline{x}_t, 0), y_{i_{t+1}} \rangle \leq 0.$$

Define $R_2(z_1, ..., z_t) = e_{i_{t+1}}$ and set $x_{t+1} = y_{i_{t+1}}$.

Theorem 2. *Let $z_1, z_2, ...$ be the allocation process induced by R_2. Then, $\overline{z}_t$ converges to the core of the game. That is, any accumulation point of the sequence $\overline{z}_t$ is in the core.*

Proof: By construction, the sequence $x_1, x_2, ...$ satisfies the condition of the approachability principle with the target set being $\{0\}$ (in $I\!R^k$). Therefore, $\min(\overline{x}_t, 0)$ converges to 0. This means that any accumulation point of $\overline{x}_t(\ell)$ is greater than or equal to 0.

The ℓ^{th} coordinate of $\overline{x}_t$, denoted $\overline{x}_t^\ell$, is $\frac{1}{t}\sum_{r=1}^t \mathbb{1}_{i_r \in S_\ell} - v(S_\ell)$. Thus, for every ℓ $\liminf_t \overline{x}_t^\ell \geq 0$. That is, for every ℓ $\liminf_t \frac{1}{t}\sum_{r=1}^t \mathbb{1}_{i_r \in S_\ell} \geq v(S_\ell)$. In other words, for every ℓ $\liminf_t \sum_{i \in S_\ell} \overline{z}_t^i \geq v(S_\ell)$.

Since all $\overline{z}_t$ are in the unit simplex, so are all the accumulation points. Let $\overline{z}$ be an accumulation point of the sequence $\overline{z}_t$, $t = 1, 2, \ldots$. The point $\overline{z}$ satisfies $\sum_{i \in S_\ell} \overline{z}^i \geq v(S_\ell)$ for every ℓ. Thus, $\overline{z}$ is in the core. ∎

The intuition behind this process is the following. The figure $\overline{x}_t^\ell$ measures the historical average surplus of the coalition S_ℓ up to stage t. At this stage the coalitions are weighted with respect to these surpluses: Those coalitions with a positive surplus are neglected while the other coalitions are assigned a weight proportional to their (negative) surplus (that is, for such a coalition, say, S_ℓ, the weight is $-\min(\overline{x}_t(\ell), 0)$). Then, a player i whose weighted actual surplus (i.e., $\sum_{\ell=1}^k \Big(-\min(\overline{x}_t^\ell, 0)\Big)\Big(\mathbb{1}_{i \in S_\ell} - v(S_\ell)\Big)$) is non-negative is chosen and is given the entire budget $v(N)$. The vector $\overline{z}_t$ is the historical distribution of the budget up to time t. Any limit point of the sequence $\overline{z}_t$ is in the core.

It is worth emphasizing that there is no guarantee that every point in the core is approached by the sequence $\overline{z}_t$.

Note that this process provides an algorithm for approximating the core: $\overline{z}_t$ approaches the core at the rate of $\frac{1}{\sqrt{t}}$. In other words, the distance of $\overline{z}_t$ to the core is $O(\frac{1}{\sqrt{t}})$. Note also that whenever v is balanced, one can find a sequence $\overline{z}_t$ whose accumulation points are in the core. Once again, since the process defined does not depend on any dual argument or separation theorem, it establishes another proof of the difficult part of the Shapley-Bondareva theorem.

4.1 The Process that Converges to the Least Core.

In constructing the allocation process that converges to the core we assumed that the game is balanced. When the game is not balanced the allocation rule R_2 is not defined. However, R_2 can be modified to fit this case.

For any ε (positive or negative), the ε-core, C_ε of the game is the set of all payoffs (i.e., vectors in Z) z such that $z(S) = \sum_{i \in S} z^i \geq v(S) - \varepsilon$ for every $S \neq \emptyset, N$. It is clear that the ε-core correspondence (as a function of ε) is monotonically increasing with respect to inclusion. The least core, denoted LC, is the intersection of all non-empty ε-cores.

Definition 3. We say that v is ε-balanced if for every k-dimensional vector $\alpha = (\alpha_\ell)_{\ell=1}^k$, with α_ℓ being a number greater than or equal to ε for every ℓ, there is a player i such that

$\langle \alpha, y_i \rangle \geq 0$.

For any ε let $\overrightarrow{-\varepsilon}$ denote the n-dimensional vector $(-\varepsilon, ..., -\varepsilon)$.

Suppose that the game v is ε-balanced. Given this information, we can modify the definition of R_2 as follows. At period t, x_{t+1} is chosen so as to satisfy

$$\langle \min(\overline{x}_t, \overrightarrow{-\varepsilon}), x_{t+1} \rangle \leq 0.$$

The approachability principle ensures that for every ℓ,

$\liminf_t \overline{x}_t^\ell \geq -\varepsilon$. That is, for every ℓ $\liminf_t \frac{1}{t} \sum_{r=1}^{t} \mathbb{1}_{i_r \in S_\ell} \geq v(S_\ell) - \varepsilon$. In other words, if the induced allocation process is $z_1, z_2, ...$, then for every ℓ $\liminf_t \sum_{i \in S_\ell} \overline{z}_t^i \geq v(S_\ell) - \varepsilon$. A similar argument to that appearing in the proof of Theorem 2 shows that any accumulation point of this allocation process is in the ε-core.

5. Another Process Converging to the Least Core

In the previous section we presented an allocation process whose accumulation points are in the ε-core, provided that the game is ε-balanced. Without this information, we could not define the corresponding allocation rule. The advantage of the process defined in this section is that it does not rely on the information regarding the ε-balancedness. There is no need to know for what ε's the game is ε-balanced. The identity of the ε_0 that satisfies $LC = C_{\varepsilon_0}$ is eventually revealed by the process.

For this reason I find this section important, although it merely involves a re-interpretation of well-known results.

5.1 Converting the Problem of Finding the Least Core to a Zero-Sum Game.

Consider a $0-1$ normalized cooperative game v (i.e., $v(i) = 0$ for every player i and $v(N) = 1$, where N is the set of all players).

In order to avoid confusion with the players of the cooperative games under discussion, the players of the non-cooperative game about to be defined will be called agents.

Let G be the following non-cooperative two-agent zero-sum game. The row agent (the maximizer) chooses $i \in N$ and the column agent (the minimizer) chooses a coalition $S \subseteq N$ providing that $S \neq \emptyset, N$. Thus, the matrix of G is $n \times 2^n - 2$. The payoff corresponding to the pair (i, S) is $\mathbb{1}_{i \in S} - v(S)$, where $\mathbb{1}$ is the characteristic function. The payoff when (i, S) is the pair of strategies chosen, is the net payoff the players in S receive beyond the worth of S, $v(S)$.

5.2 The Allocation Process that Converges to the Least Core.

We define the allocation rule R_3 using the fictitious play algorithm (Robinson (1951)) to G. At stage 1 an arbitrary player i_1 is chosen (and given the whole budget, $v(N)$). That is, the first allocation z_1 is e_{i_1} for some player i_1. Furthermore, an arbitrary coalition S_1 is chosen.

Suppose that up to time $t-1$ the players $i_1, ..., i_{t-1}$ and the coalitions $S_1, ..., S_{t-1}$ were chosen, so that for every $r \leq t-1$, $z_r = e_{i_r}$. Denoting[2] $g_r(S) = \frac{\sum_{u<r} \mathbf{1}_{i_u \in S}}{r-1} - v(S)$ and $f_r(i) = \frac{\sum_{u<r} \mathbf{1}_{i \in S_u} - v(S_u)}{r-1}$, $2 \leq r \leq t-1$, it is assumed that for every $r \leq t-1$, S_r is a coalition that minimizes $g_r(S)$ and the player i_t is chosen so as to maximize $f_r(i)$.

At time t, set S_t to be a coalition that minimizes $g_t(\cdot)$ and let the player i_t maximize $f_t(\cdot)$. That is, define $R_3(z_1, ..., z_{t-1}) = e_{i_t}$.

Theorem 3. *Let $z_1, z_2, ...$ be the allocation process induced by R_3. Then, $\overline{z}_t$ converges to the least core of the game. That is, any accumulation point of the sequence $\overline{z}_t$ is in the least core.*

Proof: Since the allocation rule R_3 follows the algorithm of Robinson (1951), the empirical distribution of the agents $i_1, ..., i_t$, which is $\overline{z}_t$, converges to an optimal strategy of the row agent. Note that the set of optimal strategies of the row agent in G is the least core of v. ∎

The interpretation of the allocation rule R_3 is the following. Given the historical distribution of wealth up to time t, the net wealth of coalition S is $g_t(S) = \frac{\sum_{u<t} \mathbf{1}_{i_u \in S}}{t-1} - v(S)$. The most deprived coalition is S_t. Thus, $S_1, S_2, ..., S_t$ is the list of all historical, most-deprived coalitions.

On the other hand, $f_r(i) = \frac{\sum_{u<t} \mathbf{1}_{i \in S_u} - v(S_u)}{t-1}$ is the average contribution of player i to the wealth of the historical most-deprived coalitions, if i is the one who receives the entire

[2] In traditional terms of cooperative game theory $g_t(S)$ is referred to as the minus of the empirical *excess* of coalition S.

budget. The player chosen at time t, i_t, maximizes this average contribution.

5.3 A Slightly Different Process.

One can deal with another non-cooperative game $\overline{G}$. Agent 1's set of actions is as in G. Agent 2 chooses a coalition S which is not the grand coalition nor the empty one, such that $v(S) > 0$. Let the payoff corresponding to (i, S) be $\frac{\mathbf{1}_{i \in S}}{v(S)}$.

We adapt the fictitious play algorithm (Robinson (1951)) to $\overline{G}$ and define the allocation rule R_4. At stage 1 an arbitrary player i_1 is chosen (and given the whole budget, $v(N)$). That is $z_1 = e_{i_1}$. Furthermore, an arbitrary coalition S_1 is chosen. Suppose that up to time $t-1$ the players $i_1, ..., i_{t-1}$ and the coalitions $S_1, ..., S_{t-1}$ were chosen and $z_r = e_{r_1}$, $r \leq t-1$. Define, $\overline{g}_t(S) = \frac{\frac{1}{t-1}\sum_{r<t} \mathbf{1}_{i_r \in S}}{v(S)}$, and $\overline{f}_t(i) = \frac{1}{t-1}\sum_{r<t} \frac{\mathbf{1}_{i \in S_r}}{v(S_r)}$.

The coalition at time t, S_t, minimizes $\overline{g}_t(S)$ and the player i_t is chosen so as to maximize $\overline{f}_t(i)$. In other words, $R_4(z_1, ..., z_{t-1}) = e_{i_t}$.

In this case the extent to which a coalition is deprived up to time t is measured by the ratio of the total cumulative wealth of a coalition and its value. Here again, the coalition chosen is one of the most deprived ones. On the other hand, the player chosen at time t, i_t, maximizes the sum of the ratios $\frac{1}{v(S)}$ over all the coalitions that were chosen in the past.

As in Theorem 3, the allocation process induced by R_4 converges to the least core.

6. Final Remarks

6.1. We described a few allocation rules that induce allocation processes that converge to well-known solution concepts. None of these rules has been obtained axiomatically. It would be interesting to find appealing axioms that are based on the dynamic aspect of the game and yield a converging allocation process.

6.2. In Section 4 we described processes that converge to the core when it is not empty. As noted, not all the points in the core are necessarily accumulation points of the process. What characterizes those core points that can be approximated by allocation processes, is a matter for further study.

7. References

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